Institute of Distance and Open Learning (IDOL)
University of Mumbai.

F.Y.B Com

MATHEMATICAL AND STATISTICAL TECHNIQUES

SEM- II

CONTENT

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SYLLABUS

Unit I

Functions Derivatives and Their Applications

Concept of real functions: constant function, linear function, x^2, e^x, a^x,log x, Demand, Supply, Total Revenue, Average Revenue, Total Cost, Average Cost and Profit function. Equilibrium Point. Derivative as rate measure. Derivatives of functions: Constant function, x^n, e^x, a^x, $log\ x$ Rules of derivatives: Scalar multiplication, sum, difference, product, quotient, simple problems. Second Order derivatives. Applications: Marginal Cost, Marginal Revenue, Elasticity of Demand. Maxima and inima for functions in Economics and Commerce.

Unit II

Interest and Annuity

Simple Interest and Compound Interest Interest Compounded more than once a year. Calculations involving up to 4 time periods. Equated Monthly Instalments (EMI) using reducing & flat interest system. Present value, Future value.

Annuity, Immediate and due: Simple problems with $A = P\left(1 + \frac{r}{100}\right)^n$ $with\ n \leq 4$.

Unit III

Bivariate Linear Correlation: Scatter Diagram, Computation of Karl Pearson's Coefficient of Correlation (Case of Bivariate Frequency Table to be excluded), Computation of Spearman's Rank Correlation Coefficient (case of repeated ranks up to 2 repetition only) Bivariate Linear Regression: Finding Regression lines by method of least squares.

Properties of Regression Coefficients – $i)\ r = \pm\sqrt{b_{yx}b_{xy}}$ $ii)$ (x, y) is a point of intersection of two regression lines.

Unit IV

Times Series: Concept and Components of time series. Estimation of Trend using Moving Average Method & Least Squares Method (only Linear Trend) Estimation of Seasonal Component using Simple Arithmetic Mean. (For Trend free data only) Concept of Forecasting using Least Squares Method.

Index Numbers: Concept and uses. Simple and Composite Index Nos. (unweighted, weighted) Laspeyre's Price Index No., Paasche's Price Index No. Fisher's Price Index No., Cost of Living Index No., Real Income, Simple Examples of Wholesale price Index no. (Examples on missing values should not be done)

Unit V

Decision Theory : Decision making situation; Decision maker, Courses of Action, States of Nature, Pay-off and Pay-off matrix; Decision making under Uncertainty, Maximum, Maximax and Laplace criteria, simple examples to find optimum decision.

Decision making under Risk Expected Monetary Value (EMV)l Decision tree, simple examples based on EMV

UNIT VI

9

FUNCTIONS, DERIVATIVES AND THEIR APPLICATIONS

OBJECTIVES

After reading this chapter you will be able to recognize.

1) Definition of function.
2) Standard Mathematical function.
3) Definition of derivative.
4) Derivatives of standard functions.
5) Second order derivatives.
6) Application of derivatives.
7) Maxima and Minima.

9.1 FUNCTIONS

If $y= f(x)$ is a function then the set of all values of x for which this function is defined is called the domain of the function f. Here x is called an independent variable and y is called the dependent variable. The set of all corresponding values of y for x in the domain is called the range of the function f.

The function f is defined from the domain to the range.

We shall discuss only those functions where the domain and the range are subsets of real numbers. Such functions are called 'real valued functions'.

9.1.1 Standard Mathematical Functions:

(1) Constant function:

The constant function is defined by

$y=f(x)= C$ where C is a constant.

The constants are denoted by real numbers or alphabets. The graph of a constant function is a straight line parallel to x-axis.

Examples:

$y=f(x)= 5$

$y=f(x)= -10$

$y=f(x)=K$

$y=f(x)=a$

(2) Linear function:

The linear function is defined by $y=f(x)= ax+ b$ where a and b are constants.

Examples: y= $f(x) = 2x+5$

y= $f(x) = -3x+10$

y= $f(x) = 5x-7$

(3) Functions with power of x:

A function $f(x) = x^n$ is called power function or function with power of x. Here x is called base and n is called power .

Examples : $f(x) = x^2$

$f(x) = x^{-5}$

$f(x) = x^{-4/3}$ $f(x) = x^{3/2}$

(4) Exponential functions :

These functions are of the type .

$f(x) = e^x$ and

$f(x) = a^x$, a >0

(5) Logarithmic function : The logarithmic function is defined by

y= $f(x) = \log_e^x$, x >0

9.1.2. Standard functions from Economics :

(1) Demand : It refers to the quantity of a product is desired by the buyers . The demand depends on the price. Therefore, there is a relationship between the price and the quantity demanded. Such relationship is called a demand function.

Hence the demand function is defined as

D= g (p) where D= demand and p= price .

Here demand is a dependent variable and the price is an independent variable.

For example, $D = 50 + 4p - 3p^2$

(2) Supply : It refers to the quantity of a product , the market can offer . The supply depends on the price. Therefore, There is a relationship between the price and the quantity supplied. Such relationship is called a supply function.

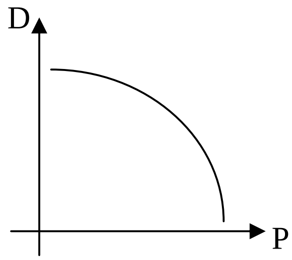
Hence the supply function is defined as S = f (p) where S = supply and p = price.

Here supply is a dependent variable and price is an independent variable.

For example, $S = 2p^2 - 6p + 25$

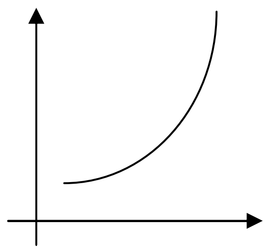
(3) Break-even Point : Equilibrium point.

(i) By the law of demand, the demand decreases when the price increases, the demand curve is a decreasing curve as shown in the figure:



D= f(p)
The demand curve

(ii) By the law of supply , the supply increases when the price increases , the supply curve is an increasing curve as shown in the figure .

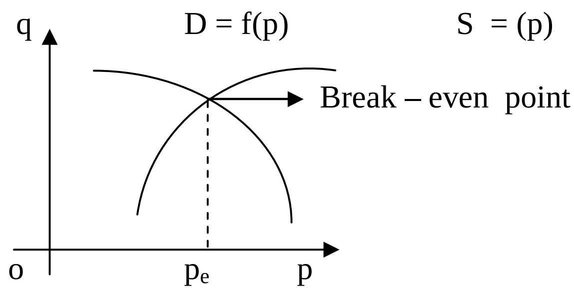


S = g(p)
The supply curve

(iii) The demand and supply curves D= f(p) and S= g(p) are intersecting at a point . The point of intersection of the demand and supply curves represents that specific price at which the demand and supply are equal. This point is called the Break-even point or equilibrium point. The corresponding price at which this point occurs is called an equilibrium price and is denoted by p_e

At equilibrium price, the amount of goods supplied is equal to the amount of goods demanded.

q= f(p) =D
q= g(p) =S



equilibrium price - Break-even point

(4) The total cost function :

The total cost function or cost function is denoted by C and it is expressed in terms of x. If C is the cost of producing x units of a product , then C is generally a function of x and is called the total cost function .

i.e. C= f (x)
For example , C= $2x^2-5x + 10$

(5) Average cost function :

The ratio between the cost function and the number of units produced is called average cost function. i.e. $AC = \frac{C}{x}$

For example , $AC = \frac{x^2 +2x +5}{x}$

(6) Total Revenue function :

The total revenue function is defined as in terms of the demand and the price per item. If D units are demanded with the selling price of p per unit , then the total revenue function R is given by

R = p x D where p = price and D= demand

For example ,

If $D = p^2 + 2p + 3$ then $R = p \times (p^2 + 2p + 3) = p^3 + 2p^2 + 3p$

(7) Average revenue :

Average revenue is defined as the ratio between the revenue and the demand and is denoted by AR.

i.e. , AR =R/D AR = p x D/D (as R= p x D)

$\therefore$ AR = p

$\therefore$ Average revenue is nothing but the selling price per unit.

(8) The Profit function :

The profit function or the total profit function is denoted by P and is defined by the difference between the total revenue and the total cost.

$\therefore$ Total Profit = Total Revenue - Total cost

i.e. P = R - C

Example 1:

Find the total profit function if the cost function $C = 40 + 15x - x^2$, x = number of items produced and the demand function is $p = 200 - x^2$

Solution :

Given $C = 40 + 15x - x^2$

$p = 200 - x^2$

$R = p \times D \quad (D = x)$

$= (200 - x^2)\, x$

$\therefore \quad R = 200x - x^3$

Profit = Revenue - Cost

$P = R - C$

$= (200x - x^3) - (40 + 15x - x^2)$

$= 200x - x^3 - 40 - 15x + x^2$

$P = 185x + x^2 - x^3 - 40$

Example 2: The total cost function is $C = 20 - 3x^2$ and the demand function is $p = 5 + 6x$. Find the profit when $x = 100$.

Solution :

Given : $C = 20 - 3x^2$

$R = p \times D \quad (D = x)$

$= p \times x$

$= (5 + 6x)\, x$

$= 5x + 6x2$

Profit = Revenue - Cost

$= R - C$

$= (5x + 6x^2) - (20 - 3x^2)$

$= 5x + 6x^2 - 20 + 3x^2$

$= 5x + 9x^2 - 20$

When $x = 100$, $P = 5(100) + 9(100)^2 - 20$
$= 500 + 90000 - 20$
$= 90480.$

9.2 DERIVATIVES

9.2.1 Derivative as rate measure :

Definition : Let $y = f(x)$ be the given function .

If $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists ,

then we say that the function $f(x)$ has derivative at x and is denoted by $f'(x)$.

i.e. , $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

The rate of change is called the " derivative " of $y = f(x)$ with respect to x and is denoted by $\frac{dy}{dx}$ or $f'(x)$.

$\frac{dy}{dx}$ = the rate of change of y with respect to x or the derivative of y with respect to x .

Note : (1) Derivative means " rate of change "

(2) The process of finding the derivative of a function is called " differentiation".

$\frac{dc}{dx}$ = the rate of change cost with respect to x .

For example $\frac{dD}{dp}$ = the rate of change of demand with respect to p .

9.2.2 Derivatives of Standard functions :

(1) If $y = x^n$, where n is a real number , then

$\frac{dy}{dx} = n x^{n-1}$

i.e. , $\boxed{\frac{dy}{dx} = \frac{d(x^n)}{dx} = nx^{n-1}}$

(2) If $y = C$, where C is a constant ,

then $\frac{dy}{dx} = 0$

i.e. , $\boxed{\frac{dy}{dx} = \frac{d(C)}{dx} = 0}$

(3) If $y = e^x$, then $\frac{dy}{dx} = e^x$

i.e. , $\boxed{\frac{dy}{dx} = \frac{d(e^x)}{dx} = e^x}$

(4) If $y = a^x$, where a is a positive real number, then

$$\frac{dy}{dx} = a^x \log a$$

i.e., $\frac{dy}{dx} = \frac{d(a^x)}{dx} = a^x \log a$

(5) If y= log x, then $\frac{dy}{dx} = \frac{1}{x}$

where $x > 0$

i.e., $\frac{dy}{dx} = \frac{d(\log x)}{dx} = \frac{1}{x}$

Examples :

(1) $y = x = x'$ $\therefore \frac{dy}{dx} = 1.x^{1-1} = 1.\,x^0 = 1$

(2) $y = x^4$ $\therefore \frac{dy}{dx} = 4.x^{4-1} = 4.\,x^3$

(3) $y = x^{10}$ $\therefore \frac{dy}{dx} = 10.x^{10-1} = 10x^9$

(4) $y = \frac{1}{x} = x^{-1}$ $\therefore \frac{dy}{dx} = -1.x^{-1-1} = -1x^{-2} = -\frac{1}{x^2}$

(5) $y = \frac{1}{x^3} = x^{-3}$ $\therefore \frac{dy}{dx} = -3\,x^{-3-1} = -3x^{-4} = \frac{-3}{x^4}$

(6) $y = \sqrt{x} = x^{1/2}$ $\therefore \frac{dy}{dx} = \frac{1}{2}\,x^{1/2-1} = \frac{1}{2}\,x^{-1/2} = \frac{1}{2\,x^{1/2}} = \frac{1}{2\sqrt{x}}$

(7) $y = x^{5/2}$ $\therefore \frac{dy}{dx} = \frac{5}{2}\,x^{5/2-1} = \frac{5}{2}\,x^{3/2}$

(8) $y = x^{-3/2}$

$\therefore \frac{dy}{dx} = -\frac{3}{2}\,x^{-3/2-1} = -\frac{3}{2}\,x^{-5/2}$

(9) $y = x^{-7/2}$

$\frac{dy}{dx} = -\frac{7}{2}\,x^{-7/2-1}$

$= -\frac{7}{2}\,x^{-9/2}$

(10) y=5 , 5 is a constant

$\therefore \frac{dy}{dx} = 0$

(11) y=K , $\frac{dy}{dx} = 0$

(12) y= log2 , $\frac{dy}{dx} = 0$

(13) y= -10 $\therefore \frac{dy}{dx} = 0$

(14) y= e^x $\therefore \frac{dy}{dx} = e^x$

(15) y= 2^x $\therefore \frac{dy}{dx} = 2^x \log 2$

(16) y= 10^x $\therefore \frac{dy}{dx} = 10^x \log 10$

(17) y= $\log x$ $\therefore \frac{dy}{dx} = \frac{1}{x}$

Exercise: 9.1

Find $\frac{dy}{dx}$ for the following :

(1) $y = x^6$
(2) $y = \frac{1}{x^2}$
(3) $y = x^{7/2}$
(4) $y = x^{-5/2}$
(5) y = 3
(6) y = log 10
(7) y = -8
(8) $y = 4^x$
(9) $y = 9^x$
(10) $y = 15^x$

Answers :

(1) $6x^5$ (2) $\frac{-2}{x^3}$ (3) $\frac{7}{2}x^{5/2}$ (4) $-\frac{5}{2}x^{-7/2}$ (5) 0 (6) 0 (7) 0 (8) $4^x \log 4$
(9) $9^x \log 9$ (10) $15^x \log 15$

9.2.3 Rules of derivatives :

Rule : 1 Addition Rule (or) Sum rule :

If y= u + v where u and v are differentiable functions of x then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

i.e, $$\boxed{\frac{dy}{dx} = \frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{du}{dx}}$$

Examples:

(1) If $y = x^2 + e^x$, find $\frac{dy}{dx}$

Solution : Given : $y = x^2 + e^x$

$$\frac{dy}{dx} = \frac{d(x^2 + e^x)}{dx}$$

$$= \frac{d(x^2)}{dx} + \frac{d(e^x)}{dx}$$

$$\therefore \frac{dy}{dx} = 2x + e^x$$

(2) If $y = x^{10} + \log x$, find $\frac{dy}{dx}$

Solution: Given : $y = x^{10} + \log x$

$$\frac{dy}{dx} = \frac{d}{dx}[x^{10} + \log x]$$

$$= \frac{d}{dx}(x^{10}) + \frac{d}{dx}(\log x)$$

$$\therefore \frac{dy}{dx} = 10x^9 + \frac{1}{x}$$

Rule:2 Subtraction Rule (or) Difference rule:

If y= u- v where u and v are differentiable functions of x then

$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

i.e, $\boxed{\frac{dy}{dx} = \frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}}$

Examples :

(1) If $y = x^5 - 2x$, find $\frac{dy}{dx}$

Solution : Given : $y = x^5 - 2^x$

$$\therefore \frac{dy}{dx} = \frac{d(x^5 - 2^x)}{dx}$$

$$= \frac{d}{dx}(x^5) - \frac{d}{dx}(2^x)$$

$$\therefore \frac{dy}{dx} = 5x^4 - 2^x \log 2$$

(2) If $y = 100 - \log x$, find $\frac{dy}{dx}$

Solution : Given : $y = 100 - \log x$

$$\frac{dy}{dx} = \frac{d}{dx}(100 - \log x)$$

$$= \frac{d}{dx}(100) - \frac{d}{dx}(\log x)$$

$$= 0 - \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{x}$$

Rule : 3 Product Rule :

If $y = uv$ where u and v are differentiable functions of x ,

then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

i.e. ,

$$\boxed{\frac{dy}{dx} = \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$$

Examples :

(1) If $y = x^4 \log x$, find $\frac{dy}{dx}$

Solution : Given $y = x^4 \log x$

$u = x^4$, $v = \log x$

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 \log x)$$

$$= x^4 \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(x^4)$$

$$= x^4 (1/x) + \log x\,(4x^3)$$

$$= x^3 + 4x^3 \log x$$

$$\therefore \frac{dy}{dx} = x^3 [\, 1 + 4 \log x \,]$$

(2) If $y = x^2 e^x$, find $\frac{dy}{dx}$

Solution : Given $y = x^2 e^x$

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^2)$$

$$= x^2 e^x + e^x(2x)$$

$$\therefore \frac{dy}{dx} = e^x [\, x^2 + 2x \,]$$

Rule :4 Quotient Rule :

If $y = \frac{u}{v}$, $v \neq 0$ where u and v are differentiable functions of x , then

$$\frac{dy}{dx} = \left[\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \right]$$

Examples :

(1) If $y = \frac{x + 4}{\log x}$, find $\frac{dy}{dx}$

Solution: Given $y = \frac{x + 4}{\log x}$

Here $u = x + 4$

$v = \log x$

$$\therefore \frac{dy}{dx} = \frac{\log x \frac{d}{dx}(x + 4) - (x + 4)\frac{d}{dx}(\log x)}{(\log x)^2}$$

$$= \frac{(\log x)(1+0) - (x + 4)(¼)}{(\log x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{(\log x) - (x + 4)(¼)}{(\log x)^2}$$

(2) If $y = \frac{e^x+5}{x^6-10}$, find $\frac{dy}{dx}$

Solution :

Given : $y = \frac{e^x+5}{x^6-10}$

Here $u = e^x + 5$; $v = x^6 - 10$

$$\therefore \frac{dy}{dx} = \frac{(x^6-10)\frac{d}{dx}(e^x+5) - (e^x+5)\frac{d}{dx}(x^6-10)}{(x6 - 10)^2}$$

$$\therefore \frac{dy}{dx} = \frac{(x^6-10)(e^x) - (e^x+5)(6x^5)}{(x^6-10)^2}$$

(3) If $y = \frac{x^3 - 1}{x^3+1}$, find $\frac{dy}{dx}$

Solution :

Given : $y = \frac{x^3 - 1}{x^3 + 1}$

Here $u = x^3 - 1$

$v = x^3 + 1$

$$\therefore \frac{dy}{dx} = \frac{(x^3+1)\frac{d}{dx}(x^3-1) - (x^3-1)\frac{d}{dx}(x^3+1)}{(x^3+1)^2}$$

$$= \frac{(x^3 + 1)(3x^2) - (x^3-1)(3x^2)}{(x^3+1)^2}$$

$$= \frac{3x^2[(x^3+1) - (x^3-1)]}{(x^3+1)^2}$$

$$= \frac{3x^2\ [x^3 + 1 - x^3 + 1]}{(x^3 + 1)^2}$$

$$= \frac{3x^2\ [2]}{(x^3 + 1)^2}$$

$$\frac{dy}{dx} = \frac{6x^2}{(x+1)^2}$$

Rule 5 : Scalar multiplication rule or constant multiplied by a function rule.

If y = cu , c is a constant , where u is a differentiable function of x ,then

$$\frac{dy}{dx} = c\ \frac{du}{dx}$$

i.e.,
$$\boxed{\frac{dy}{dx} = \frac{d}{dx}(cu) = c\ \frac{du}{dx}}$$

Examples :

(1) If y= 5 x^3 , find $\frac{dy}{dx}$

Solution : Given : y= $5x^3$

$$\therefore\ \frac{dy}{dx} = \frac{d}{dx}(5\,x^3)$$

$$= 5\ \frac{d}{dx}(x^3) = 5(3x^2) = 15\,x^2$$

(2) If y= 10 log x , find $\frac{dy}{dx}$

Solution : Given : y = 10 log x

$$\therefore\ \frac{dy}{dx} = \frac{d}{dx}(10 \log x)$$

$$= 10\ \frac{d}{dx}(\log x)$$

$$= 10\ (1/x)$$

$$= \frac{10}{x}$$

9.2.4 List of formulae :

y = $f(x)$	$\frac{dy}{dx} = f'(x)$
1. x^n	nx^{n-1}
2. C , C = constant	0
3. e^x	e^x
4. a^x	$a^x \log a$
5. log x	$1/x$
6. x	1
7. $\sqrt{x}$	$1/ 2\sqrt{x}$
8. $1/x$	$-1/x^2$

9.2.5 List of Rules :

(1) $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

(2) $\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$

(3) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

(4) $\frac{d}{dx}(u/v) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

(5) $\frac{d}{dx}(cu) = c\frac{du}{dx}$, c = constant.

9.2.6 Examples :

Find $\frac{dy}{dx}$ for each of the following :

Ex: (1) $y= x^6 + 4e^x + \log x + 10$

Solution:

$\frac{dy}{dx} = \frac{d}{dx}(x^6 + 4e^x + \log x + 10)$

$= \frac{d}{dx}(x6) + \frac{d}{dx}(4e^x) + \frac{d}{dx}(\log x) + \frac{d}{dx}(10)$

$= \frac{d}{dx}(x6) + 4\frac{d}{dx}(e^x) + \frac{d}{dx}(\log x) + \frac{d}{dx}(10)$

$= 6x^5 + 4e^x + 1/x + 0$

$\therefore \frac{dy}{dx} = 6x^5 + 4e^x + 1/x$

Ex : (2) $y= 5x^4 - 3e^x + 4\sqrt{x} + 2^x$

Solution :

$\frac{dy}{dx} = \frac{d}{dx}(5x^4 - 3e^x + 4\sqrt{x} + 2^x)$

$= \frac{d}{dx}(5x^4) - \frac{d}{dx}(3e^x) + \frac{d}{dx}(4\sqrt{x}) + \frac{d}{dx}(2^x)$

$= 5\frac{d}{dx}(x^4) - 3\frac{d}{dx}(e^x) + 4\frac{d}{dx}(\sqrt{x}) + \frac{d}{dx}(2^x)$

$= 5(4x^3) - 3e^x + 4(1/2\sqrt{x}) + 2^X \log_2$

$= 20x^3 - 3e^x + 2/\sqrt{x} + 2^X \log_2$

Ex: (3) $y= x^{3/2} + 4\log x - 10x^2 + 15$

Solution:

$\frac{dy}{dx} = \frac{d}{dx}(x^{3/2} + 4\log x - 10x^2 + 15)$

$$= \frac{d}{dx}(x^{3/2}) + \frac{d}{dx}(4 \log x) - \frac{d}{dx}(10x^2) + \frac{d}{dx}(15)$$

$$= \frac{3}{2} x^{3/2-1} + 4\left(\frac{1}{x}\right) - 10(2x) + 0$$

$$\frac{dy}{dx} = \frac{3}{2} x^{1/2} + \frac{4}{x} - 20x.$$

Ex: (4) y= $(x + e^x)(5+\log x)$

Solution: Here u= $x + e^x$

v= $5+ \log x$

$$\frac{dy}{dx} = (x + e^x)\frac{d}{dx}(5+\log x) + (5+\log x)\frac{d}{dx}(x + e^x)$$

$$= (x + e^x)(0+1/x) + (5+ \log x)(1+e^x)$$

$$\therefore \frac{dy}{dx} = (x + e^x)(1/x) + (5+ \log x)(1+e^x)$$

Ex: (5) y= $(x^{10})(10^x)$

Solution : Here u=x^{10} , v = 10^x

$$\therefore \frac{dy}{dx} = x^{10}\frac{d}{dx}(10^x) + 10^x\frac{d}{dx}(x^{10}) = x^{10}(10^x \log 10) + 10^X(10x^9)$$

Ex:(6) y= $(\sqrt{x} + e^x)(2x^3+7)$

Solution: Here u= $\sqrt{x} + e^x$, v= $2x^3+7$

$$\frac{dy}{dx} = (\sqrt{x} + e^x)\frac{d}{dx}(2x^3+7) + (2x^3+7)\frac{d}{dx}(\sqrt{x} + e^x)$$

$$\therefore \frac{dy}{dx} = (\sqrt{x} + e^x)(6x^2) + (2x^3+7)(1/2\sqrt{x} + e^x)$$

Ex: (7) y = $\dfrac{x^2 + 5x + 6}{x+ 7}$

Solution : Here u= $x^2 + 5x + 6$

v= $x+ 7$

$$\therefore \frac{dy}{dx} = \frac{(x+ 7)\frac{d}{dx}(x^2 + 5x + 6) - (x^2 + 5x + 6)\frac{d}{dx}(x+ 7)}{(x+ 7)^2}$$

$$= \frac{(x+ 7)(2x +5) - (x^2 + 5x + 6)(1)}{(x+ 7)^2}$$

$$= \frac{2x^2 + 19x + 35 - x^2 -5x -6}{(x+ 7)^2}$$

$$\therefore \frac{dy}{dx} = \frac{x^2 + 14x + 29}{(x+ 7)^2}$$

Ex : 8 y = $\dfrac{10 e^x + 5 \log x}{x^3+ 12}$

Solution : Here $u = 10e^{x} + 5\log x$

$v = x^3 + 12$

$$\therefore \frac{dy}{dx} = \frac{(x^3+12)\frac{d}{dx}(10e^{x} + 5\log x) - (10e^{x} + 5\log x)\frac{d}{dx}(x^3+12)}{(x^3+12)^2}$$

$$\frac{dy}{dx} = \frac{(x^3+12)(10e^{x} + 5/x) - (10e^{x} + 5\log x)(3x^2)}{(x^3+12)^2}$$

Ex:9 $y = \frac{4^x + 6}{2x^2 + 5}$

Solution : Here $u = 4^x + 6$, $v = 2x^2 + 5$

$$\therefore \frac{dy}{dx} = \frac{(2x^2+5)\frac{d}{dx}(4^x+6) - (4^x+6)\frac{d}{dx}(2x^2+5)}{(2x^2+5)^2}$$

$$= \frac{(2x^2+5)(4^x \log 4) - (4^x+6)(4x)}{(2x^2+5)^2}$$

Ex: 10 $y = 2\sqrt{x} + 16e^{x} + (6^{x}) + 20x$

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}(2\sqrt{x} + 16e^{x} + 6^{x} + 20x)$$

$$= 2\,(1/2\sqrt{x}) + 16e^{x} + 6^{x}\log 6 + 20$$

$$\therefore \frac{dy}{dx} = 1/\sqrt{x} + 16e^{x} + 6^{x}\log 6 + 20$$

Exercise : 9.2

Differentiate the following with respect to x.

(1) $y = x^8 - 6e^{x} + 4x^{3/2} - 3x^2 + 5$

(2) $y = 6\log x - 3^{x} + 2e^{x} + 10\sqrt{x} + 2$

(3) $y = 5x^4 - 12x^3 + 18e^{x} + 10^{x} - 25$

(4) $y = 8^{x}(5x^3 + 3x + 1)$

(5) $y = (10x^2 + 2x + 5)(\sqrt{x} + e^{x})$

(6) $y = (2x^3 + 3x^2)(5\log x + 14)$

(7) $y = (x + \log x)(x^5 - 4x^2 + 10)$

(8) $y = (8x^5 - 6x^{5/2} + 1)(40\sqrt{x} + 2e^{x})$

(9) $y = (e^{x} + 2\log x + 2)(6^{x} + 2x^2 + 5)$

(10) $y = \frac{x^2 + 1}{x^4 - 1}$

(11) $y = \dfrac{2^x + 4\sqrt{x}}{2e^x + 5}$

(12) $y = \dfrac{x^3 - x^2 + 2}{x^2 - 4}$

(13) $y = \dfrac{x + \sqrt{x}}{\sqrt{x} - 1}$

(14) $y = \dfrac{3\log x + 5}{x^5 + 2x}$

(15) $y = \dfrac{e^x - \sqrt{x}}{2\sqrt{x} + 1}$

Answers :

(1) $8x^7 - 6e^x + 6x^{1/2} - 6x$

(2) $6/x - 3^x \log 3 + 2e^x + 5/\sqrt{x}$

(3) $20\, x3 - 36\, x2 + 18e^x + 10^x \log 10$

(4) $8^x (15x^2 + 3) + (5x^3 + 3x + 1)(8^x \log 8)$

(5) $(10x^2 + 2x + 5)(1/2\sqrt{x} + e^x) + (\sqrt{x} + e^x)(20x + 2)$

(6) $(2x^3 + 3x^2)(5/x) + (5\log x + 14)(6x^2 + 6x)$

(7) $(x + \log x)(5x^4 - 8x) + (x^5 - 4x^2 + 10)(1 + 1/x)$

(8) $(8x^5 - 6x^{5/2} + 1)(20/\sqrt{x} + 2e^x) + (40\sqrt{x} + 2e^x)(40x^4 - 15x^{3/2})$

(9) $(e^x + 2\log x + 2)(6^x \log 6 + 4x) + (6^x + 2x^2 + 5)(e^x + 2/x)$

(10) $\dfrac{-2x^5 - 4x^3 - 2x}{(x^4 - 1)^2}$

(11) $\dfrac{(2e^x + 5)(2^x \log 2 + 2/\sqrt{x}) - (2^x + 4\sqrt{x})(2e^x)}{(2e^x + 5)^2}$

(12) $\dfrac{x^4 - 12x^2 + 4x}{(x^2 - 4)^2}$

(13) $\dfrac{\sqrt{x} - 1 - (1 + x)/2\sqrt{x}}{(\sqrt{x} - 1)^2}$

(14) $\dfrac{(x^5 + 2x)(3/x) - (3\log x + 5)(5x^4 + 2)}{(x^5 + 2x)^2}$

(15) $\dfrac{(2\sqrt{x} + 1)(e^x - 1/2\sqrt{x}) - (e^x - \sqrt{x})(1/\sqrt{x})}{(2\sqrt{x} + 1)^2}$

9.3 SECOND ORDER DERIVATIVES

If y= $f(x)$ is differentiable function with respect to x , then

$\frac{dy}{dx} = f'(x)$ is called first order derivative of y with respect to x and

$\frac{d^2y}{dX2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called the second order derivative of y with respect to X

Notation :

(i) First order derivative :

$$\frac{dy}{dx} = f'(x) = y_1 = y'$$

(ii) Second order derivative :

$$\frac{d^2y}{dx^2} = f''(x) = y_2 = y''$$

9.3.1 Examples :

(1) If $y = x^3 - 6x^2 + 19x + 100$, find $\frac{d^2y}{dx^2}$

Solution :

Given : $y = x^3 - 6x^2 + 19x + 100$

$$\frac{dy}{dx} = 3x^2 - 12x + 19$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2 - 12x + 19)$$

$$\frac{d^2y}{dx^2} = 6x - 12$$

(2) If $y = e^x + 2x^3 + 5x^2 + 4$, find $\frac{d^2y}{dx^2}$

Solution :

Given : $y = e^x + 2x^3 + 5x^2 + 4$

$$\frac{dy}{dx} = e^x + 6x^2 + 10x$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(dy/dx)$$

$$= \frac{d}{dx}(e^x + 6x^2 + 10x)$$

$$\frac{d^2y}{dx^2} = e^x + 12x + 10$$

(3) If y= $f(x)= x^5 - 6x^3 + 2x^2 + 10x + 5$, find $f''(x)$

Solution:

Given $f(x) = x^5 - 6x^3 + 2x^2 + 10x + 5$

$f'(x) = 5x^4 - 18x^2 + 4x + 10$

$f''(x) = 20x^3 - 36x + 4$

(4) If $y = x + 1/x$, $x \neq 0$, find d^2y/dx^2

Solution: Given: $y = x + 1/x$

$dy/dx = 1 + (-1/x^2) = 1 - 1/x^2$

$$\frac{d^2y}{dx^2} = 0 - (-2/x^3) = \frac{2}{x^3}$$

Exercise:9.3

Find $\frac{d^2y}{dx^2}$ for each of the following:

(1) $y = 8x^5 - 16x^4 + 4x^3 + x + 2$
(2) $y = 2x^3 - 5x^2 + 12x + 15$
(3) $y = x + 25/x$
(4) $y = 2x^2 + e^x + 5x + 12$
(5) $y = x^2 + x + \log x$

Answers:
1) $160x^3 - 192x^2 + 24x$
2) $12x - 10$
3) $\frac{50}{x^3}$
4) $4 + e^x$

5) $2 - \frac{1}{x^2}$

9.4 APPLICATIONS OF DERIVATIVES

Applications to Economics:

9.4.1 The Total Cost function:

(i) Total cost function $C = f(x)$
(ii) Average cost function $AC = \frac{C}{x}$
(iii) Marginal cost function:

The rate of change of cost with respect to the number of units produced is called the Marginal cost and is denoted by MC.

i.e, $MC = \frac{dC}{dx}$

9.4.2 The Total Revenue Function:

(i) Total Revenue Function R = p x D
(ii) Average Revenue Function AR= p
(iii) Marginal Revenue function:
The rate of change of total revenue with respect to the demand D is called
the Marginal revenue function and is denoted by MR.
i.e $MR = \frac{dR}{dD}$

9.4.3 Elasticity:

Let D be the demand and p be the price. The quantity $\frac{-P}{D}\frac{dD}{dP}$ is

called elasticity of demand with respect to the price and is denoted by η (η = eta : Greek alphabet).

i.e, $\eta = \frac{-p}{D}\frac{dD}{dp}$

If η = 0, demand D is Constant and the demand is said to be perfectly elastic.

If 0 < η < 1, the demand is said to be inelastic

If η = 1, the demand is directly proportional to the price.

If η > 1, the demand is said to be elastic.

9.4.4 Relation between the Marginal Revenue and elasticity of demand.

Let R = Total revenue
p = price
D = demand

$\therefore R = pD$

$$MR = \frac{dR}{dD}$$

$$= \frac{d}{dD}(pD)$$

$$= p\frac{d}{dD}(D) + D.\frac{d}{dD}(p) \quad \text{(by product rule)}$$

$$= p(1) + D\frac{dp}{dD}$$

$$MR = p + D\frac{dp}{dD}$$

$$\eta = -\frac{p}{D}\frac{dD}{dp}$$

$$\therefore D\frac{dp}{dD} = \frac{-p}{\eta}$$

$$MR = p + \left(\frac{-p}{\eta}\right)$$

$$MR = p\left[1 - \frac{1}{\eta}\right]$$

$$MR = AR\left[1 - \frac{1}{\eta}\right] \qquad (AR = p)$$

9.4.5 Examples:

Ex: (1) The cost of producing x items is given by

$2x^2 + 5x + 20$. Find the total cost, average cost and marginal cost when x= 10.

Solution:

Let $C = f(x) = 2x^2 + 5x + 20$

$$AC = \frac{C}{x} = \frac{2x^2 + 5x + 20}{x}$$

$$MC = \frac{dC}{dx}$$

$$= \frac{d}{dx}(2x^2 + 5x + 20)$$

$MC = 4x + 5$

when $x = 10$

$C = 2(10)^2 + 5(10) + 20$

$C = 270$

$$AC = \frac{C}{x} = \frac{270}{10} = 27$$

$MC = 4(10) + 5 = 45$

Ex: (2)

The demand function is given by $\rho = 50 + 6D + 4D^2$. Find the total revenue, average revenue and the marginal revenue when the demand is 5 units.

Solution:

Given : $\rho = 50 + 6D + 4D^2$

$R = \rho \times D$

$= (50 + 6D + 4D^2)(D)$

$R = 50D + 6D^2 + 4D^3$

$AR = \rho = 50 + 6D + 4D^2$

$MR = \dfrac{dR}{dD}$

$= \dfrac{d}{dD}(50D + 6D^2 + 4D^3)$

$MR = 50 + 12D + 12D^2$

When $D = 5$

$R = 50(5) + 6(5)^2 + 4(5)^3$

$= 250 + 150 + 500$

$R = 900$

$AR = 50 + 6(5) + 4(5)^2$

$= 50 + 30 + 100$

$= 180$

$MR = 50 + 12(5) + 12(5)^2$

$= 50 + 60 + 300$

$= 410$

Ex : (3)

The total revenue function is given by $R = 20D - D^2$, D = Demand. Find the demand function. Also find AR when MR = 0 .

Solution :

Given $R = 20D - D^2$

$\rho D = 20D - D^2 \quad (\therefore R = \rho D)$

$\therefore \rho = \dfrac{20D - D^2}{D}$

$\rho = 20 - D$

$\therefore$ The demand function is $\rho = 20 - D$.

Now, $MR = \dfrac{dR}{dD}$

$= \dfrac{d}{dD}(20D - D^2)$

$MR = 20 - 2D$

Given that MR = 0.

$\therefore \quad 0 = 20 - 2D$

$2D = 20$

$\therefore \quad D = 10$

$AR = \rho = 20 - D$

$\therefore AR = 20 - 10$

$AR = 10$

Ex : (4)

The demand function is given by

$D = 25 - 2\rho - \rho^2$

Find the elasticity of demand when the price is 4.

Solution :

Given : $D = 25 - 2\rho - \rho^2$

$\therefore \dfrac{dD}{dp} = 0 - 2 - 2\rho$

$\dfrac{dD}{dp} = -2 - 2\rho$

$$\therefore \eta = \frac{-\rho}{D} \quad \frac{dD}{d\rho}$$

$$= \frac{(-\rho)}{25 - 2\rho - \rho^2} \; (-2 - 2\rho)$$

$$\eta = \frac{p(2 + 2p)}{25 - 2p - p^2}$$

When $p = 4$,

$$\eta = \frac{4(2 + 8)}{25 - 8 - 16}$$

$$= \frac{4(10)}{1}$$

$$\therefore \eta = 40$$

Ex: 5

The demand function is given by $D = \dfrac{p + 3}{2p - 1}$ where D = Demand and p = price. Find the elasticity of demand when the price is 8.

Solution:

Given $D = \dfrac{p + 3}{2p - 1}$

$$\therefore \frac{dD}{dp} = \frac{(2p - 1)\,\frac{d}{dp}(p + 3) - (p + 3)\,\frac{d}{dp}(2p - 1)}{(2p - 1)^2}$$

$$= \frac{(2p - 1)(1) - (p + 3)(2)}{(2p - 1)^2}$$

$$= \frac{2p - 1 - 2p - 6}{(2p - 1)^2}$$

$$\frac{dD}{dp} = \frac{-7}{(2p - 1)^2}$$

$$\therefore \eta = \frac{-p}{D} \quad \frac{dD}{dp}$$

$$= \frac{(-p)}{[(p+3)/(2p-1)]} \left[\frac{-7}{(2p-1)^2} \right]$$

$$= \frac{7p\,(2p-1)}{(p+3)(2p-1)^2}$$

$$\eta = \frac{7p}{(p+3)(2p-1)}$$

When p = 8

$$\eta = \frac{7(8)}{(8+3)[2(8)-1]}$$

$$= \frac{56}{(11)(15)}$$

$$= \frac{56}{165}$$

$\eta = 0.34$

Ex : (6) If MR = 45, AR = 75, Find η

Solution:

Given MR = 45

AR = 75

η = ?

∴ MR = AR [1 - 1/η]

45 = 75 [1 - 1/ η]

$$\frac{45}{75} = 1 - 1/\eta$$

$$\frac{3}{5} = 1 - 1/\eta$$

$0.6 = 1 - 1/\eta$

$1/\eta = 1 - 0.6$

$1/\eta = 0.4$

$\eta = 1/0.4$

$= 2.5$

Ex : (7)

If AR = 95 and η = 7/ 2 , Find MR.

Solution : Given AR = 95
η = 7/2 = 3.5
MR = ?

MR = AR [1 - 1 / η]

= 95 [1 - 1 / 3.5]

= 95 [1 - 0.29]

= 95 [0.71]

MR = 67.45

Exercise : 9.4

(1) The cost of producing x items is given by $x^3 + 4x + 15$. Find the total cost, average cost and marginal cost when x = 6.

(2) The total cost function is given by $C = x^3 + 2x^2 + 5x + 30$. Find the total cost, average cost and marginal cost when x = 10.

(3) The demand function is given by $p = 20 - 8D + 3D^2$. Find the total revenue, average revenue and marginal revenue when the demand is 4 units.

(4) The total revenue function is given by $R = 30D - 2D^2 + D^3$. Find the demand function. Also find total revenue, average revenue and marginal revenue when the demand is 5 units.

(5) The demand function is given by $D = -28 - 5p + 2p^2$ Find the elasticity of demand when the price is 3.

(6) The demand function is given by $D = \frac{2p + 5}{p - 3}$ Where D = Demand and p = price. Find the elasticity of demand when price is 6.

(7) If AR = 65 and η = 3, find MR.

(8) If MR = 85 and η = 4.5 , find AR.

(9) If MR = 55 and AR = 98 , find η .

(10) If the price is 65 and the elasticity of demand is 5.2, find the marginal revenue.

Answers:

(1) C = 255 ; AC = 42.5 ; MC = 112

(2) C = 1280 ; AC = 128 ; MC = 345

(3) R = 144 ; AR = 36 ; MR = 100

(4) $p = 30 - 2D + D^2$
R = 225 ; AR = 45 ; MR = 85

(5) $\eta = 0.64$

(6) $\eta = 1.29$

(7) MR = 43.3

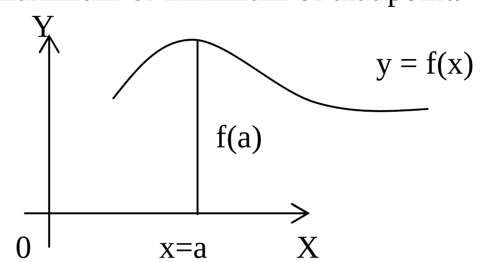
(8) AR = 108.97

(9) $\eta = 2.28$

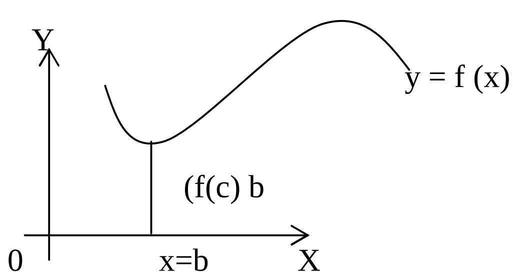
(10) MR = 52.5

9.5 MAXIMA AND MINIMA.

Let $y = f(x)$ be the given function. A curve $f(x)$ is said to have a maximum or minimum point (extreme point), if $f(x)$ attains either a maximum or minimum of that point.



Maximum at x = a



Maximum at x = b

In the first figure, x = a is the point where the curve f (x) attains a maximum. In the second figure, x = b is the point where the curve f (x) attains a minimum

9.5.1 Conditions for Maximum & Minimum:

1. Condition for Maximum:

   (i) $f'(x) = 0$

   (ii) $f''(x) < 0$ at x = a

2. Condition for Minimum:

   (i) $f'(x) = 0$

   (ii) $f''(x) > 0$ at x = b

9.5.2. To find the Maximum and Minimum values of f(x) :

Steps:

(i) Find $f'(x)$ and $f''(x)$.

(ii) put $f'(x) = 0$, solve and get the values of x.

(iii) Substitute the values of x in $f''(x)$.

If $f''(x) < 0$, then $f(x)$ has maximum value at x = a. If $f''(x) > 0$, then $f(x)$ has minimum value at x= b.

(iv) To find the maximum and minimum values, put the points x = a and x = b in $f(x)$.

Note: Extreme values of $f(x)$ = Maximum and minimum values of $f(x)$.

9.5.3. Examples :

Ex: (1) Find the extreme values of $f(x) = x^3 - 3x^2 - 45x + 25$.

Solution:

Given : $f(x) = x^3 - 3x^2 - 45x + 25$

$\therefore f'(x) = 3x^2 - 6x - 45$

$f''(x) = 6x - 6$.

Since f(x) has maximum or minimum value,

$f'(x) = 0$

$\therefore \quad 3x^2 - 6x - 45 = 0$

$3(x^2 - 2x - 15) = 0$

$\therefore \quad x^2 - 2x - 15 = 0$

$x^2 - 5x + 3x - 15 = 0$

$x(x - 5) + 3(x - 5) = 0$

$(x - 5)(x + 3) = 0$

$x - 5 = 0$ or $x + 3 = 0$

$\therefore \quad x = 5$ or $x = -3$.

When $x = 5$, $f''(5)=6(5)-6=24>0$

$\therefore$ f(x) has minimum at x = 5.

When $x = -3$, $f''(-3) = 6(-3) - 6$

$= -24 < 0$

$\therefore$ f(x) has maximum at x = -3.

To find the maximum and minimum values of f(x) :

put x = 5 and x = -3 in f(x) .

$\therefore f(5) = 5^3 - 3(5)^2 - 45(5) + 25$

$= 125 - 75 - 225 + 25$

$= -150$

$f(-3) = (-3)^3 - 3(-3)^2 - 45(-3) + 25$

$= -27 - 27 + 135 + 25$

$= 106$

$\therefore$ Maximum value = 106 at x = -3

Minimum value = -150 at x = 5.

Ex: (2) Find the maximum and minimum values of $f(x) = x + \dfrac{16}{x}, x \neq 0$.

Solution : Given : $f(x) = x + (16/x)$

$\therefore \quad f'(x) = 1 + 16\,(-1/x^2)$

$f'(x) = 1 - \dfrac{16}{x^2}$

$\therefore \quad f''(x) = 0 - 16\left(\dfrac{-2}{x^3}\right)$

$f''(x) = \dfrac{32}{x^3}$

Since $f(x)$ has maximum or minimum value, $f'(x) = 0$.

$\therefore \quad 1 - \dfrac{16}{x^2} = 0$

$\dfrac{x^2 - 16}{x^2} = 0$

$x^2 - 16 = 0$

$x^2 - 4^2 = 0$

$(x - 4)(x + 4) = 0$

$x = 4$ or $x = -4$

when $x = 4$, $f''(4) = \dfrac{32}{(4)^3} = \dfrac{32}{64} = \dfrac{1}{2} > 0$

$\therefore$ $f(x)$ has minimum at $x = 4$.

when $x = -4$, $f''(-4) = \dfrac{32}{(-4)^3} = \dfrac{32}{-64} = -\dfrac{1}{2} < 0$

$\therefore$ $f(x)$ has maximum at $x = -4$.

Now to find the extreme values of $f(x)$: put $x = 4$ and $x = -4$ in $f(x)$

$\therefore \quad f(4) = 4 + \dfrac{16}{4} = 4 + 4 = 8$

$f(-4) = -4 + \dfrac{16}{(-4)} = -4 - 4 = -8$

$\therefore$ Maximum value $= -8$ at $x = -4$

Minimum value $= 8$ at $x = 4$.

Ex: (3) Divide 80 into two parts such that the sum of their squares is a minimum.

Solution:

Let x and 80 - x be the two required numbers.

∴ By the given condition,

$f(x) = x^2 + (80- x)^2$

$f(x)= x^2 + 80^2 - 2(80)(x) + x^2$

$f(x) = 2x^2 - 160x + 6400$

∴ $f'(x)= 4x- 160$

$f''(x) = 4$

Since $f(x)$ has minimum,

$f'(x)= 0$

$4x-160= 0$

$4x=160$

$x=\frac{160}{4}=40.$

∴ $f''(x) = 4 > f''(40)= 4>0.$

∴ $f(x)$ has minimum at $x= 40$.

∴ The required numbers are
40 and 80-40=40

∴ The required parts of 80 are
40 and 40.

Ex: (4)

A manufacturer can sell x items at a price of Rs. (330-x) each. The cost of producing x items is Rs. $(x^2 + 10x+12)$. Find x for which the profit is maximum.

Solution:

Given that the total cost function is
$C= x^2+ 10x+ 12.$

Selling price p= 330-x

Revenue is R= p x D

$= p \times x \quad (D= x)$

$= (330-x)\, x$

$= 330x- x^2$

$\therefore$ Profit= Revenue- Cost

$\therefore$ P= R- C

$= (330x-x^2) - (x^2+10x+ 12)$

$= 330x-x^2-x^2-10x-12$

$P= 320x-2x^2- 12$

$\therefore \frac{dp}{dx} = 320-4x$

$\frac{d^2p}{dx^2} = -4 < 0$

$\therefore$ The profit is maximum.

Since the profit is maximum,

$\frac{dp}{dx} = 0$

$\therefore$ 320-4x= 0

$\therefore$ 4x=320

x=80.

Hence the profit is maximum when 80 items are sold.

Ex: (5) The total cost function is

$C= x^3- 9x^2 + 24x+ 70.$

Find x for which the total cost is minimum.

Solution:

Let $C = f(x)= x^3 - 9x^2 + 24x+ 70$

$\therefore C' = f'(x) = 3x^2 - 18x+24$

$C'' = f''(x) = 6x-18$

Since $f(x)$ has minimum, $f'(x) = 0$

$\therefore 3x^2 = 18x+ 24=0$

$\therefore 3(x^2- 6x+8) = 0$

$\therefore x^2 - 6x +8=0$

$x^2 - 2x-4x+8=0$

$x(x-2)- 4(x-2)=0$

$(x-2)(x-4)= 0$

$x-2=0$ or $x-4=0$

$x=2$ or $x=4$

When x=4

$f''(x) = 6x-18$
$f''(4) = 6(4)-18$
$= 6>0$
$\therefore$ $f(x)$ has minimum at x=4.

$\therefore$ The total cost is minimum at x=4.

Ex: (6) The total revenue function is given by

$$R = 4x^3 - 72x^2 + 420x + 800.$$

Find x for which the total revenue is maximum.

Solution:
Let $R = f(x) = 4x^3 - 72x^2 + 420x+800$

$\therefore R' = f'(x) = 12x^2 - 144x+420$

$R'' = f''(x) = 24x-144$

Since $f(x)$ has maximum,
$f'(x) = 0$
$\therefore 12x^2 -144x+420=0$
$\therefore 12(x^2 - 12x+35)=0$
$\therefore x^2 - 12x+35=0$
$x^2 - 5x-7x+35=0$
$x(x-5)-7(x-5)=0$
$(x-5)(x-7)=0$
$x-5=0$ or $x-7=0$
$\therefore x=5$ or $x=7$

When x=5

$R'' = f''(x) = 24x-144$
$= 24(5)-144$
$= -24<0$

$\therefore$ $f(x)$ has maximum at x=5

$\therefore$ The total revenue is maximum at x=5

Exercise : 9.5

(1) Find the extreme values of $f(x)=2x^3 - 6x^2 - 48x+90$.

(2) Find the maximum and minimum values of $f(x) = x+(9/x), x\neq 0$

(3) Find the extreme values of $f(x)= 4x^3-12x^2 - 36x+25$

(4) Find the extreme values of $f(x) = x + (36/x)$, $x \neq 0$.

(5) Divide 120 into two parts such that their product is maximum.

(6) Divide 70 into two parts such that the sum of their squares is a minimum.

(7) A manufacturer sells x items at a price of Rs. (400-x) each. The cost of producing x items is Rs ($x^2 + 40x+52$). Find x for which the profit is maximum.

(8) The cost function is given by $C= x^3- 24x^2 + 189x+120$. Find x for which the cost is minimum,

(9) The total revenue function is given by

$$R= 2x^3 - 63 x^2+ 648x+250.$$

Find x for which the total revenue is maximum.

(10) The total cost of producing x units is Rs.($x^2+ 2x+5$) and the price is Rs. (30-x) per unit. Find x for which the profit is maximum.

Answers:

(1) Maximum value= 146 at x=-2
Minimum value = -70 at x=4

(2) Maximum value= -6 at x=-3
Minimum value = 6 at x= 3

(3) Maximum value= 45 at x=-1
Minimum value= -83 at x=3

(4) Maximum value = -12 at x=-6
Minimum value = 12 at x=6

(5) 60,60
(6) 35,35
(7) 90
(8) 9
(9) 9
(10) 7

❖❖❖

Unit VII

10

SIMPLE INTEREST AND COMPOUND INTEREST

OBJECTIVES

After reading this chapter you will be able to:

- Define interest, principal, rate of interest, period.
- Find simple interest (SI), rate of S.I., period of investment.
- Find Compound Interest (CI), rate of C.I., Amount accumulated at the end of a period.
- Compound interest compounded yearly, half-yearly, quarterly or monthly.

10.1 INTRODUCTION

In every day life individuals and business firms borrow money from various sources for different reasons. This amount of money borrowed has to be returned from the lender in a stipulated time by paying some *interest* on the amount borrowed. In this chapter we are going to study the two types of interests viz. simple and compound interest. We start with some definitions and then proceed with the formula related to both the types of interests.

10.2 Definitions Of Terms Used In This Chapter

Principal: The sum borrowed by a person is called its *principal*. It is denoted by ***P***.

Period: The time span for which money is lent is called *period*. It is denoted by ***n***.

Interest: The amount paid by a borrower to the lender for the use of money borrowed for a certain period of time is called *Interest*. It is denoted by ***I***.

Rate of Interest: This is the interest to be paid on the amount of Rs. 100 per annum (i.e. per year). This is denoted by ***r***.

Total Amount: The sum of the principal and interest is called as the total amount (*maturity value*) and is denoted by A. Thus, $A = P + I$.
i.e. Interest paid $I = A - P$.

10.2 SIMPLE INTEREST

The interest which is payable on the principal only is called as *simple interest* (S.I.). For example the interest on Rs. 100 at 11% after one year is Rs.11 and the amount is 100 + 11 = Rs. 111.

It is calculated by the formula: $I = \frac{Pnr}{100} = P \text{ x } n \text{ x } \frac{r}{100}$

Simple Interest = Prinicpal x period x rate of interest

Amount at the end of n^{th} year = $\mathbf{A = P + I = P +} \frac{Pnr}{100} = \boldsymbol{P}\left(1 + \frac{nr}{100}\right)$

Remark: The period *n* is always taken in 'years'. If the period is given in months/days, it has to be converted into years and used in the above formula. For example, if period is 4 months then we take $n = 4/12 = 1/3$ or if period is 60 days then $n = 60/365$.

***Example* 1:** If Mr. Sagar borrows Rs. 500 for 2 years at 10% rate of interest, find (i) simple interest and (ii) total amount.

Ans: Given P = Rs. 500, $n = 2$ and $r = 10$ %

(i) $I = \frac{Pnr}{100} = \frac{500 \text{ x } 2 \text{ x } 10}{100}$ = Rs. 100

(ii) $A = P + I = 500 + 100$ = Rs. 600

10.2.1 Problems involving unknown factors in the formula $I = \frac{Pnr}{100}$

The formula $I = \frac{Pnr}{100}$ remaining the same, the unknown factor in the formula is taken to the LHS and its value is computed. For example, if rate of interest is unknown then the formula is rewritten as $r = \frac{I \text{ x } 100}{\text{P x } n}$.

***Example* 2:** If Mr. Prashant borrows Rs. 1000 for 5 years and pays an interest of Rs. 300, find rate of interest.

Ans: Given $P = 1000$, $n = 5$ and I = Rs. 300

Now, $I = \frac{Pnr}{100} \quad \Rightarrow r = \frac{I \text{ x } 100}{\text{P x } n} = \frac{300 \text{ x } 100}{1000 \text{ x } 5} = 6$

Thus, the rate of interest is 6%.

***Example* 3:** Find the period for Rs. 2500 to yield Rs. 900 in simple interest at 12%.

Ans: Given P = Rs. 2500, $I = 900$, $r = 12\%$

Now, $I = \frac{Pnr}{100} \quad \Rightarrow n = \frac{I \text{ x } 100}{P \text{ x } r} = \frac{900 \text{ x } 100}{2500 \text{ x } 12} = 3$

Thus, the period is 3 years.

***Example* 4:** Find the period for Rs. 1000 to yield Rs. 50 in simple interest at 10%.
Ans: Given P = Rs. 1000, I = 50, r = 10%

Now, $I = \frac{Pnr}{100}$ $\Rightarrow n = \frac{I \text{ x } 100}{P \text{ x } r} = \frac{50 \text{ x } 100}{1000 \text{ x } 10} = 0.5$

Thus, the period is 0.5 years i.e. 6 months.

***Example* 5:** Mr. Akash lent Rs. 5000 to Mr. Prashant and Rs. 4000 to Mr. Sagar for 5 years and received total simple interest of Rs. 4950. Find (i) the rate of interest and (ii) simple interest of each.
Ans: Let the rate of interest be r.

S.I. for Prashant = $\frac{5000 \text{ x } 5 \text{ x } r}{100} = 250r$ … (1)

and S.I. for Sagar = $\frac{4000 \text{ x } 5 \text{ x } r}{100} = 200r$ … (2)

from (1) and (2), we have,
total interest from both = $250r + 200r$
= $450r$
But total interest received be Mr. Akash = Rs. 4950

$\therefore 450r = 4950$ $\Rightarrow r = \frac{4950}{450} = 11$

$\therefore$ the rate of interest = 11%

***Example* 6:** The S.I. on a sum of money is one-fourth the principal. If the period is same as that of the rate of interest then find the rate of interest.

Ans: Given $I = \frac{P}{4}$ and $n = r$

Now, we know that $I = \frac{Pnr}{100}$

$\therefore \frac{P}{4} = \frac{P \text{ x } r \text{ x } r}{100}$ $\Rightarrow$ $\frac{100}{4} = r^2$

$\therefore r^2 = 25$ $\Rightarrow r = 5$.
$\therefore$ the rate of interest = 5%

***Example* 7:** If Rs. 8400 amount to Rs. 11088 in 4 years, what will Rs. 10500 amount to in 5 years at the same rate of interest?
Ans:
(i) Given n = 4, P = Rs. 8400, A = Rs. 11088
$\therefore I = A - P$ = 11088 – 8400 = Rs. 2688
Let r be the rate of interest.

Now, $I = \frac{Pnr}{100}$ $\Rightarrow 2688 = \frac{8400 \text{ x } 4 \text{ x } r}{100}$

$\therefore r$ = 8%

(ii) To find A when n = 5, P = Rs. 10500, r = 8

$A = P\left(1 + \frac{nr}{100}\right) = 10500 \text{ x } \left(1 + \frac{5 \text{ x } 8}{100}\right) = 10500 \text{ x } \frac{140}{100} = 14700$

$\therefore$ the required amount = Rs. 14,700

***Example* 8**: Mr. Shirish borrowed Rs. 12,000 at 9% interest from Mr. Girish on January 25, 2007. The interest and principal is due on August 10, 2007. Find the interest and total amount paid by Mr. Shirish.

Ans: Since the period is to be taken in years, we first count number of days from 25^{th} January to 10^{th} August, which is **197** days.

January	6
February	28
March	31
April	30
May	31
June	30
July	31
August	10
Total	**197**

Now, $I = \frac{Pnr}{100} = 12000 \times \frac{197}{365} \times \frac{9}{100}$

$\therefore$ I **= Rs. 582.9**

Total amount = $P + I$ = 12000 + 582.9

$\therefore A$ **= Rs. 12,582.9**

Check your progress 10.1

1. Find the SI and amount for the following data giving principal, rate of interest and number of years:

(*i*) 1800, 6%, 4 years. (*ii*) 4500, 8%, 5 years

(*iii*) 7650, 5.5%, 3 years. (*iv*) 6000, 7.5%, 6 years

(*v*) 25000, 8%, 5 years (*vi*) 20000, 9.5%, 10 years.

Ans: (*i*) 432, 2232 (*ii*) 1800, 6300, (*iii*) 1262.25, 8912.25

(*iv*) 2700, 8700 (*v*) 10000, 35000 (*vi*) 19000, 39000

2. Find the S.I. and the total amount for a principal of Rs. 6000 for 3 years at 6% rate of interest.

Ans: 1080, 7080

3. Find the S.I. and the total amount for a principal of Rs. 3300 for 6 years at 3½ % rate of interest.

Ans: 693, 3993

4. Find the S.I. and the total amount for a principal of Rs. 10550 for 2 years at 10¼ % rate of interest.

Ans: 2162.75, 12712.75

5. Find the rate of interest if a person invests Rs. 1000 for 3 years and receives a S.I. of Rs. 150.

Ans: 5%

6. Find the rate of interest if a person invests Rs. 1200 for 2 years and receives a S.I. of Rs. 168.

Ans: 7%

7. A person invests Rs. 4050 in a bank which pays 7% S.I. What is the balance of amount of his savings after (*i*) six months, (*ii*) one year?

Ans: 141.75, 283.5

8. A person invests Rs. 3000 in a bank which offers 9% S.I. After how many years will his balance of amount will be Rs. 3135?

Ans: 6 months

9. Find the principal for which the SI for 4 years at 8% is 585 less than the SI for 3½ years at 11%.

Ans: 9000

10. Find the principal for which the SI for 5 years at 7% is 250 less than the SI for 4 years at 10%.

Ans: 5000

11. Find the principal for which the SI for 8 years at 7.5% is 825 less than the SI for 6½ years at 10.5%.

Ans: 10000

12. Find the principal for which the SI for 3 years at 6% is 230 more than the SI for 3½ years at 5%.

Ans: 46000

13. After what period of investment would a principal of Rs. 12,350 amount to Rs. 17,043 at 9.5% rate of interest?

Ans: 4 years

14. A person lent Rs. 4000 to Mr. *X* and Rs. 6000 to Mr. *Y* for a period of 10 years and received total of Rs. 3500 as S.I. Find (*i*) rate of interest, (*ii*) S.I. from Mr. *X*, Mr. *Y*.

Ans: 3.5%, 1400, 2100

15. Miss Pankaj Kansra lent Rs. 2560 to Mr. Abhishek and Rs. 3650 to Mr. Ashwin at 6% rate of interest. After how many years should he receive a total interest of Rs. 3726?

Ans: 10 years

16. If the rate of S.I. on a certain principal is same as that of the period of investment yields same interest as that of the principal, find the rate of interest.

Ans: 10%

17. If the rate of S.I. on a certain principal is same as that of the period of investment yields interest equal to one-ninth of the principal, find the rate of interest.

Ans: $3\frac{1}{3}$ years

18. Find the principal and rate of interest if a certain principal amounts to Rs. 2250 in 1 year and to Rs. 3750 in 3 years.

Ans: 1500, 50%

19. Find the principal and rate of interest if a certain principal amounts to Rs. 3340 in 2 years and to Rs. 4175 in 3 years. **Ans:** 1670, 50%

20. If Rs. 2700 amount Rs. 3078 in 2 years at a certain rate of interest, what will Rs. 7200 amount to in 4 years at the same rate on interest?

Ans: 7%, 9216

21. At what rate on interest will certain sum of money amount to three times the principal in 20 years?

Ans: 15%

22. Mr. Chintan earns as interest Rs. 1020 after 3 years by lending Rs. 3000 to Mr. Bhavesh at a certain rate on interest and Rs. 2000 to Mr. Pratik at a rate on interest 2% more than that of Mr. Bhavesh. Find the rates on interest.

Ans: 6%, 8%

23. Mr. Chaitanya invested a certain principal for 3 years at 8% and received an interest of Rs. 2640. Mr. Chihar also invested the same amount for 6 years at 6%. Find the principal of Mr. Chaitanya and the interest received by Mr. Chihar after 6 years.

Ans: 11000, 3960

24. Mr. Ashfaque Khan invested some amount in a bank giving 8.5% rate of interest for 5 years and some amount in another bank at 9% for 4 years. Find the amounts invested in both the banks if his total investment was Rs. 75,000 and his total interest was Rs. 29,925.

Ans: 45000, 30000

25. Mrs. Prabhu lent a total of Rs. 48,000 to Mr. Diwakar at 9.5% for 5 years and to Mr. Ratnakar at 9% for 7 years. If she receives a total interest of Rs. 25,590 find the amount she lent to both.

Ans: 18000, 30000

10.3 COMPOUND INTEREST

The interest which is calculated on the amount in the previous year is called ***compound interest***.

For example, the compound interest on Rs. 100 at 8% after one year is Rs. 8 and after two years is 108 + 8% of 108 = Rs. 116.64

If P is the principal, r is the rate of interest p.a. then the amount at the end of n^{th} year called as ***compound amount*** is given by the formula:

$$A = P\left(1+\frac{r}{100}\right)^{n}$$

The ***compound interest*** is given by the formula:

$$CI = A - P$$

Note:

1. The interest may be compounded annually (yearly), semi-annually (half yearly), quarterly or monthly. Thus, the general formula to calculate the amount at the end of n years is as follows:

$$A = P\left(1+\frac{r}{p \times 100}\right)^{np}$$

Here ***p***: number of times the interest is compounded in a year.
***p* = 1** if interest is compounded **annually**
***p* = 2** if interest is compounded **semi-annually** (**half-yearly**)
***p* = 4** if interest is compounded **quarterly**
***p* = 12** if interest is compounded **monthly**

2. It is easy to calculate amount first and then the compound interest as compared with finding interest first and then the total amount in case of simple interest.

***Example* 9:** Find the compound amount and compound interest of Rs. 1000 invested for 10 years at 8% if the interest is compounded annually.

Ans: Given $P = 1000$, $r = 8$, $n = 10$.
Since the interest is compounded annually, we have

$$A = P\left(1+\frac{r}{100}\right)^{n} = 1000 \text{ x } \left(1+\frac{8}{100}\right)^{10} = 1000 \text{ x } 2.1589 = \text{Rs. } 2158.9$$

***Example* 10**: Find the principal which will amount to Rs. 11,236 in 2 years at 6% compound interest compounded annually.

Ans: Given A = Rs. 11236, $n = 2$, $r = 6$ and $P = ?$

Now, $A = P\left(1+\frac{r}{100}\right)^{n}$

$\therefore 11236 = P\left(1+\frac{6}{100}\right)^{2} = P \text{ x } 1.1236$

$\therefore P = \frac{11236}{1.1236} = 10{,}000$

$\therefore$ the required principal is Rs. 10,000.

***Example* 11:** Find the compound amount and compound interest of Rs. 1200 invested for 5 years at 5% if the interest is compounded (*i*) annually, (*ii*) semi annually, (*iii*) quarterly and (*iv*) monthly.

Ans: Given P = Rs. 1200, $r = 5$, $n = 5$

Let us recollect the formula $A = P\left(1+\frac{r}{p \text{ x } 100}\right)^{np}$

(*i*) If the interest is compounded annually, $p = 1$:

$$\boldsymbol{A} = P\left(1+\frac{r}{100}\right)^{n} = 1200 \text{ x } \left(1+\frac{5}{100}\right)^{5} = 1200 \text{ x } 1.2763 = \textbf{Rs. 1531.56}$$

CI $= A - P = 1531.56 - 1200 =$ **Rs. 331.56**

(*ii*) If the interest is compounded semi-annually, $p = 2$:

$$\boldsymbol{A} = P\left(1+\frac{r}{2 \text{ x } 100}\right)^{2n} = 1200 \text{ x } \left(1+\frac{5}{200}\right)^{10} = 1200 \text{ x } 1.28 = \textbf{Rs. 1536}$$

CI $= A - P = 1536 - 1200 =$ **Rs. 336.**

(*iii*) If the interest is compounded quarterly, p = 4:

$$\boldsymbol{A} = P\left(1+\frac{r}{4 \text{ x } 100}\right)^{4n} = 1200 \text{ x } \left(1+\frac{5}{400}\right)^{20} = 1200 \text{ x } 1.2820 =$$

Rs. 1538.4

CI = *A* – *P* = 1538.4 – 1200 = **Rs. 338.4**

(*iv*) If the interest is compounded monthly, p = 12:

$$\boldsymbol{A} = P\left(1+\frac{r}{12 \text{ x } 100}\right)^{12n} = 1200 \text{ x } \left(1+\frac{5}{1200}\right)^{60} = 1200 \text{ x } 1.2834 = \textbf{Rs. 1540}$$

CI = *A* – *P* = 1540 – 1200 = **Rs. 340**

***Example* 12:** Mr. Santosh wants to invest some amount for 4 years in a bank. Bank *X* offers 8% interest if compounded half yearly while bank *Y* offers 6% interest if compounded monthly. Which bank should Mr. Santosh select for better benefits?

Ans: Given n = 4.

Let the principal Mr. Santosh wants to invest be P = Rs. 100

From Bank *X*: ***r* = 8** and interest is compounded half-yearly, so ***p* = 2**.

$$\therefore A = P\left(1+\frac{r}{2 \text{ x } 100}\right)^{2n} = 100 \text{ x } \left(1+\frac{8}{200}\right)^{4} = 116.9858 \qquad \ldots (1)$$

From Bank *Y*: ***r* = 6, *p* = 12**

$$\therefore A = P\left(1+\frac{r}{12 \text{ x } 100}\right)^{12n} = 100 \text{ x } \left(1+\frac{6}{1200}\right)^{48} = 127.0489 \qquad \ldots (2)$$

Comparing (1) and (2), Dr. Ashwinikumar should invest his amount in bank *Y* as it gives more interest at the end of the period.

***Example* 13:** In how many years would Rs. 75,000 amount to Rs. 1,05,794.907 at 7% compound interest compounded semi-annually?

Ans: Given A = Rs. 105794.907, P = Rs. 75000, r = 7, p = 2

$$A = P\left(1+\frac{r}{2 \text{ x } 100}\right)^{2n}$$

$$\therefore\ 105794.907 = 75000 \text{ x } \left(1+\frac{7}{200}\right)^{2n}$$

$$\therefore\ \frac{105794.907}{75000} = (1.035)^{2n}$$

$$\therefore 1.41059876 = (1.035)^{2n}$$

$$\therefore\ (1.035)^{10} = (1.035)^{2n} \qquad \Rightarrow\ 2n = 10$$

Thus, ***n* = 5**

***Example* 14:** A certain principal amounts to Rs. 4410 after 2 years and to Rs.4630.50 after 3 years at a certain rate of interest compounded annually. Find the principal and the rate of interest.

Ans: Let the principal be P and rate of interest be r.

Now, we know that $A = P\left(1+\frac{r}{100}\right)^n$

From the given data we have,

$4410 = P\left(1+\frac{r}{100}\right)^2$ and $4630.5 = P\left(1+\frac{r}{100}\right)^3$

$\therefore 4410 = P(1 + 0.01\boldsymbol{r})^2$ … (1)

$4630.5 = P(1 + 0.01\boldsymbol{r})^3$ … (2)

Do not write '1 + 0.01*r*' as 1.01*r*

Dividing (2) by (1), we have

$\frac{4630.5}{4410} = \frac{P(1+0.01r)^3}{P(1+0.01r)^2} \Rightarrow 1.05 = 1 + 0.01\boldsymbol{r}$

$\therefore\ 0.05 = 0.01r$

Thus, $\boldsymbol{r = 5\%}$

***Example* 15:** Find the rate of interest at which a sum of Rs. 2000 amounts to Rs. 2690 in 3 years given that the interest is compounded half yearly. ($\sqrt[6]{1.345} = 1.05$)

Ans: Given P = Rs. 2000, A = Rs. 2680, $n = 3$, $p = 2$

Now, $A = P\left(1+\frac{r}{2 \text{ x } 100}\right)^{2n}$

$\therefore 2690 = 2000 \text{ x } \left(1+\frac{r}{200}\right)^6$

$\therefore \frac{2690}{2000} = \left(1+\frac{r}{200}\right)^6 \Rightarrow 1.345 = \left(1+\frac{r}{200}\right)^6$

$\therefore \sqrt[6]{1.345} = 1 + \frac{r}{200} \qquad \Rightarrow 1.05 = 1 + \frac{r}{200}$

$\therefore r = 0.05 \text{ x } 200 = 10\%$

Thus, the rate of compound interest is **10 %**.

***Example* 16:** If the interest compounded half yearly on a certain principal at the end of one year at 8% is Rs. 3264, find the principal.

Ans: Given CI = Rs. 3264, $n = 1$, $p = 2$ and $r = 8$

Now, $CI = A - P = P\left(1+\frac{8}{200}\right)^2 - P$

i.e. $3264 = P[(1.04)^2 - 1] = 0.0816P$

$\therefore P = \frac{3264}{0.0816} = 40000$

Thus, the principal is **Rs. 40,000.**

Check your progress 10.2

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1. Compute the compound amount and interest on a principal of Rs. 21,000 at 9% p.a. after 5 years.

Ans: 32,311.10, 11,311.10

2. Compute the compound amount and interest on a principal of Rs. 6000 at 11% p.a. after 8 years.

Ans: 13827.23, 7827.23

3. Compute the compound amount and compound interest of Rs. 5000 if invested at 11% for 3 years and the interest compounded *i*) annually, (*ii*) semi annually, (*iii*) quarterly and (*iv*) monthly.

Ans: (*i*) 6838.16, 1838.16 (*ii*) 6894.21, 189421
(*iii*) 6923.92, 1923.92 (*iv*) 6944.39, 1944.39

4. Compute the compound amount and compound interest of Rs. 1200 if invested at 9% for 2 years and the interest compounded *i*) annually, (*ii*) semi annually, (*iii*) quarterly and (*iv*) monthly.

Ans: (*i*) 1425.72, 225.72 (*ii*) 1431.02, 231.02
(*iii*) 1433.8, 233.8 (*iv*) 1435.7, 235.7

5. Miss Daizy invested Rs. 25,000 for 5 years at 7.5% with the interest compounded semi-annually. Find the compound interest at the end of 5 years. **Ans:** 11,126.10

6. Mr. Dayanand borrowed a sum of Rs. 6500 from his friend at 9% interest compounded quarterly. Find the interest he has to pay at the end of 4 years? **Ans:** 2779.54

7. Mr. Deepak borrowed a sum of Rs. 8000 from his friend at 8% interest compounded annually. Find the interest he has to pay at the end of 3 years? **Ans:** 2077.70

8. Mr. Deshraj borrowed Rs. 1,25,000 for his business for 3 years at 25% interest compounded half yearly. Find the compound amount and interest after 3 years. **Ans:** 2,53,410.82; 12,8410.82

9. Mrs. Hemlata bought a Sony Digital Camera for Rs. 15,800 from Vijay Electronics by paying a part payment of Rs. 2,800. The remaining amount was to be paid in 3 months with an interest of 9% compounded monthly on the due amount. How much amount did Mrs. Hemlata paid and also find the interest.

Ans: 13294.70, 294.70

10. Mr. Irshad bought a Kisan Vikas Patra for Rs. 10000, whose maturing value is Rs. 21,000 in 4½ years. Calculate the rate of interest if the compound interest is compounded quarterly.

Ans: 16.8%

11. What sum of money will amount to Rs. 11236 in 2 years at 6% p.a. compound interest? **Ans:** 10,000

12. Find the principal which will amount to Rs. 13468.55 in 5 years at 6% interest compounded quarterly. [$(1.015)^{20} = 1.346855$]

Ans: 10000

13. Find the principal which will amount to Rs. 30626.075 in 3 years at 7% interest compounded yearly. **Ans:** 25000

14. Find the principal if the compound interest payable annually at 8% p.a. for 2 years is Rs. 1664. **Ans:** 10000

15. If Mr. Sagar wants to earn Rs. 50000 after 4 years by investing a certain amount in a company at 10% rate of interest compounded annually, how much should he invest? **Ans:** 34150.67

16. Find after how many years will Rs. 4000 amount to Rs. 4494.40 at 6% rate of interest compounded yearly. **Ans:** $n = 2$

17. Find after how many years Rs. 10,000 amount to Rs. 12,155 at 10% rate of interest compounded half-yearly. **Ans:** $n = 1$

18. Find the rate of interest at which a principal of Rs.10000 amounts to Rs. 11236 after 2 years. **Ans:** 6%

19. Find the rate of interest at which a principal of Rs.50000 amounts to Rs. 62985.6 after 3 years. ($\sqrt[3]{1.259712} = 1.08$) **Ans:** 8%

20. Mrs. Manisha Lokhande deposited Rs. 20,000 in a bank for 5 years. If she received Rs.3112.50 as interest at the end of 2 years, find the rate of interest p.a. compounded annually. **Ans:** 7.5%

21. A bank *X* announces a super fixed deposit scheme for its customers offering 10% interest compounded half yearly for 6 years. Another bank *Y* offers 12% simple interest for the same period. Which bank's scheme is more beneficial for the customers? **Ans:** Bank *X*

22. *ABC* bank offers 9% interest compounded yearly while *XYZ* bank offers 7% interest compounded quarterly. If Mr. Arunachalam wants to invest Rs. 18000 for 5 years, which bank should he choose?

Ans: Bank *ABC*

23. Mangesh borrowed a certain amount from Manish at a rate of 9% for 4 years. He paid Rs. 360 as simple interest to Manish. This amount he invested in a bank for 3 years at 11% rate of interest compounded quarterly. Find the compound interest Mangesh received from the bank after 3 years. **Ans:** 1384.78

24. On a certain principal for 3 years the compound interest compounded annually is Rs. 1125.215 while the simple interest is Rs. 1050, find the principal and the rate of interest.

Ans: 5000, 7%

25. On a certain principal for 4 years the compound interest compounded annually is Rs. 13923 while the simple interest is Rs. 12000, find the principal and the rate of interest. **Ans:** 30000, 10%.

26. Which investment is better for Mr. Hariom Sharma (i) 6% compounded half yearly or (ii) 6.2% compounded quarterly?

Ans:

27. Which investment is better for Mr. Suyog Apte (i) 9% compounded yearly or (ii) 8.8% compounded quarterly?

Ans:

28. A bank *X* offers 7% interest compounded semi-annually while another bank offers 7.2% interest compounded monthly. Which bank gives more interest at the end of the year?

Ans:

29. Mr. Nitin Tare has Rs. 10000 to be deposited in a bank. One bank offers 8% interest p.a. compounded half yearly, while the other offers 9% p.a. compounded annually. Calculate the returns in both banks after 3 years. Which bank offers maximum return after 3 years?

Ans:

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Unit VII

11

ANNUITIES AND EMI

OBJECTIVES

After reading this chapter you will be able to:

- Define annuity, future value, present value, EMI, Sinking Fund.
- Compute Future Value of annuity due, Present Value of an ordinary annuity.
- Compute EMI of a loan using reducing balance method and flat interest method.
- Compute Sinking Fund (periodic payments).

11.1 INTRODUCTION

In the previous chapter we have seen how to compute compound interest when a lump sum amount is invested at the beginning of the investment. But many a time we pay (or are paid) a certain amount not in lump sum but in periodic installments. This series of equal payments done at periodic intervals is called as *annuity*.

Let us start the chapter with the definition of an *annuity*.

11.2 ANNUITY

A series of equal periodic payments is called ***annuity***. The payments are of ***equal size*** and ***at equal time interval***.

The common examples of annuity are: monthly recurring deposit schemes, premiums of insurance policies, loan installments, pension installments etc. Let us understand the terms related to annuities and then begin with the chapter.

Periodic Payment:
The amount of payment made is called as *periodic payment*.

Period of Payment:
The time interval between two payments of an annuity is called as the *period of payment*.

Term of an annuity:
The time of the beginning of the first payment period to the end of the last payment period is called as *term of annuity*. An annuity gets *matured* at the end of its term.

11.3 TYPES OF ANNUITIES

Though we will be discussing two types of annuities in detail, let us understand different types of annuities based on the duration of the term or on the time when the periodic payments are made. On the basis of the closing of an annuity, there are three types of annuities:

1. **Certain Annuity**:
   Here the duration of the annuity is fixed (or certain), hence called *certain annuity*. We will be learning such annuities in detail.

2. **Perpetual Annuity**:
   Here the annuity has no closing duration, i.e. it has indefinite duration. Practically there are rarely any perpetuities.

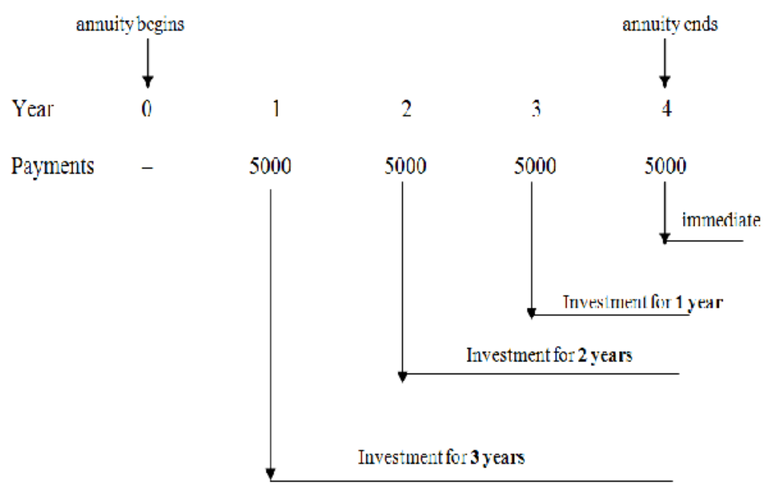
3. **Contingent Annuity**:
   Here the duration of the annuity depends on an event taking place. An example of contingent annuity is *life annuity*. Here the payments are to be done till a person is alive, like, pension, life insurance policies for children (maturing on the child turning 18 years) etc.

On the basis of when the periodic payments are made we have two types of annuities: ordinary annuity and annuity due.

11.3.1 Immediate (Ordinary) Annuity:

The annuity which is paid at the *end of each period* is called as *immediate* (*ordinary*) *annuity*. The period can be monthly, quarterly or yearly etc. For example, stock dividends, salaries etc.

Let us consider an example of an investment of Rs. 5000 each year is to be made for four years. If the investment is done at the end of each year then we have the following diagrammatic explanation for it:



11.3.2 Present Value:

The sum of all periodic payments of an annuity is called its *present value*. In simple words, it is that sum which if paid *now* will give the same amount which the periodic payments would have given at the end of the decided period. It is the one time payment of an annuity.
The formula to find the present value (*PV*) is as follows:

$$PV = \frac{P}{\left(\frac{r}{p \text{ x } 100}\right)}\left[1 - \frac{1}{\left(1 + \frac{r}{p \text{ x } 100}\right)^{np}}\right]$$

Where
P: periodic equal payment
r: rate of interest p.a.
p: period of annuity

Let $\boldsymbol{i} = \frac{\boldsymbol{r}}{\boldsymbol{p} \textbf{ x 100}}$, the rate per period, then the above formula can be rewritten as follows:

$$PV = \frac{P}{i}\left[1 - \frac{1}{(1+i)^{np}}\right]$$

11.3.3 Future Value (Accumulated value):

The sum of all periodic payments along with the interest is called the *future value* (*accumulated amount*) of the annuity.

The formula to find the future value (*A*) of an immediate annuity is as follows:

$$\boldsymbol{A} = \frac{P}{\frac{r}{p \text{ x } 100}}\left[\left(1 + \frac{r}{p \text{ x } 100}\right)^{np} - 1\right]$$

$$\boldsymbol{A} = \frac{P}{i}\left[(1+i)^{np} - 1\right]$$

Here,
P: periodic equal payment
r: rate of interest p.a.
p: period of annuity i.e. yearly, half yearly, quarterly or monthly
and $i = \frac{r}{p \text{ x } 100}$

***Example* 1:** Find the future value after 2 years of an immediate annuity of Rs. 5000, the rate of interest being 6% p.a compounded annually.

Ans: Given $n = 2$, P = Rs. 5000, $r = 6$ and $p = 1 \Rightarrow i = \frac{6}{100} = 0.06$

$$A = \frac{P}{i}\left[(1+i)^{np} - 1\right] = \frac{5000}{0.06}\left[(1+0.06)^2 - 1\right] = 5000\left[\frac{1.1236 - 1}{0.06}\right]$$

$\therefore \boldsymbol{A}$ = 5000 x 2.06 = **Rs. 10300**

***Example* 2**: Find the amount for an ordinary annuity with periodic payment of Rs. 3000, at 9% p.a. compounded semi-annually for 4 years.

Ans: Given $n = 4$, P = Rs. 3000, $r = 9$ and $p = 2 \Rightarrow i = \frac{9}{2 \text{ x } 100} = 0.045$

Now, $A = \frac{P}{i}\left[(1+i)^{np} - 1\right] = \frac{3000}{0.045}[(1+0.045)^{2 \text{ x } 4} - 1] = \frac{3000}{0.045}$ x 0.4221

Thus, ***A* = Rs. 28,140**

***Example* 3**: Mr. Ravi invested Rs. 5000 in an annuity with quarterly payments for a period of 2 years at the rate of interest of 10%. Find the accumulated value of the annuity at the end of 2nd year.

Ans: Given n = 2, P = Rs.5000, r = 10 and p = 4 $\Rightarrow i = \frac{10}{4 \text{ x } 100} = 0.025$

Now, $A = \frac{P}{i}\left[(1+i)^{np} - 1\right] = \frac{5000}{0.025}\left[(1.025)^{2 \text{ x } 4} - 1\right] = \frac{5000}{0.025}$ x 0.2184

Thus, ***A* = Rs. 43,680**

***Example* 4:** Mr. Ashok Rane borrowed Rs. 20,000 at 4% p.a. compounded annually for 10 years. Find the periodic payment he has to make.

Ans: Given PV = Rs. 20,000; n = 10; p = 1 and r = 4 $\Rightarrow i = 0.04$

Now to find the periodic payments P we use the following formula:

$$PV = \frac{P}{i}\left[1 - \frac{1}{(1+i)^{np}}\right]$$

$$\therefore 20000 = \frac{P}{0.04}\left[1 - \frac{1}{(1+0.04)^{10}}\right] = \frac{P}{0.04} \text{ x } 0.3244$$

$$\therefore P = \frac{20000 \text{ x } 0.04}{0.3244} = 2466.09$$

Thus, Mr. Rane has to make the periodic payments of Rs. 2466.09

***Example* 5:** Find the future value of an immediate annuity after 3 years with the periodic payment of Rs. 12000 at 5% p.a. if the period of payments is (i) yearly, (ii) half-yearly, (iii) quarterly and (iv) monthly.

Ans: Given P = Rs. 1200, n = 3, r = 5

(i) period p = 1 then $i = \frac{5}{100} = 0.05$

$$A = \frac{P}{i}\left[(1+i)^{n} - 1\right] = \frac{12000}{0.05}\left[(1+0.05)^{3} - 1\right] = \frac{12000}{0.05}[1.1576 - 1]$$

$\therefore$ A = 12000 x 3.1525 = **Rs. 37,830**

(ii) period p = 2 then $i = \frac{5}{2 \text{ x } 100} = 0.025$

$$A = \frac{P}{i}\left[(1+i)^{2n} - 1\right] = \frac{12000}{0.025}\left[(1+0.025)^{6} - 1\right] = \frac{12000}{0.025} \text{ x } 0.1597$$

$\therefore A$ = 12000 x 6.388 = **Rs. 76,656**

(iii) period p = 4 then $i = \frac{5}{4 \text{ x } 100} = 0.0125$

$A = \frac{P}{i}\left[(1+i)^{4n} - 1\right] = \frac{12000}{0.0125}[(1 + 0.0125)^{12} - 1] = \frac{12000}{0.0125}$ x 0.16075

$\therefore A$ = 12000 x 12.86 = **Rs. 1,54,320**

(iv) period p = 12 then $i = \frac{5}{12 \text{ x } 100}$ = 0.00417

$A = \frac{P}{i}\left[(1+i)^{12n} - 1\right] = \frac{12000}{0.00417}[(1 + 0.00417)^{36} - 1] = \frac{12000}{0.00417}$ x 0.1615

$\therefore A$ = 1200 x 38.729 = **Rs. 4,64,748**

***Example* 6:** Mr. Nagori invested certain principal for 3 years at 8% interest compounded half yearly. If he received Rs.72957.5 at the end of 3rd year, find the periodic payment he made. $[(1.04)^6 = 1.2653]$

Ans: Given n = 3, r = 8, p = 2 $\Rightarrow i = \frac{8}{2 \text{ x } 100}$ = 0.04

Now, $A = \frac{P}{i}\left[(1+i)^{np} - 1\right]$

$\therefore$ 72957.5 = $\frac{P}{0.04}[(1 + 0.04)^6 - 1] = \frac{P}{0.04}$ x 0.2653

$\therefore$ 72957.5 = P[6.6325]

$\therefore P = \frac{72957.5}{6.6325}$ = 11000

Thus, the periodic payment is **Rs. 11,000**

11.4 SINKING FUND

The fund (money) which is kept aside to accumulate a certain sum in a fixed period through periodic equal payments is called as **sinking fund**.

We can consider an example of a machine in a factory which needs to be replaced after say 10 years. The amount for buying a new machine 10 years from now may be very large, so a proportionate amount is accumulated every year so that it amounts to the required sum in 10 years. This annual amount is called as *sinking fund*. Another common example is of the *maintenance tax* collected by any Society from its members.

A sinking fund being same as an annuity, the formula to compute the terms is same as that we have learnt in section **11.3.3**

***Example* 7:** A company sets aside a sum of Rs. 15,000 annually to enable it to pay off a debenture issue of Rs. 1,80,000 at the end of 10 years. Assuming that the sum accumulates at 6% p.a., find the surplus after paying off the debenture stock.

Ans: Given P = Rs. 15000, n = 10, r = 6 $\Rightarrow i$ = 0.06

$$\therefore A = \frac{P}{i}\left[(1+i)^n - 1\right] = \frac{15000}{0.06} \text{ x } [(1 + 0.06)^{10} - 1] = \frac{15000}{0.06} \text{ x } 0.7908$$

$\therefore$ ***A* = Rs. 1,97,700**

Thus, the surplus amount after paying off the debenture stock is
= 197712 – 180000 = **Rs. 17712.**

***Example* 8:** Shriniketan Co-op Hsg. Society has 8 members and collects Rs. 2500 as maintenance charges from every member per year. The rate of compound interest is 8% p.a. If after 4 years the society needs to do a work worth Rs. 100000, are the annual charges enough to bear the cost?
Ans: Since we want to verify whether Rs. 2500 yearly charges are enough or not we assume it to be *P* and find its value using the formula:

$$A = \frac{P}{i}\left[(1+i)^n - 1\right]$$

Here A = Rs. 100000, $n = 4$, $r = 8 \Rightarrow i = 0.08$

$$\therefore P = \frac{A \text{ x } i}{(1+i)^n - 1} = \frac{100000 \text{ x } 0.08}{(1+0.08)^4 - 1} = 22192$$

Thus, the annual payment of all the members i.e. 8 members should be Rs. 22192.

$\therefore$ the annual payment per member $= \frac{22192}{8}$ = Rs. 2774

This payment is less than Rs. 2500 which the society has decided to take presently. Thus, the society should increase the annual sinking fund.

11.5 EQUATED MONTHLY INSTALLMENT (EMI)

Suppose a person takes a loan from a bank at a certain rate of interest for a fixed period. The equal payments which the person has to make to the bank per month are called as ***e**quated **m**onthly **i**nstallments* in short EMI.

Thus, EMI is a kind of annuity with **period of payment being monthly** and the **present value being the sum borrowed**.
We will now study the method of finding the EMI using **reducing balance method** and **flat interest method.**

(a) Reducing balance method:
Let us recall the formula of finding the present value of an annuity.

$$PV = \frac{P}{i}\left[1 - \frac{1}{(1+i)^{np}}\right]$$

The equal periodic payment (P) is our EMI which is denoted it by E.

The present value (*PV*) is same as the sum (*S*) borrowed.

Also the period being monthly ***p* = 12**, ***i*** = $\frac{r}{1200}$ as we are interested in finding **monthly** installments and ***n*** is period in **years**.
Substituting this in the above formula we have:

$$S = \frac{E}{i}\left[1 - \frac{1}{(1+i)^{12n}}\right]$$

Thus, if *S* is the sum borrowed for *n* years with rate of interest *r* % p.a. then the *EMI* is calculated by the formula:

$$E = \frac{S \text{ x } i}{1 - \frac{1}{(1+i)^{12n}}}$$

(b) Flat Interest Method:

Here the amount is calculated using Simple Interest for the period and the EMI is computed by dividing the amount by total number of monthly installments.

Let ***S*** denote the sum borrowed, ***r*** denote the rate of interest and ***n*** denote the duration in years, then as we know the amount using simple interest formula is ***A*** = ***S***$\left(1 + \frac{nr}{100}\right)$**.** The total number of monthly installments for duration of ***n*** years is **12*n***. Hence the *EMI* is calculated as

$$\boldsymbol{E} = \frac{A}{12n}$$

***Example* 9:** Mr. Sudhir Joshi has taken a loan of Rs. 10,00,000 from a bank for 10 years at 11% p.a. Find his EMI using (a) reducing balance method and (b) Flat interest method.

Ans: Given *S* = Rs. 1000000, *n* = 10, *r* = 11 $\Rightarrow i = \frac{11}{1200} = 0.0092$

(a) **Using flat interest method**:

$$A = S\left(1 + \frac{nr}{100}\right) = 1000000\left(1 + \frac{110}{100}\right) = 2100000$$

Thus, $E = \frac{A}{12n} = \frac{2100000}{120} = 17{,}500$ … (1)

(b) **Using reducing balance method**:

Now, $E = \frac{S \text{ x } i}{1 - \frac{1}{(1+i)^{12n}}} = \frac{1000000 \text{ x } 0.0092}{1 - \frac{1}{(1+0.0092)^{120}}} = 13797.65$

$\therefore$ ***E* = Rs. 13,798** approximately … (2)

Comparing (1) and (2), *we can see that the EMI using flat interest method is higher than by reducing balance method*.

***Example* 10:** Mr. Prabhakar Naik has borrowed a sum of Rs. 60,000 from a person at 6% p.a. and is due to return it back in 4 monthly installments. Find the *EMI* he has to pay and also prepare the amortization table of repayment.

Ans: Given S = Rs. 60,000; n = 4 months;

$$r = 6\% \quad \Rightarrow i = \frac{6}{1200} = 0.005$$

$$\text{Now, } E = \frac{S \text{ x } i}{1-\frac{1}{(1+i)^n}} = \frac{60000 \text{ x } 0.005}{1-\frac{1}{(1+0.005)^4}} = \frac{300}{0.01975}$$

$$\therefore \boldsymbol{E} = \textbf{Rs. 15,187.97}$$

Now, we will prepare the ***amortization table*** i.e. the table of repayment of the sum borrowed using reducing balance method.

In the beginning of the 1st month the outstanding principal is the sum borrowed i.e. Rs. 60000 and the EMI paid is Rs. 15187.97

The interest on the outstanding principal is 0.005 x 60000 = Rs. 300 … (1)
Thus, the principal repayment is 15187.97 – 300 = Rs. 14887.97 … (2)

The outstanding principal (**O/P**) in the beginning of the 2nd month is now 60000 – 14887.97 = 45112.03.

Note:

- (1) is called the ***interest part*** of the EMI and (2) is called as the ***principal part*** of the EMI.
- As the tenure increases the interest part reduces and the principal part increases.

This calculation can be tabulated as follows:

Month	**O/P**	**EMI**	**Interest Part**	**Principal Part**
	(*a*)	**(*b*)**	**(*c*) = (*a*) x *i***	**(*b*) - (*c*)**
1	60000	15187.97	300	14887.97
2	45112.03	15187.97	225.56	14962.45
3	30141.02	15187.97	150.75	15037.22
4	15111.80	15187.97	75.56	15112.41

In the beginning of the 4th month the outstanding principal is Rs. 15111.80 but the actual principal repayment in that month is Rs. 15112.41. This difference is due to rounding off the values to two decimals, which leads the borrower to pay 61 paise more!!

***Example* 11:** Mr. Shyam Rane has borrowed a sum of Rs. 100000 from a bank at 12% p.a. and is due to return it back in 5 monthly installments. Find the *EMI* he has to pay and also prepare the amortization table of repayment.

Ans: Given S = Rs. 100000; n = 5 months;

$r = 12\%$ p.a. $= \frac{12}{12} = 1\%$ p.m $\quad \Rightarrow i = 0.01$

Now, $$E = \frac{S \text{ x } i}{1-\frac{1}{(1+i)^n}} = \frac{100000 \text{ x } 0.01}{1-\frac{1}{(1+0.01)^5}} = \frac{1000}{0.0485343} = 20603.98$$

The amortization table is as follows:

Month	O/P	EMI	Interest Part	Principal Part
	(*a*)	**(*b*)**	**(*c*) = (*a*) x *i***	**(*b*) - (*c*)**
1	100000	20603.98	1000	19603.98
2	80396.02	20603.98	803.96	19800.02
3	60596	20603.98	605.96	19998.02
4	40597.98	20603.98	405.98	20198
5	**20399.98**	20603.98	204	**20399.98**

Check your progress

1. An overdraft of Rs. 50,000 is to be paid back in equal annual installments in 20 years. Find the installments, if the interest is 12% p.a. compounded annually. [$(1.12)^{20} = 9.64629$]
2. A man borrows Rs. 30,000 at 6% p.a. compounded semi-annually for 5 years. Find the periodic payments he has to make.
3. What periodic payments Mr. Narayanan has to make if he has borrowed Rs. 1,00,000 at 12% p.a. compounded annually for 12 years? [$(1.12)^{12} = 3.896$]
4. Find the future value of an immediate annuity of Rs. 1200 at 6% p.a. compounded annually for 3 years.
5. Find the future value of an immediate annuity of Rs. 500 at 8% p.a. compounded p.m. for 5 years.
6. Find the accumulated value after 2 years if a sum of Rs. 1500 is invested at the end of every year at 10% p.a. compounded quarterly.
7. Find the accumulated amount of an immediate annuity of Rs. 1000 at 9% p.a. compounded semi-annually for 4 years.
8. Find the future value of an immediate annuity of Rs. 2800 paid at 10% p.a. compounded quarterly for 2 years. Also find the interest earned on the annuity.
9. Find the sum invested and the accumulated amount for an ordinary annuity with periodic payment of Rs. 2500, at the rate of interest of 9% p.a. for 2 years if the period of payment is (a) yearly, (b) half-yearly, (c) quarterly or (d) monthly.
10. Find the sum invested and the accumulated amount for an ordinary annuity with periodic payment of Rs. 1500, at the rate of interest of

10% p.a. for 3 years if the period of payment is (a) yearly, (b) half-yearly, (c) quarterly or (d) monthly.

11. Mr. Banerjee wants to accumulate Rs. 5,00,000 at the end of 10 years from now. How much amount should he invest every year at the rate of interest of 9% p.a. compounded annually?

12. Find the periodic payment to be made so that Rs. 25000 gets accumulated at the end of 4 years at 6% p.a. compounded annually.

13. Find the periodic payment to be made so that Rs. 30,000 gets accumulated at the end of 5 years at 8% p.a. compounded half yearly.

14. Find the periodic payment to be made so that Rs. 2000 gets accumulated at the end of 2 years at 12% p.a. compounded quarterly.

15. Find the rate of interest if a person depositing Rs. 1000 annually for 2 years receives Rs. 2070.

16. Find the rate of interest compounded p.a. if an immediate annuity of Rs. 50,000 amounts to Rs. 1,03,000 in 2 years.

17. Find the rate of interest compounded p.a. if an immediate annuity of Rs. 5000 amounts to Rs. 10400 in 2 years.

18. What is the value of the annuity at the end of 5 years, if Rs. 1000 p.m. is deposited into an account earning interest 9% p.a. compounded monthly? What is the interest paid in this amount?

19. What is the value of the annuity at the end of 3 years, if Rs. 500 p.m. is deposited into an account earning interest 6% p.a. compounded monthly? What is the interest paid in this amount?

20. Mr. Ashish Gokhale borrows Rs. 5000 from a bank at 8% compound interest. If he makes an annual payment of Rs. 1500 for 4 years, what is his remaining loan amount after 4 years?

(**Hint**: find the amount using compound interest formula for 4 years and then find the accumulated amount of annuity, the difference is the remaining amount.)

21. Find the present value of an immediate annuity of Rs. 10,000 for 3 years at 6% p.a. compounded annually.

22. Find the present value of an immediate annuity of Rs. 100000 for 4 years at 8% p.a. compounded half yearly.

23. Find the present value of an immediate annuity of Rs. 1600 for 2 years at 7% p.a. compounded half yearly.

24. A loan is repaid fully with interest in 5 annual installments of Rs. 15,000 at 8% p.a. Find the present value of the loan.

25. Mr. Suman borrows Rs. 50,000 from Mr. Juman and agreed to pay Rs. 14000 annually for 4 years at 10% p.a. Is this business profitable to Mr. Juman?

(**Hint:** Find the PV of the annuity and compare with Rs. 50000)

26. Mr. Paradkar is interested in saving a certain sum which will amount to Rs. 3,50,000 in 5 years. If the rate of interest is 12% p.a., how much should he save yearly to achieve his target?

27. Mr. Kedar Pethkar invests Rs. 10000 per year for his daughter from her first birthday onwards. If he receives an interest of 8.5% p.a., what is the amount accumulated when his daughter turns 18?

28. Dr. Wakankar, a dentist has started his own dispensary. He wants to install a machine chair which costs Rs. 3,25,000. The machine chair is also available on monthly rent of Rs. 9000 at 9% p.a. for 3 years. Should Dr. Wakankar buy it in cash or rent it?

29. A sum of Rs. 50,000 is required to buy a new machine in a factory. What sinking fund should the factory accumulate at 8% p.a. compounded annually if the machine is to be replaced after 5 years?

30. The present cost of a machine is Rs. 80,000. Find the sinking fund the company has to generate so that it could buy a new machine after 10 years, whose value then would be 25% more than of today's price. The rate of compound interest being 12% p.a. compounded annually.

31. Mr. Mistry has to options while buying a German wheel alignment machine for his garage: (a) either buy it at Rs. 1,26,000 or (b) take it on lease for 5 years at an annual rent of Rs. 30,000 at the rate of interest of 12% p.a.. Assuming no scrap value for the machine which option should Mr. Mistry exercise?

32. Regency Co-op. Hsg. Society which has 50 members require Rs. 12,60,000 at the end of 3 years from now for the society repairs. If the rate of compound interest is 10% p.a., how much fund the society should collect from every member to meet the necessary sum?

33. Mr. Lalwaney is of 40 years now and wants to create a fund of Rs. 15,00,000 when he is 60. What sum of money should he save annually so that at 13% p.a. he would achieve his target?

34. If a society accumulates Rs. 1000 p.a. from its 200 members for 5 years and receives 12% interest then find the sum accumulated at the end of the fifth year. If the society wants Rs. 13,00,000 for society maintenance after 5 years, is the annual fund of Rs. 1000 per member sufficient?

35. How much amount should a factory owner invest every year at 6% p.a. for 6 years, so that he can replace a mixture-drum (machine) costing Rs. 60,000, if the scrap value of the mixture-drum is Rs. 8,000 at the end of 6 years.

36. If a society accumulates Rs. 800 p.a. from its 100 members for 3 years and receives 9% interest then find the sum accumulated at the end of the third year. If the society wants Rs. 2,50,000 for society maintenance after 3 years, is the annual fund of Rs. 800 per member sufficient?

37. Mr. Kanishk wants clear his loan of Rs. 10,00,000 taken at 12% p.a. in 240 monthly installments. Find his EMI using reducing balance method.

38. Using the reducing balance method find the EMI for the following:

Loan amount (in Rs.)	**rate of interest (in % p.a.)**	**period of loan (in yrs.)**
i) 1000	6	5
ii) 50000	6	10
iii) 8000	7	6
iv) 12000	9	10
v) 1000	9.5	10
vi) 1100000	12.5	20

39. Mr. Vilas Khopkar has taken a loan of Rs. 90,000 at 11% p.a. Find the EMI using (a) reducing balance method and (b) Flat interest method, if he has to return the loan in 4 years.

40. Find the EMI using reducing balance method on a sum of Rs. 36,000 at 9%, to be returned in 6 monthly installments.

41. Find the EMI using reducing balance method on a sum of Rs. 72,000 at 12%, to be returned in 12 installments.

42. Mr. Sachin Andhale has borrowed Rs. 10,000 from his friend at 9% p.a. and has agreed to return the amount with interest in 4 months. Find his EMI and also prepare the amortization table.

43. Mr. Arvind Kamble has borrowed Rs. 30,000 from his friend at 14% p.a. If he is to return this amount in 5 monthly installments, find the installment amount, the interest paid and prepare the amortization table for repayment.

44. Mrs. Chaphekar has taken a loan of Rs. 1,25,000 from a bank at 12% p.a. If the loan has to be returned in 3 years, find the EMI, Mrs. Chaphekar has to pay. Prepare the amortization table of repayment of loan and find the interest she has to pay.

45. A loan of Rs. 75,000 is to be returned with interest in 4 installments at 15% p.a. Find the value of the installments.

46. A loan of Rs. 60,000 is to be returned in 6 equal installments at 12% p.a. Find the amount of the installments.

47. Find the sum accumulated by paying an EMI of Rs. 11,800 for 2 years at 10% p.a.

48. Find the sum accumulated by paying an EMI of Rs. 1,800 for 2 years at 12% p.a.

49. Find the sum accumulated by paying an EMI of Rs. 12,000 for 3 years at 9% p.a.

50. Find the sum accumulated by paying an EMI of Rs. 11,000 for 8 years at 9.5% p.a.

Hints & Solutions to Check your progress

(1) 6694 **(2)** 3517 **(3)** 16,144 **(4)** 3820.32
(5) 36555.65 **(6)** 13104 **(7)** 9380 **(8)** 24461

(9) **(10)**

Period	Sum Invested	Accumulated Amount
Yearly	5000	5225
Half-yearly	10000	10695.5
Quarterly	20000	21648
Monthly	60000	65471

Period	Sum Invested	Accumulated Amount
Yearly	4500	4965
Half-yearly	9000	10203
Quarterly	18000	20693
Monthly	54000	62635

(11) 32910 **(12)** 5715 **(13)** 2498.72 **(14)** 225
(15) 7% **(16)** 6% **(17)** 8% **(18)** 75424, 15424
(19) 19688, 1688 **(20)** 4719 **(21)** 26730
(22) 673274.5 **(23)** 5877 **(24)** 59890.65 **(25)** 44378, Yes
(26) 97093.4 **(27)** 393229.95 **(28)** 283021.25, take it on rent
(29) 12523 **(30)** 17698.42 **(31)** 108143.28 < 126000, Mr. Mistry should use the second option. **(32)** 16245 **(33)** 18530
(34) 1270569.47, not sufficient **(35)** 7454.86 **(36)** 2,62,248; yes
(37) 11,011
(38)

Loan amount (in Rs.)	rate of interest (in % p.a.)	period of loan (in yrs.)	EMI (in Rs.)
i) 1000	6	5	19
ii) 50000	6	10	555
iii) 8000	7	6	136
iv) 12000	9	10	152
v) 1000	9.5	10	13
vi) 1100000	12.5	20	12498

(39) 2326, 2700 **(40)** 6158.48 **(41)** 6397.11

(42)

Month	**O/P**	**EMI**	**Interest Part**	**Principal Part**
	(*a*)	**(*b*)**	**(*c*) = (*a*) x *i***	**(*b*) - (*c*)**
1	10000	2547.05	75	2472.05
2	7527.95	2547.05	56.45	2490.6
3	5037.35	2547.05	37.78	2509.27
4	2528.08	2547.05	18.96	2528.09

(43)

Month	**O/P**	**EMI**	**Interest Part**	**Principal Part**
	(*a*)	**(*b*)**	**(*c*) = (*a*) x *i***	**(*b*) - (*c*)**
1	30000	6212.23	351	5861.23
2	24138.77	6212.23	282.42	5929.81
3	18208.96	6212.23	213.04	5999.19
4	12209.77	6212.23	142.85	6069.38
5	6140.39	6212.23	71.84	6140.39

(45) 19339.57 **(46)** 16353 **(47)** 3,12,673.60
(48) 48552.24 **(49)** 4,93,832.6 **(50)** 15,72,727

❖❖❖

12

CORRELATION AND REGRESSION

OBJECTIVES

- To understand the relationship between two relevant characteristics of a statistical unit.
- Learn to obtain the numerical measure of the relationship between two variables.
- Use the mathematical relationship between two variables in order to estimate value of one variable from the other.
- Use the mathematical relationship to obtain the statistical constants line means and S.D.'s

12.1 INTRODUCTION

In the statistical analysis we come across the study of two or more relevant characteristics together in terms of their interrelations or interdependence. e.g. Interrelationship among production, sales and profits of a company. Inter relationship among rainfall, fertilizers, yield and profits to the farmers.

Relationship between price and demand of a commodity When we collect the information (data) on two of such characteristics it is called bivariate data. It is generally denoted by (X,Y) where X and Y are the variables representing the values on the characteristics.

Following are some examples of bivariate data.

a) Income and Expenditure of workers.
b) Marks of students in the two subjects of Maths and Accounts.
c) Height of Husband and Wife in a couple.
d) Sales and profits of a company.

Between these variables we can note that there exist some sort of interrelationship or cause and effect relationship. i.e. change in the value of one variable brings out the change in the value of other variable also.Such relationship is called as correlation.

Therefore, correlation analysis gives the idea about the nature and extent of relationship between two variables in the bivariate data.

12.2 TYPES OF CORRELATION:

There are two types of correlation.

a) Positive correlation. and b) Negative correlation.

12.2.1 Positive correlation: When the relationship between the variables *X* and Y is such that increase or decrease in X brings out the increase or decrease in *Y* also, i.e. there is direct relation between X and Y, the correlation is said to be positive. In particular when the 'change in X equals to change in *Y*' the correlation is perfect and positive. e.g. Sales and Profits have positive correlation.

12.2.2Negative correlation: When the relationship between the variables X and Y is such that increase or decrease in X brings out the decrease or increase in Y, i.e. there is an inverse relation between X and Y, the correlation is said to be negative. In particular when the 'change in X equals to change in Y' but in opposite direction the correlation is perfect and negative. e.g. Price and Demand have negative correlation.

12.3 MEASUREMENT OF CORRELATION

The extent of correlation can be measured by any of the following methods:

- Scatter diagrams
- Karl Pearson's co-efficient of correlation
- Spearman's Rank correlation

12.3.1 Scatter Diagram: The Scatter diagram is a chart prepared by plotting the values of X and Y as the points (X,Y) on the graph. The pattern of the points is used to explain the nature of correlation as follows. The following figures and the explanations would make it clearer.

(*i*) *Perfect Positive Correlation*:

If the graph of the values of the variables is a straight line with positive slope as shown in Figure **4.1**, we say there is a *perfect positive correlation* between *X* and *Y*. Here $r = 1$.

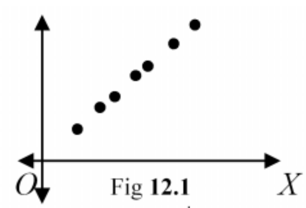


Fig **12.1**

(*ii*) *Imperfect Positive Correlation*:

If the graph of the values of *X* and *Y* show a band of points from lower left corner to upper right corner as shown in Figure **4.2**, we say that there is an *imperfect positive correlation*. Here $0 < r < 1$.

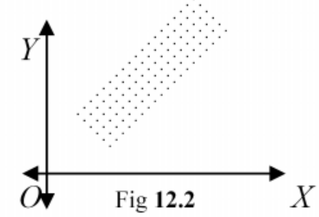


Fig **12.2**

(*iii*) *Perfect Negative Correlation*:

If the graph of the values of the variables is a straight line with negative slope as shown in Figure **4.3**, we say there is a *perfect negative correlation* between *X* and *Y*. Here $r = -1$.

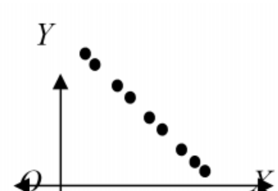


Fig. **12.3**

(*iv*) *Imperfect Negative Correlation*:

If the graph of the values of *X* and *Y* show a band of points from upper left corner to the lower right corner as shown in Figure **4.4**, then we say that there is an *imperfect negative correlation*. Here $-1 < r < 0$

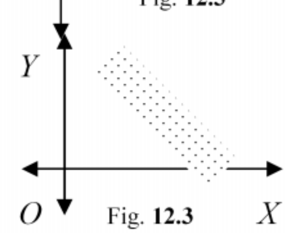


Fig. **12.3**

(*v*) *Zero Correlation*:

If the graph of the values of *X* and *Y* do not show any of the above trend then we say that there is a *zero correlation* between *X* and *Y*. The graph of such type can be a straight line perpendicular to the axis, as shown in Figure **4.5** and **4.6,** or may be completely scattered as shown in Figure **4.7**. Here $r = 0$.

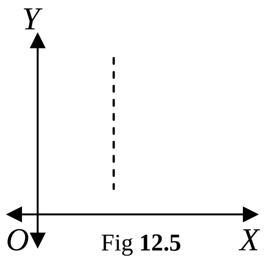


Fig **12.5**

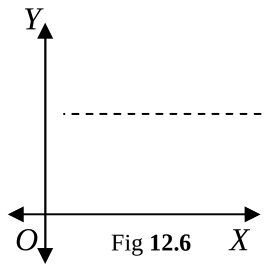


Fig **12.6**

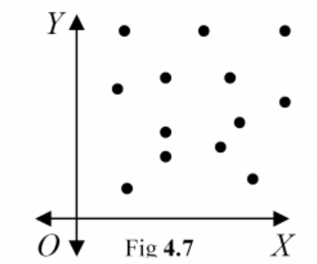


Fig **4.7**

The Figure **4.5** show that the increase in the values of *Y* has no effect on the value of *X*, it remains the same, hence zero correlation. The Figure **4.6** show that the increase in the values of *X* has no effect on the value of *Y*, it remains the same, hence zero correlation. The Figure **4.7** show that the points are completely scattered on the graph and show no particular trend, hence there is no correlation or zero correlation between *X* and *Y*.

12.3.2 Karl Pearson's co-efficient of correlation.

This co-efficient provides the numerical measure of the correlation between the variables X and Y. It is suggested by Prof. Karl Pearson and calculated by the formula

$$r = \frac{Cov(x, y)}{\sigma_x . \sigma_y}$$

Where, Cov(x,y) : Covariance between *x* & *y*

σ_x.: Standard deviation of *x* & σ_y: Standard deviation of *y*

Also, $Cov(x,y) = \frac{1}{n}\Sigma(x-\bar{x})(y-\bar{y}) = \frac{1}{n}\Sigma xy - \bar{x}\ \bar{y}$

$$S.D.(x) = \sigma x = \sqrt{\frac{1}{n}\Sigma(x-\bar{X})^2} \quad = \sqrt{\frac{1}{n}\Sigma x^2 - \bar{X}^2} \qquad \text{and}$$

$$S.D.(y) = \sigma y = \sqrt{\frac{1}{n}\Sigma(y-\bar{Y})^2} \quad \sqrt{\frac{1}{n}\Sigma y^2 - \bar{Y}^2}$$

Remark : We can also calculate this co-efficient by using the formula given by

$$r = \frac{\frac{1}{n}\Sigma(x-\bar{X})(y-\bar{Y})}{\sqrt{\frac{1}{n}\Sigma(x-\bar{X})^2}\sqrt{\frac{1}{n}\Sigma(y-\bar{Y})^2}} = \frac{\frac{\Sigma xy}{n} - \bar{X}\bar{Y}}{\sqrt{\left(\frac{\Sigma x^2}{n} - \bar{X}^2\right)\left(\frac{\Sigma y^2}{n} - \bar{Y}^2\right)}}$$

The Pearson's Correlation co-efficient is also called as the 'product moment correlation co-efficient'

Properties of correlation co-efficient ‘r’

The value of ‘r’ can be positive (+) or negative(-)

The value of ‘r’ always lies between –1 & +1, i.e. –1< r<+1]

Significance of ‘r’ equals to –1, +1 & 0

When ‘r’= +1; the correlation is perfect and positive.

‘r’= -1; the correlation is perfect and negative.

and when there is no correlation ‘r’= 0

SOLVED EXAMPLES :

Example.1: Calculate the Karl Pearson’s correlation coefficient from the following.

X:	12	10	20	13	15
Y:	7	14	6	12	11

Solution: Table of calculation,

X	Y	XxY	X^2	Y^2
12	7	84	144	49
10	14	140	100	196
20	6	120	400	36
13	12	156	169	144
15	11	165	225	121
$\Sigma x = 70$	$\Sigma y = 50$	$\Sigma xy = 665$	$\Sigma x^2 = 1038$	$\Sigma y^2 = 546$

And n= 5

The Pearson’s correlation coefficient r is given by,

$$r = \frac{Cov(x, y)}{\sigma_x . \sigma_y}$$

Where,

$$\bar{x} = \frac{\Sigma x}{n} = \frac{70}{5} = 14 \qquad \bar{y} = \frac{\Sigma y}{n} = \frac{50}{5} = 10$$

$$Cov(x,y) = \frac{\Sigma xy}{n} - \bar{X}\bar{Y} \qquad \sigma_x = \sqrt{\frac{1}{n}\Sigma x^2 - \bar{X}^2} \qquad \sigma_y = \sqrt{\frac{1}{n}\Sigma y^2 - \bar{Y}^2}$$

$$= \frac{665}{5} - 14x10 \qquad = \sqrt{\frac{1038}{5} - 14^2} \qquad = \sqrt{\frac{546}{5} - 10^2}$$

$$= 133 - 140 \qquad = \sqrt{11.6} = 3.40 \qquad \sqrt{9.2} = 3.03 = \quad -07$$

∴ Cov(x,y) = -7 σ_x = 3.40 and σ_y = 3.03

Substituting the values in the formula of **r** we get

$$r = \frac{-7}{3.40x3.03} = -0.68$$

∴ r = - 0.68

Example 2: Let us calculate co-efficient of correlation between Marks of students in the Subjects of Maths & Accounts. in a certain test conducted.

Table of calculation:

Marks InMaths X	Marks In Accounts Y	XY	X^2	Y^2
28	30	840	784	900
25	40	1000	625	1600
32	50	1600	1024	2500
16	18	288	256	324
20	25	500	400	625
15	12	180	225	144
19	11	209	361	121
17	21	357	289	441
40	45	1800	1600	2025
30	35	1050	900	1225
$\Sigma x = 242$	$\Sigma y = 287$	$\Sigma xy = 7824$	$\Sigma x^2 = 6464$	Σy^2 9905

n=10

Now Pearson's co-efficient of correlation is given by the fomula,

$$r = \frac{Cov(x,y)}{\sigma_x . \sigma_y}$$

Where,

$$\bar{x} = \frac{\Sigma x}{n} = \frac{242}{10} = 24.2 \qquad \bar{y} = \frac{\Sigma y}{n} = \frac{287}{10} = 28.7$$

$$\text{Cov}(x,y) = \frac{\Sigma xy}{n} - \bar{X}\bar{Y} = \frac{7824}{10} - 24.2\text{x}28.7 = 782.4 - 694.54$$

$$\sigma_x = \sqrt{\frac{1}{n}\Sigma x^2 - \bar{X}^2} \qquad \sigma_x = \sqrt{\frac{6464}{10} - 24.2^2} = \sqrt{60.76}$$

$$\sigma_y = \sqrt{\frac{1}{n}\Sigma y^2 - \bar{Y}^2} \qquad \sigma_y = \sqrt{\frac{9905}{10} - 28.7^2} = \sqrt{166.81}$$

Cov(x,y) = 87.86, σ_x = 7.79 and σ_y = 12.91

∴ Cov(x,y) = 87.86 σ_x = 7.79 and σ_y = 12.91

Substituting the values in the formula of **r** we get

$$r = \frac{87.86}{7.79 x 12.91} = \mathbf{0.87}$$

∴ r = 0.87

12.3 RANK CORRELATION

In many practical situations, we do not have the scores on the characteristics, but the ranks (preference order) decided by two or more observers. Suppose, a singing competition of 10 participants is judged by two judges A and B who rank or assign scores to the participants on the basis of their performance. Then it is quite possible that the ranks or scores assigned may not be equal for all the participants. Now the difference in the ranks or scores assigned indicates that there is a difference of openion between the judges on deciding the ranks. The rank correlation studies the association in this ranking of the observations by two or more observers. The measure of the extent of association in rank allocation by the two judges is calculated by the co-efficient of Rank correlation 'R'. This co-efficient was developed by the British psychologist Edward Spearman in 1904.

Mathematically, Spearman's rank correlation co-efficient is defined as,

$$R = 1 - \frac{6\Sigma d^2}{n(n^2 - 1)}$$

Where d= rank difference and n= no of pairs.

Remarks: We can note that, the value of 'R' always lies between –1 and +1

The positive value of 'R' indicates the positive correlation (association) in the rank allocation. Whereas, the negative value of 'R' indicates the negative correlation (association) in the rank allocation.

SOLVED EXAMPLES:

Example 3

a) <u>When ranks are given:-</u>

Data given below read the ranks assigned by two judeges to 8 participennts. Calculate the co-efficient of Rank correlation.

Participant No.	Ranks by Judge		Rank diff Square d^2
	A	B	
1	5	4	$(5-4)^2 = 1$
2	6	8	4
3	7 `	1	36
4	1	7	36
5	8	5	9
6	2	6	16
7	3	2	1
8	4	3	1
N = 8		Total	$104 = \Sigma d^2$

Spearman's rank correlation co-efficient is given by

$$R=1-\frac{6\Sigma d^2}{n(n^2-1)}$$

Substituting the values from the table we get,

$$R= 1- \frac{6x\ 104}{8(8^2-1)} = -0.23$$

The value of correlation co-efficient is - 0.23. This indicates that there is negative association in rank allocation by the two judges A and B

b) <u>When scores are given:-</u>

Example 4

The data given below are the marks given by two Examiners to a set of 10 students in a aptitude test. Calculate the Spearman's Rank correlation co-efficient, 'R'

Student No	Marka by Examiner		Ranks		Rank difference square
	A	B	R_A	R_B	D^2
1	85	80	2	2	0
2	56	60	8	7	1
3	45	50	10	10	0
4	65	62	6	6	0
5	96	90	1	1	0
6	52	55	9	8	1
7	80	75	3	4	1
8	75	68	5	5	0
9	78	77	4	3	1
10	60	53	7	9	1
N = 10				Total	$5 = \Sigma d^2$

Now the Spearman's rank correlation co-efficient is given by

$$R= 1- \frac{6\Sigma d^2}{n(n^2-1)}$$

Substituting the values from the table we get,

$$R= 1-\frac{6x5}{10(10^2-1)}$$

$$= 1-0.03$$

$$= 0.97$$

The value of correlation co-efficient is + 0.97. This indicates that there is positive association in assessment of two examiners, A and B.

c) <u>Case of repeated values:-</u>

It is quite possible that the two participants may be assigned the same score by the judges. In such cases Rank allocation and calculation of rank correlation can be explained as follows.

Example: The data given below scores assigned by two judges for 10 participants in the singing competition. Calculate the Spearman's Rank correlation co-efficient.

Participant No	Score assigned By Judges		Ranks		Rank difference square
	A	B	R_A	R_B	D^2
1	28	35	**9 (8.5)**	6	$(8.5-6)^2$ =6.25
2	40	26	3	**10(9.5)**	42.25
3	35	42	**5 (4.5)**	3	2.25
4	25	26	10	**9 (9.5)**	0.25
5	28	33	**8 (8.5)**	7	2.25
6	35	45	**4 (4.5)**	2	6.25
7	50	32	1	8	49
8	48	51	2	1	1
9	32	39	6	4	4
10	30	36	7	5	4
N = 10				Total	Σd^2 =117.5

Explanation:- In the column of A and B there is repeatation of scores so while assigning the ranks we first assign the ranks by treating them as different values and then for rereated scores we assign the average rank. e.g In col A the score 35 appears 2 times at number 4 and 5 in the order of ranking so we calculate the average rank as <u>(4+5)/2 = 4.5.</u>
Hence the ranks assigned are 4.5 each. The other repeated scores can be ranked in the same manner.

Note: In this example we can note that the ranks are in fraction e.g. 4.5, which is logically incorrect or meaningless. Therefore in the calculation of 'R' we add a correction factor (C.F.) to Σd^2 calculated as follows.

Table of correction factor (C.F.)

Value Repeated	Frequency M	$m(m^2-1)$
35	2	$2x(2^2-1)=6$
28	2	6
26	2	6
	Total	$\Sigma m(m^2-1)=18$

Now $C.F. = \frac{\Sigma(m^3-m)}{12}$ = 18/12 =1.5

$\therefore \Sigma d^2$ = 117.5+1.5= 119

We use this value in the calculation of 'R'

Now the Spearman's rank correlation co-efficient is given by

$$R = 1 - \frac{6\Sigma d^2}{n(n^2-1)}$$

Substituting the values we get, $R = 1 - \frac{6x\ 119}{10(10^2-1)}$ = 1-0.72 = 0.28

EXERCISE I

1. What is mean by correlation? Explain the types of correlation with suitable examples.

2. What is a scatter diagram? Draw different scattered diagrams to explain the correlation between two variables x and y.

3. State the significance of 'r' = +1, –1 and 0.

4. Calculate the coefficient of correlation r from the following data.

X:	18	12	16	14	10	15	17	13
Y:	9	13	20	15	11	24	26	22

5. The following table gives the price and demand of a certain commodity over the period of 8 months. Calculate the Pearson's coefficient of correlation.

Price:	15	12	23	25	18	17	11	19
Demand	45	30	60	65	48	45	28	50

6. Following results are obtained on a certain bivariate data.

(i) n = 10 Σx= 75 Σy = 70 Σx^2 = 480
Σy^2 = 600 Σxy = 540

(ii) n = 15 Σx= 60 Σy = 85 Σx^2 = 520
Σy^2 =1200 Σxy = -340

Calculate the Pearson's correlation coefficient in each case.

7. Following data are available on a certain bi-variate data :

(i) $\Sigma(x-\bar{x})(y-\bar{y})=120$, $\Sigma(x-\bar{x})^2 = 150$ $\Sigma(y-\bar{y})^2 = 145$

(ii) $\Sigma(x-\bar{x})(y-\bar{y})= -122$, $\Sigma(x-\bar{x})^2 = 136$ $\Sigma(y-\bar{y})^2 = 148$

Find the correlation coefficient.

8. Calculate the Pearson's coefficient of correlation from the given information on a bivariate series:

No of pairs: 25
Sum of x values:300
Sum of y values:375
Sum of squares of x values: 9000
Sum of squares of y values:6500
Sum of the product of x and y values:4000.

9. The ranks assigned to 8 participants by two judges are as followes.Calculate the Spearman's Rank correlation coefficient 'R'.

Participant No:	1	2	3	4	5	6	7	8
Ranks by JudgeI:	5	3	4	6	1	8	7	2
JudgeII :	6	8	3	7	1	5	4	2

10.Calculate the coefficient of rank correlation from the data given below.

X:	40	33	60	59	50	55	48
Y:	70	60	85	75	72	82	69

11. Marks given by two Judges to a group of 10 participants are as follows. Calculate the coefficient of rank correlation.

Marks by Judge										
A:	52	53	42	60	45	41	37	38	25	27
Judge B:	65	68	43	38	77	48	35	30	25	50.

12. An examination of 8 applicants for a clerical post was by a bank. The marks obtained by the applicants in the subjects of Mathematics and Accountancy were as follows. Calculate the rank correlation coefficient.

Applicant:	A	B	C	D	E	F	G	H
Marks in Maths:	15	20	28	12	40	60	20	80
Marks in Accounts:	40	30	50	30	20	10	25	60

12.4 REGRESSION ANALYSIS

As the correlation analysis studies the nature and extent of interrelationship between the two variables X and Y, regression analysis helps us to estimate or approximate the value of one variable when we know the value of other variable. Therefore we can define the 'Regression' as the estimation (prediction) of one variable from the other

variable when they are correlated to each other. e.g. We can estimate the Demand of the commodity if we know it's Price.
Why are there two regressions?
When the variables X and Y are correlated there are two possibilities,

(i) Variable X depends on variable y. in this case we can find the value of x if know the value of y. This is called regression of x on γ.

(ii) Variable γ depends on variable X. we can find the value of y if know the value of X. This is called regression of y on x. Hence there are two regressions,

(a) Regression of X on Y; (b) Regression of *X* on Y.

8.4.1 Formulas on Regression equation,

Regression of X **on** Y	**Regression of** X **on** Y
Assumption: X depends on Y The regression equation is $(x-\bar{x}) = b_{xy}(y-\bar{y})$ b_{xy}= Regression co-efficient of X on Y = $\frac{Cov(x,y)}{V(y)}$	Y depends on X The regression equation is $(y-\bar{y}) = b_{yx}(x-\bar{x})$ b_{yx}= Regression co-efficient of Y on X = $\frac{Cov(x,y)}{V(x)}$

Where,

$$Cov(x,y) = \frac{1}{n}\Sigma(x-\bar{x})(y-\bar{y}) = \frac{1}{n}\Sigma xy-\bar{x}\,\bar{y}$$

$$V(x) = \frac{1}{n}\Sigma(x-\bar{x})^2 \quad \text{and} \quad V(y) = \frac{1}{n}\Sigma(y-\bar{y})^2$$

$$V(x) = \frac{1}{n}\Sigma x^2 - \bar{x}^2 \quad \text{and} \quad V(y) = \frac{1}{n}\Sigma y^2-\bar{y}^2$$

Use: To find X Use: To find γ

SOLVED EXAMPLES

Example 1:

Obtain the two regression equatione and hence find the value of x when y=25

Data:-

X	Y	X^2	Y^2	XxY
8	15	64	225	120
10	20	100	400	200
12	30	144	900	360
15	40	225	1600	600
20	45	400	2025	900
$\Sigma x = 65$	$\Sigma y = 150$	$\Sigma x^2 = 933$	$\Sigma y^2 = 5150$	$\Sigma xy = 2180$

And n= 5

Now the two regression equations are,

$$(x-\bar{x}) = b_{xy}(y-\bar{y}) \quad \text{-------x on y (i)}$$
$$(y-\bar{y}) = b_{yx}(x-\bar{x}) \quad \text{------- y on x (ii)}$$

Where,

$$\bar{x} = \frac{1}{n}\Sigma x = \frac{65}{5} = 13 \quad \text{and} \quad \bar{y} = \frac{1}{n}\Sigma y = \frac{150}{5} = 30$$

Also,

$$Cov(x,y,) = \frac{1}{n}\Sigma xy - \bar{x}\,\bar{y} = \frac{2180}{5} - 13x30 = 436-390$$

$\therefore$ **Cov(x,y)** = 46

$$V(x) = \frac{1}{n}\Sigma x^2 - \bar{x}^2 = \frac{933}{5} - 13^2 = 186.6 - 169$$

V(x) = 17.6

$$V(y) = \frac{1}{n}\Sigma y^2 - \bar{y}^2 = \frac{5150}{5} - 30^2 = 1030 - 900$$

V(y) =130

Now we find,

Regression co-efficient of X on Y

$$b_{xy} = \frac{Cov(x,y)}{V(y)} = \frac{46}{130}$$

$\therefore$ b_{xy} = 0.35 and

Regression co-efficient of X on Y

$$b_{yx} = \frac{Cov(x,y)}{V(x)} = \frac{46}{17.6}$$

b_{yx} = 2.61

Now substituting the values of $\bar{x}$, $\bar{y}$, b_{xy} and b_{yx} in the regression equations we get,

$$(x-13) = 0.35(y-30) \quad \text{-------x on y (i)}$$
$$(y-30) = 2.61(x-13) \quad \text{------- y on x (ii)}$$

as the two regression equations.

Now to estimate x when y =25, we use the regression equation of x on y

$\therefore$ (x-13) = 0.35(25-30)

$\therefore$ **x = 13 -1.75 = 11.25**

Remark:

From the above example we can note some points about Regression coefficients.

- Both the regression coefficients carry the same sign (+ or -)
- Both the regression coefficients can not be greater than 1 in number
  (e.g. -1.25 and -1.32) is not possible.
- Product of both the regression coefficients b_{xy} and b_{yx} must be < 1
  i.e. $b_{xy} \times b_{yx} < 1$ Here 0.35x2.61 = 0.91< 1 ***(Check this always)***

Example 2:

Obtain the two regression equations and hence find the value of y when x=10

Data:-

X	Y	XxY	X^2	Y^2
12	25	300	144	625
20	18	360	400	324
8	17	136	64	289
14	13	182	196	169
16	15	240	256	225
$\Sigma x= 70$	$\Sigma y=88$	$\Sigma xy=1218$	$\Sigma x^2=1060$	$\Sigma y^2=1632$

And n= 5

Now the two regression equations are,

$$(x-\bar{x}) = b_{xy}(y-\bar{y}) \quad \text{-------x on y (i)}$$

$$(y-\bar{y}) = b_{yx}(x-\bar{x}) \quad \text{------- y on x (ii)}$$

Where,

$$\bar{x} = \frac{1}{n}\Sigma x = \frac{70}{5} = 14 \qquad \text{and} \qquad \bar{y} = \frac{1}{n}\Sigma y = \frac{88}{5} = 17.6$$

Also,

$$\text{Cov}(x,y,) = \frac{1}{n}\Sigma xy - \bar{x}\,\bar{y} = \frac{1218}{5} - 14\text{x}17.6 = 243.6 - 246.4$$

$\therefore$ **Cov(x,y)** = -2.8

$$V(x) = \frac{1}{n}\Sigma x^2 - \bar{x}^2 = \frac{1060}{5} - 14^2 = 212 - 196$$

V(x) = 16

$$V(y) = \frac{1}{n}\Sigma y^2 - \bar{y}^2 = \frac{1632}{5} - 17.6^2 = 326.4 - 309.76$$

V(y) = 16.64

Now we find,

Regression co-efficient of X on Y

$$b_{xy} = \frac{Cov(x,y)}{V(y)} = \frac{2.8}{16.64}$$

$\therefore$ b_{xy} = - 0.168

Regression co-efficient of X on Y

$$b_{yx} = \frac{Cov(x,y)}{V(x)} = -\frac{2.8}{16.64}$$

b_{yx} = 0.175

Now substituting the values of $\bar{x}$, $\bar{y}$, b_{xy} and b_{yx} in the regression equations we get,

$$(x-14) = -0.168(y-17.6) \quad \text{-------x on y (i)}$$
$$(y-17.6)=-0.175(x-14) \quad \text{------- y on x (ii)}$$

as the two regression equations.

Now to estimate y when x =10, we use the regression equation of y on x

$$\therefore \ (y-17.6) =-0.175(10-14)$$
$$\therefore \ \mathbf{y =17.6 +0.7 = 24.3}$$

Example 3:

The following data give the experience of machine operators and their performance rating given by the number of good parts turned out per 100 pieces.

Operator:	1	2	3	4	5	6	7	8
Experience: (in years)	16	12	18	4	3	10	5	12
Performance: Rating	87	88	89	68	78	80	75	83

Obtain the two regression equations and estimate the permance rating of an operator who has put 15 years in service.

Solution: We define the variables,

X: Experience y: Performance rating

Table of calculations:

X	Y	Xy	x^2	Y^2
16	87	1392	256	7569
12	88	1056	144	7744
18	89	1602	324	7921
4	68	272	16	4624
3	78	234	9	6084
10	80	800	100	6400
5	75	375	25	5625
12	83	996	144	6889
$\Sigma x= 80$	$\Sigma y= 648$	$\Sigma xy= 6727$	$\Sigma x^2= 1018$	$\Sigma y^2= 52856$

Now the two regression equations are,

$$(x-\bar{x}) = b_{xy}(y-\bar{y}) \quad \text{-------x on y (i)}$$
$$(y-\bar{y}) = b_{yx}(x-\bar{x}) \quad \text{------- y on x (ii)}$$

Where,

$$\bar{x} = \frac{1}{n}\Sigma x = \frac{80}{8} = 10 \quad \text{and} \quad \bar{y} = \frac{1}{n}\Sigma y = \frac{648}{8} = 81$$

Also,

$Cov(x,y,) = \frac{1}{n}\Sigma xy - \bar{x}\ \bar{y}$
$= \frac{6727}{8} - 10x81$
$= 840.75 - 810$
∴ **Cov(x,y)** = 30.75

$V(x) = \frac{1}{n}\Sigma x^2 - \bar{x}^2$
$= \frac{1018}{8} - 10^2$
$= 127.25 - 100$
V(x) = 27.25

$V(y) = \frac{1}{n}\Sigma y^2 - \bar{y}^2$
$= \frac{52856}{8} - 81^2$
$= 6607 - 6561$
V(y) = 46

Now we find,

Regression co-efficient of X on Y

$b_{xy} = \frac{Cov(x,y)}{V(y)}$

$= \frac{30.75}{46}$

∴ $b_{xy} = 0.67$ and

Regression co-efficient of X on Y

$b_{yx} = \frac{Cov(x,y)}{V(x)}$

$= \frac{30.75}{27.25}$

$b_{yx} = 1.13$

Now substituting the values of $\bar{x}$, $\bar{y}$, b_{xy} and b_{yx} in the regression equations we get,

(x-10) = 0.67(y-81) -------x on y (i)
(y-81)=1.13(x-10) ------- y on x (ii)

as the two regression equations.
Now to estimate Performance rating (y) when Experience (x) = 15, we use the regression equation of y on x

∴ (y-81) =1.13(15-10)
∵ y = 81+ 5.65 = 86.65

Hence the estimated performance rating for the operator with 15 years of experience is approximately 86.65 i.e approximately 87

12.4.2 Regression coefficients in terms of correlation coefficient.

We can also obtain the regression coefficients b_{xy} and b_{yx} from standard deviations, σ_x., σ_y and correlation coefficient 'r' using the formulas

$$b_{xy} = r\frac{\sigma_x}{\sigma_y} \quad \text{and} \quad b_{yx} = r\frac{\sigma_y}{\sigma_x}$$

Also consider,

$$b_{xy} \text{ x } b_{yx} = r\frac{\sigma_x}{\sigma_y}\ r\frac{\sigma_y}{\sigma_x} = r^2 \qquad \text{i.e. } r = \sqrt{b_{xy} \text{ x } b_{yx}}$$

Hence the correlation coefficient 'r' is the geometric mean of the regression coefficients, b_{xy} and b_{yx}

Example 5:

You are given the information about advertising expenditure and sales:

	Exp. on Advertisiment (Rs. In Lakh)	Sales (Rs. In Lakh)
Mean	10	90
S.D.	3	12

Coefficient of correlation between sales and expenditure on Advertisement is 0.8.Obtain the two regression equations.
Find the likely sales when advertisement budget is Rs. 15 Lakh.

Solution: We define the variables,

X: Expenditure on advertisement

Y: Sales achieved.

Therefore we have,

$\bar{x}$ = 10, $\bar{y}$ =90, 6x= 3, 6y = 12 and r = 0.8

Now, using the above results we can write the two regression equations as

$$(x-\bar{x}) = r\frac{\sigma_x}{\sigma_y}(y-\bar{y}) \quad \text{-------x on y (i)}$$

$$(y-\bar{y}) = r\frac{\sigma_y}{\sigma_x}(x-\bar{x}) \quad \text{------- y on x (ii)}$$

Substituting the values in the equations we get,

$$(x-10) = 0.8\frac{3}{12}(y-90)$$

i.e x- 10 = 0.2 (y-90) -------x on y (i)

also $(y-90) = 0.8\frac{12}{3}(x-10)$

i.e. y-90 = 3.2 (x-10) -------y on x (ii)

Now when expenditure on advertisement (x) is 15, we can find the sales from eqn (ii) as,

y-90 = 3.2 (15-10)

$\therefore$ y = 90 + 16 = 106

Thus the likely sales are Rs.106 Lakh.

Example 6: Comput the two regression equations on the basis of the following information:

	X	Y
Mean	40	45
Standard deviation	10	9

Karl Pearson's coefficient of correlation between x and y = 0.50.
Also estimate the value of x when y = 48 using the appropriate equation.

Solution: We have,

$$\bar{x} = 40, \quad \bar{y} = 45, \quad \sigma x = 10, \sigma y = 9 \text{ and } r = 0.5$$

Now, we can write the two regression equations as

$$(x-\bar{x}) = r\frac{\sigma_x}{\sigma_y}(y-\bar{y}) \quad \text{-------x on y (i)}$$

$$(y-\bar{y}) = r\frac{\sigma_y}{\sigma_x}(x-\bar{x}) \quad \text{------- y on x (ii)}$$

Substituting the values in the equations we get,

$$(x-40) = 0.5\frac{10}{9}(y-45)$$

i.e $\quad$ x- 40 = 0.55 (y-45) -------eqn of x on y (i)

and $\quad (y-45) = 0.5\frac{9}{10}(x-40)$

i.e. $\quad$ y-45 = 0.45(x-40) -------eqn of y on x (ii)

Now when y is 48, we can find x from eqn (i) as,

x-40 = 0.55(48-45)

$\therefore$ x = 40 +1.65 = 41.65

Example 7:

Find the marks of a student in the Subject of Mathematics who have scored 65 marks in Accountancy Given,

Average marks in Mathematics	70
Accountancy	80
Standard Deviation of marks in Mathematics	8
in Accountancy	10

Coefficient of correlation between the marks of Mathematics and marks of Accountancy is 0.64.

Solution: We define the variables,

X: Marks in Mathematics

Y: Marks in Accountancy

Therefore we have,

$$\bar{x} = 70, \quad \bar{y} = 80, \quad \sigma_x = 8, \sigma_y = 10 \text{ and } r = 0.64$$

Now we want to approximate the marks in Mathematics (x), we obtain the regression equation of x on y, which is given by

$$(x-\bar{x}) = r\frac{\sigma_x}{\sigma_y}(y-\bar{y}) \quad \text{-------x on y (i)}$$

Substituting the values we get,

$$(x-70) = 0.64\frac{8}{10}(y-80)$$

i.e $\quad$ x- 70 = 0.57 (y-80)

Therefore, when marks in Accountancy (Y) = 65

x- 70 = 0.57(65-80)

$\therefore$ x = 70-2.85 = 67.15 $\quad$ i.e. 67 appro.

Use of regression equations to find means $\bar{x}$, $\bar{y}$ S.D.s σ_x, σ_y and correlation coefficient 'r'

As we have that, we can obtain the regression equations from the values of Means, standard deviations and correlation coefficients 'r', we can get back these values from the regression equations.

Now, we can note that the regression equation is a linear equation in two variables x and y. Therefore, the linear equation of the type Ax+By+C = 0 or y = a+bx represents a regression equation.

e.g. 3x+5y-15 = 0 and 2x+7y+10 = 0 represent the two regression equations.

The values of means $\bar{x}$, $\bar{y}$ can be obtain by solving the two equations as the simultaneous equations.

Example 8:

From the following regression equation, find **means** $\bar{x}$, $\bar{y}$, σ_x, σ_y and 'r'

$$3x-2y-10 = 0, \quad 24x-25y+145 = 0$$

Solution: The two regression equations are,

$$3x-2y-10 = 0 \text{ --------(i)}$$

$$24x-25y+145 = 0 \text{ ---(ii)}$$

Now for $\bar{x}$ and $\bar{y}$ we solve the two equations as the simultaneous equations.

Therefore, by (i) x 8 and (ii) x1, we get

$$24x-16y-80 = 0$$

$$24x-25y+145 = 0$$

$$- \quad + \quad -$$

$$9y-225 = 0 \qquad y = \frac{225}{9} = 25$$

Putting y = 25 in eqn (i), we get

$$3x-2(25)-10 = 0$$

$$3x - 60 = 0 \qquad x = \frac{60}{3} = 20$$

Hence $\bar{x} = 20$ and $\bar{y} = 25$.

Now to find 'r' we express the equations in the form y=a+bx

So, from eqns (i) and (ii)

$$y = \frac{3x}{2} - \frac{10}{2} \quad \text{and} \quad y = \frac{24x}{25} + \frac{145}{25}$$

$$\therefore \mathbf{b_1 = \frac{3}{2} = 1.5} \qquad \therefore \mathbf{b_2 = \frac{24}{25} = 0.96}$$

Since, $\mathbf{b_1 > b_2}$ **(i.e. $\mathbf{b_2}$** is smaller in number irrespective of sign + or -)

∴ Equation (ii) is regression of y on x and $\mathbf{b_{yx} = 0.96}$

Hence eqn (i) is regression of x on y and $b_{xy} = 1/1.5 = 0.67$

Now we find, $r = \sqrt{b_{xy} \times b_{yx}}$ i.e. $r = \sqrt{0.67 \times 0.96} = +0.84$

(The sign of 'r' is same as the sign of regression coefficients)

Example 9:
Find the means values of x,y, and r from the two regression equations. 3x+2y-26=0 and σx+y-31=0. Also find σ_x when σ_y = 3.
Solution: The two regression equations are,

3x+2y-26=0 -------- (i)
6x+y-31=0 ----------(ii)

Now for x and y we solve the two equations as the simultaneous equations.
Therefore, by (i) x 2 and (ii) x1, we get

6x+4y-52 = 0
6x+ y-31 = 0
- - +

3y-21 = 0

$\therefore y = \frac{21}{3} = 7$

Putting y = 7 in eqn (i), we get

3x+2(7)-26 = 0
3x – 12 = 0 $\quad x = \frac{12}{3} = 4.$

Hence x = 4 and y = 7.
Now to find 'r' we express the equations in the form y=a+bx
So, from eqns (i) and (ii)

$y = -\frac{3}{2}x - \frac{26}{2}$ and $y = -\frac{6}{1}x + \frac{31}{1}$

$\therefore \mathbf{b_1 = -\frac{3}{2} = -1.5}$ $\quad \therefore \mathbf{b_2 = \frac{6}{1} = -6}$

since, **$b_1 < b_2$ (i.e. b_1** is smaller in number irrespective of sign + or -)
∴ Equation (i) is regression of y on x and b_{yx} = - 1.5
Hence, eqn (ii) is regression of x on y and b_{xy} = - 1/6 = - 0.16
Now we find, . $r = \sqrt{b_{xy} \times b_{yx}}$ $\quad r = \sqrt{0.16 x 1.5}$ = = - 0.16

Note: ***The sign of 'r' is same as the sign of regression coefficients***

Now to find 6x when 6y = 3, we use the formula,

$$b_{yx} = r\frac{\sigma_x}{\sigma_y}$$

$$\mathbf{-1.5 = -}\frac{0.16x3}{6x}$$

$$\therefore \mathbf{6x =} \frac{0.48}{1.5} \mathbf{= 0.32}$$

Hence means $\bar{x} = 4$, $\bar{y} = 7$, r = - 0.16 and 6x = 0.32.

EXERCISES

1. What is mean by Regression? Explain the use of regression in the statistical analysis.

2. Why are there two Regressions? Justify.

3. State the difference between Correlation and Regression.

4. Obtain the two regression equations from the data given bellow.

X:	7	4	6	5	8
Y:	6	5	9	8	2

Hence estimate y when x = 10.

5. The data given below are the years of experience (x) and monthly wages (y) for a group of workers. Obtain the two regression equations and approximate the monthly wages of a workers who have completed 15 years of service.

Experience: In years	11	7	9	5	8	6	10
Monthly wages: (in '000Rs.)	10	8	6	8	9	7	11

6. Following results are obtained for a bivariate data. Obtain the two regression equations and find y when x = 12

$n = 15 \quad \Sigma x = 130 \quad \Sigma y = 220 \quad \Sigma x^2 = 2288 \quad \Sigma y^2 = 5506 \quad \Sigma xy = 3467$

7. Marks scored by a group of 10 students in the subjects of Maths and Stats in a class test are given below.Obtain a suitable regression equation to find the marks of a student in the subject of Stats who have scored 25 marks in Maths.

Student no:	1	2	3	4	5	6	7	8	9	10
Marks in Maths	13	18	9	6	14	10	20	28	21	16
Marks in Stats:	12	25	11	7	16	12	24	25	22	20

8. The data given below are the price and demand for a certain commodity over a period of 7 years. Find the regression equation of Price on Demand and hence obtain the most likely demand for the in the year 2008 when it's price is Rs.23.

Year:	2001	2002	2003	2004	2005	2006	2007	
Price(in RS):	15	12	18	22	19	21	25	
Demand (100 units)		89	86	90	105	100	110	115

9. For a bivariate data the following results were obtained

$\bar{x}$ = 53.2 , $\bar{y}$ = 27.9 , 6x = 4.8, σy = σ.4 and r =0.75

Obtain the two regression equations, find the most probable value of x when y =25.

10. A sample of 50 students in a school gave the following statistics about Marks of students in Subjects of Mathematics and Science,

Subjects:	Mathematics	Science
Mean	58	79
S.D.	12	18

Coefficient of correlation between the marks in Mathematics and marks in Science is 0.8. Obtain the two regression equations and approximate the marks of a student in the subject of Mathematics whose score in Science is 65.

11. It is known that the Advertisement promotes the Sales of the company. The company's previous records give the following results.

	Expenditure on Advertisement (Rs. In Lakh)	Sales (Rs. In Lakh)
Mean	15	190
S.D.	6	20

Coefficient of correlation between sales and expenditure on Advertisement is 0.6. Using the regression equation find the likely sales when advertisement budget is Rs.25 Lakh.

12. Find the values of x,y, and r from the two regression equations given bellow. 3x+2y-26=0 and 6x+y-31=0. Also find 6x when σy = 3.

13. Two random variables have the regression equations: 5x+7y-22=0 and 6x+2y-20=0. Find the mean values of x and y. Also find S.D. of x when S.D. of y = 5.

14. The two regression equations for a certain data were y = x+5 and 16x = 9y-94. Find values of $\bar{x}$, $\bar{y}$ and r. Also find the S.D. of y when S.D. of x is 2.4.

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UNIT IX

13

TIME SERIES

OBJECTIVES

From this chapter student should learn analysis of data using various methods. Methods involve moving average method and least square method seasonal fluctuations can be studied by business for casting method.

13.1 INTRODUCTION

Every business venture needs to know their performance in the past and with the help of some predictions based on that, would like to decide their strategy for the present By studying the past behavior of the characteristics, the nature of variation in the value can be determined. The values in the past can be compared with the present values of comparisons at different places during formulation of future plan and policies. This is applicable to economic policy makers, meteorological department, social scientists, political analysis. Forecasting thus is an important tool in Statistical analysis. The statistical data, particularly in the field of social science, are dynamic in nature. Agricultural and Industrial production increase every year or due to improved medical facilities, there is decline in the death rate over a period of time. There is increase in sales and exports of various products over a period of years. Thus, a distinct change (either increasing or decreasing) can be observed in the value of time-series.

A time series is a sequence of value of a phenomenon arranged in order of their occurrence. Mathematically it can expressed as a function, namely $y = f(t)$ where t represents time and y represents the corresponding values. That is, the value y_1, y_2, y_3 …… of a phenomenon with respect to time periods t_1, t_2, t_3 …. Form a Time Series.

Forecasting techniques facilities prediction on the basic of a data available from the past. This data from the past is called a time series. A set of observations, of a variable, taken at a regular (fixed and equal) interval of time is called time series. A time series is a bivariate data, with time as the independent variable and the other is the variable under consideration. There are various forecasting method for time series which enable us to study the variation or trends and estimate the same for the future.

13.2 IMPORTANCE OF TIME SERIES ANALYSIS

The analysis of the data in the time series using various forecasting model is called as time analysis. The importance of time series analysis is due to the following reasons:

- *Understanding the past behavior*
- *Planning the future action*
- *Comparative study*

13.3 COMPONENTS OF TIME SERIES

The fluctuation in a time series are due to one or more of the following factors which are called "components" of time series.

(a) Secular Trend :
The general tendency of the data, either to increase, to decrease or to remain constant is called Secular Trend. It is smooth, long term movement of the data. The changes in the values are gradual and continuous. An increasing demand for luxury items like refrigerators or colour T.V. sets reflect increasing trend. The production of steel, cement, vehicles shows a rising trend. On the other hand, decreasing in import of food grains is an example of decreasing trend. The nature of the trend may be linear or curvilinear, in practice, curvilinear trend is more common.

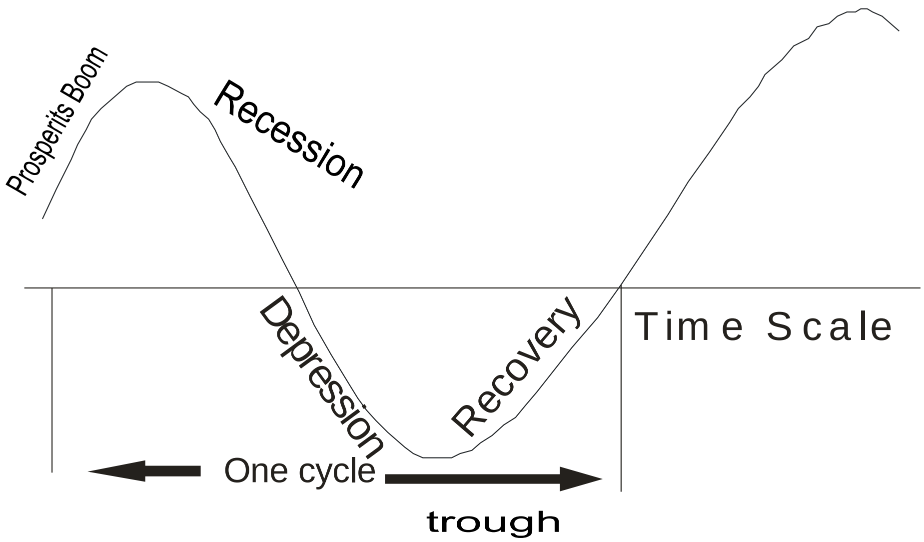
Trend in due to long term tendency. Hence it can be evaluation if the time series is a available over a long duration.

(b) Seasonal Variation :
The regular, seasonal change in the time series are called "Seasonal Variation". It is observed that the demand for umbrellas, raincoats reaches a peak during monsoon or the advertisement of cold drinks, ice creams get a boom in summer. The demand for greeting cards, sweets, increase during festival like Diwali, Christmas. In March, there is maximum withdrawal of bank deposits for adjustment of income-tax payment, so also variation tax-saving schemes shoot up during this period.

The causes, for these seasonal fluctuations, are thus change in weather conditions, the traditions and customs of people etc. The seasonal component is measured to isolate these change from the trend component and to study their effect, so that, in any business, future production can be planned accordingly and necessary adjustments for seasonal change can be made

(c) Cyclical Variation :
These are changes in time series, occurring over a period which is more than a year. They are recurring and periodic in nature. The period may not be uniform. These fluctuations are due to changes in a business cycle.

There are four important phases of any business activity viz. prosperity, recession, depression and recovery. During prosperity, the business flourishes and the profit reaches a maximum level. Thereafter, in recession, the profit decreases, reaching a minimum level during depression. After some time period, the business again recovers (recovery) and it is followed by period of prosperity. The variation in the time series due to these phases in a business cycle are called “Cyclical Variation:.



The knowledge of cyclic variations is important for a businessman to plan his activity or design his policy for the phase of recession or depression. But one should know that the factors affecting the cyclical variations are quite irregular, difficult to identify and measure. The cyclical variation are denoted by

(d) Irregular Variation :

The changes in the time series which can not be predicated and are erratic in nature are called “Irregular Variation”. Usually, these are short term changes having signification effect on the time series during that time interval. These are caused by unforeseen event like wars, floods, strikes, political charges, etc. During Iran-Iraq war or recent Russian revolution, prices of petrol and petroleum product soared very high. In recent budget, control on capital issued was suddenly removed. As an effect, the all Indian-Index of share market shooted very high, creates all time records. If the effect of other components of the time series is eliminated, the remaining variation are called “Irregular or Random Variations”. No forecast of these change can be made as they do not reflect any fixed pattern.

MODELS FOR ANALYSIS OF TIME SERIES

The purpose of studying time series is to estimate or forecast the value of the variable. As there are four components of the time series, these are to be studied separately. There are two types of models which are used to express the relationship of the components of the time series. They are additive model and multiplicative model.

O = Original Time Series
T = Secular Trend
S = Seasonal Variations
C = Cyclical Variations and
I = Irregular Variations

In Additive model, it is assumed that the effect of the individual components can be added to get resultant value of the time series, that is the components are independent of one another. The model can be expressed as

$$O = T + S + C + I$$

In multiplicative model, it is assumed that the multiplication of the individual effect of the components result in the time series, that is, the components are due to different causes but they are not necessarily independent, so that changes in any one of them can affect the other components. This model is more commonly used. It is expressed as

$$O = T \times S \times C \times I$$

If we want to estimate the value in time series, we have to first estimate the four components and them combine them to estimate the value of the time series. The irregular variations can be found. However, we will restrict ourselves, to discuss method of estimating the first components, namely Secular Trend.

13.4 METHODS TO FIND TREND

There are various method to find the trend. The major methods are as mentioned below:

I. Free Hand Curve.
II. Method of Semi – Averages.
III. Method of Moving Averages.
IV. Method of Least Squares.

we will study only the method of moving average and least squares.

13.4.1.Method of Moving Averages

This is a simple method in which we take the arithmetic average of the given times series over a certain period of time. These average move over period and are hence called as moving averages. The time interval for the average is taken as 3 years, 4 years or 5 years and so on. The average are thus called as 3 yearly, 4 yearly and 5 yearly moving average. The moving averages are useful in smoothing the fluctuations caused to the variable. Obviously larger the time interval of the average more is the smoothing. We shall study the odd yearly (3 and 5) moving average first and then the 4 yearly moving average.

Odd Yearly Moving Average

In this method the total of the value in the time series is taken for the given time interval and is written in front of the middle value. The average so taken is also written in front of this middle value. This average is the trend for that middle year. The process is continued by replacing the first value with the next value in the time series and so on till the trend for the last middle value is calculated. Let us understand this with example:

Example 1:
Find 3 years moving averages and draw these on a graph paper. Also represent the original time series on the graph.

Year	1999	2000	2001	2002	2003	2004	2005	2006	2007
Production (in thousand unit)	12	15	20	18	25	32	30	40	44

Solution:
We calculate arithmetic mean of first three observations viz. 12, 15 and 20, then we delete 12 and consider the next one so that now, average of 15, 20 and 18 is calculated and so on. These averages are placed against the middle year of each group, viz. the year 2000, 2001 and so on. Note moving averages are not obtained for the year 1999 and2007.

Year	**Production (in thousand unit)**	**3 Years Total**	**3yrly.Moving Average**
1999	12		
2000	15	12 + 15 + 20 = 47	47 / 3 = 15.6
2001	20	15 + 20 + 18 = 53	53 / 3 = 17.6
2002	18	20 + 18 + 25 = 63	63 / 3 = 21.0
2003	25	18 + 25 + 32 = 75	75 / 3 = 25.0
2004	32	25 + 32 + 30 = 87	87 / 3 = 29.0
2005	30	32 + 30 + 40 = 102	102 / 3 = 34.0
2006	40	30 + 40 + 44 = 114	114 / 3 = 38.0
2007	44		

Example 2:
Find 5 yearly moving average for the following data.

Year	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006
Sales (in lakhs of Rs.)	51	53	56	57	60	55	59	62	68	70

Solution:
We find the average of first five values, namely 51, 53, 56, 57 and 60. Then we omit the first value 51 and consider the average of next five values, that is, 53, 56, 57, 60 and 55. This process is continued till we get the average of the last five values 55, 59, 62, 68 and 70. The following table is prepared.

Year	Sales (in lakhs of Rs.)	5 Years Total	Moving Average (Total / 5)
1997	51	….	….
1998	53	….	….
1999	56	51 + 53 + 56 + 57 +60 = 277	55.4
2000	57	53 + 56 + 57 + 60 + 55 = 281	56.2
2001	60	56 + 57 + 60 + 55 + 59 = 287	57.4
2002	55	57 + 60 + 55 + 59 + 62 = 293	58.6
2003	59	60 + 55 + 59 + 62 + 68 = 304	60.8
2004	62	55 + 59 + 62 + 68 + 70 = 314	62.8
2005	68	….	….
2006	70	….	….

Example 3:
Determine the trend of the following time series using 5 yearly moving averages.

Year	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990	1991
Exports in‘000Rs	78	84	80	83	86	89	88	90	94	93	96

Solution: The time series is divided into overlapping groups of five years, their 5 yearly total and average are calculated as shown in the following table.

Year	Export (Y)	5 – yearly total (T)	5 – yearly moving average: (T/5)
1981	78		
1982	84		
1983	80		
1984	83	78+84+80+83+86 = 411	411 / 5 = 82.2
1985	86	84+80+83+86+89 = 422	422 / 5 = 84.4
1986	89	80+83+86+89+88 = 426	85.2
1987	88	83+86+89+88+90 = 436	87.2
1988	90	86+89+88+90+94 = 447	89.4
1989	94	89+88+90+94+93 = 454	90.8
1990	93	88+90+94+93+96 = 461	92.2
1991	96		

<u>Observations:</u>

I. In case of the 5 – yearly moving average, the total and average for the first two and the last two in the time series is not calculated. Thus, the moving average of the first two and the last two years in the series cannot be computed.

II. To find the 3 – yearly total (or 5 – yearly total) for a particular years, you can subtract the first value from the previous year’s total, and add the next value so as to save your time!

Even yearly moving averages

In case of even yearly moving average the method is slightly different as here we cannot find the middle year of the four years in consideration. Here we find the total for the first four years and place it between the second and the third year value of the variable. These totals are again sunned into group of two, called as centered total and is placed between the two totals. The 4 – yearly moving average is found by dividing these centered totals by 8. Let us understand this method with an example

Example 4: Calculate the 4 yearly moving averages for the following data.

Year	1991	1992	1993	1994	1995	1996	1997	1998	1999
Import in'000Rs	15	18	20	24	21	25	28	26	30

Ans: The table of calculation is show below. Student should leave one line blank after every to place the centered total in between two years.

Years	**Import (Y)**	**4 – yearly total**	**4–yearly centered total**	**4-yearly moving averages:**
1991	15	-	-	-
1992	18	-	-	-
		77		
1993	20		77+83=160	160/8= 20
		83		
1994	24		83+90=173	173/8=21.6
		90		
1995	21		90+98=188	188/8=23.5
		98		
1996	25		98+100=198	198/8=24.8
		100		
1997	28		100+109=209	209/8=26.1
		109		
1998	26	-	-	-
1999	30	-	-	-

Example 5:

Find the moving average of length 4 for the following data. Represent the given data and the moving average on a graph paper.

Year	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007
Sales (in thousand unit)	60	69	81	86	78	93	102	107	100	109

Solution: We prepare the following table.

Year	Sale (in thousand unit)	4 Yearly Totals	Centred Total	Moving / Avg. Central = Total / 8
1998	60			
1999	69			
		60 + 69 + 81 + 86 = 296		
2000	81		296 + 314 = 610	76.25
		69 + 81 + 86 + 78 = 314		
2001	86		314 + 338 = 652	81.5
		81 + 86 + 78 + 93 = 338		
2002	78		338 + 359 = 697	87.125
		86 + 78 + 93 + 102 = 359		
2003	93		359 + 380 = 739	92.375
		78 + 93 + 102 + 107 = 380		
2004	102		380 + 402 = 782	97.75
		93 + 102 + 107 + 100 = 402		
2005	107		402 + 418 = 820	102.5
		102 + 107 + 100 + 109 = 418		
2006	100			
2007	109			

Note that 4 yearly total are written between the years 1999-2000, 2000-01, 2001-02 etc. and the central total are written against the years 2000, 2001, 2002 etc. so also the moving average are considered w.r.t. years; 2000, 2001 and so on. The moving averages are obtained by dividing the certain total by 8.

The graph of the given set of values and the moving averages against time representing the trend component are shown below. Note that the moving averages are not obtained for the years 1998, 1999, 2006 and 2007. (i.e. first and last two extreme years).

When the values in the time series are plotted, a rough idea about the type of trend whether linear or curvilinear can be obtained. Then, accordingly a linear or second degree equation can be fitted to the values. In this chapter, we will discuss linear trend only.

13.4.2. LEAST SQUARES METHOD:

Let y = a + bx be the equation of the straight line trend where a, b are constant to be determined by solving the following normal equations,

$$\sum y = na + b\sum x$$
$$\sum xy = a\sum x + b\sum x^2$$

where y represents the given time series.
We define x from years such that $\sum x = 0$. So substituting $\sum x = 0$ in the normal equation and simplifying, we get

$$b = \frac{\sum xy}{\sum x^2} \quad \text{and} \quad a = \frac{\sum y}{n}$$

Using the given set of values of the time series, a, b can be calculated and the straight line trend can be determined as y = a + bx. This gives the minimum sum of squares line deviations between the original data and the estimated trend values. The method provides estimates of trend values for all the years. The method has mathematical basis and so element of personal bias is not introduced in the calculation. As it is based on all the values, if any values are added, all the calculations are to be done again.

Odd number of years in the time series

When the number of years in the given time series is add, for the middle year we assume the value of x = 0. For the years above the middle year the value given to x are …, -2, -1 while those after the middle year are values 1,2, … and so on.

Even number of years in the time series

When the number of years in the time series is even, then for the upper half the value of x are assumed as…., -5, -3, -1. For the lower half years, the values of x are assumed as 1, 3, 5, …. And so on.

Example 6:

Fit a straight line trend for the following data giving the annual profits (in lakhs of Rs.) of a company. Estimate the profit for the year 1999.

Years	1992	1993	1994	1995	1996	1997	1998
Profit	30	34	38	36	39	40	44

Solution: Let y = a + bx be the straight line trend.
The number of years is seven, which is sold. Thus, the value of x is taken as 0 for the middle years 1995, for upper three years as -3, -2, -1 and for lower three years as 1, 2, 3.

The table of computation is as shown below:

Years	**Profit (y)**	**x**	**xy**	**x^2**	**Trend Value: $Y_t = a + bx$**
1992	30	-3	-90	9	31.41
1993	34	-2	-68	4	33.37
1994	38	-1	-38	1	35.33
1995	36	0	0	0	37.29
1996	39	1	39	1	39.25
1997	40	2	80	4	41.21
1998	44	3	132	9	43.17
Total	$\sum y = 261$	$\sum x = 0$	$\sum xy = 55$	$\sum x^2 = 28$	

From the table : $n = 7$, $\sum xy = 55$, $\sum x^2 = 28$, $\sum y = 261$

There fore $b = \frac{\sum xy}{\sum x^2} = \frac{55}{28} = 1.96$ and $a = \frac{\sum y}{n} = \frac{261}{7} = 37.29$

Thus, the straight line trend is $y = 37.29 + 1.96x$.

The trend values in the table for the respective years are calculated by substituting the corresponding value of x in the above trend line equation.

For the trend value for 1992: $x = -3$:

$y_{1992} = 37.29 + 1.96 (-3) = 37.29 - 5.88 = 31.41$

Similarly, all the remaining trend values are calculated.
(A short-cut method in case of odd number of years to find the remaining trend values once we calculate the first one, is to add the value of b to the first trend value to get the second trend value, then to the second trend value to get the third one and so on. This is because the difference in the values of x is 1.)

To estimate the profit for the years 1999 in the trend line equation, we substitute the prospective value of x, if the table was extended to 1999. i.e. we put $x = 4$, the next value after $x = 3$ for the year 1998.

$\therefore y_{1999} = 37.29 + 1.96 (4) = 45.13$

There fore the estimated profit for the year 1999 is Rs. 45.13 lakhs.

Example 7:
Fit straight line trend by the method of lease squares for the following data representing production in thousand units. Plot the data and the trend line on a graph paper. Hence or otherwise estimate the trend for the years 2007.

Year	1999	2000	2001	2002	2003	2004	2005
Production (in thousand unit)	14	15	17	16	17	20	23

Solution:
Here, the total number of years is 7, an odd number. So we take the center as 1986 the middle-most year and define x as year 2002. The values of x will be -3, -2, -1, 0, 1, 2, 3.

Prepare the following table to calculate the required summations. Note that the trend values can be written in the table only after calculation of a and b.

Year	**Production (y)**	**x**	**x^2**	**x y**	**Trend Values**
1999	14	-3	9	-42	13.47
2000	15	-2	4	-30	14.79
2001	17	-1	1	-17	16.11
2002	16	0	0	0	17.43
2003	17	1	1	17	18.75
2004	20	2	4	40	20.07
2005	23	3	3	69	21.39
	122		28	37	

Here, n = 7, $\sum y = 122$, $\sum x^2 = 28$, $\sum xy = 37$
Now, a and b are calculated as follows:

$$a = \frac{\sum y}{n} = \frac{122}{7} = 17.4286 \approx 17.43$$

$$b = \frac{\sum xy}{\sum x^2} = \frac{37}{28} = 1.3214 \approx 1.32$$

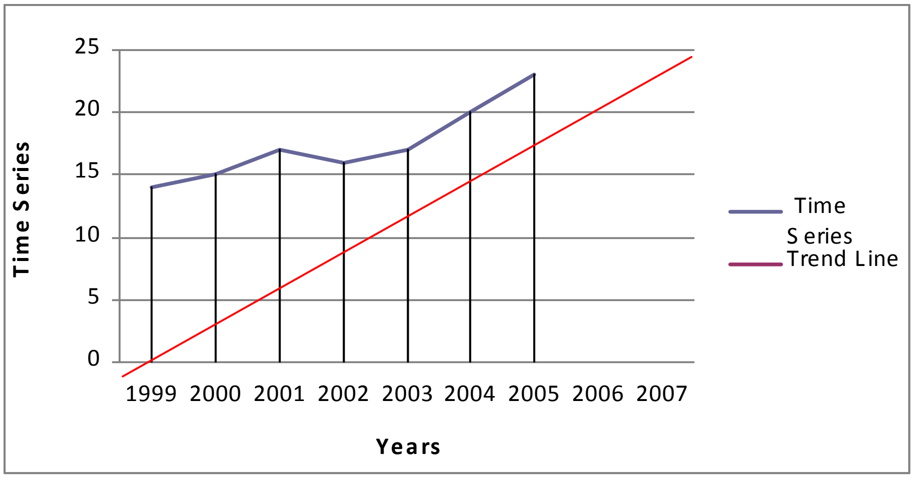
So, the equation is used to find trend values.
y = a + b x
i.e. y = 17.43 + 1.32x
The equation is used to find trend values.

For the year 1999, x = -3, substituting the value od x, we get,
y = 17.43 + 1.32 (-3) = 17.43 – 3.96 = 13.47
to find the remaining trend values we can make use of the property of a straight line that as all the values of x are equidistant with different of one unit (-3, -2, -1, ---- and so on), the estimated trend value will also be equidistant with a difference of b unit.

In this case as b = 1.32, the remaining trend values for x = -2, -1, 0, --- etc. are obtained by adding b = 1.32 to the previous values. So, the trend values are 13.47, 14.79, 16.11, 17.43, 18.75, 20.07 and 21.39.

Now to estimate trend for the year 2007, x = 5, substituting in the equation
y = 17.43 + 1.32x
= 17.43 + 1.32 (5) = 24.03

So, the estimated trend value for the year 2007 is 24,030 unit.
For graph of time series, all points are plotted. But for the graph of trend line, any two trend values can be plotted and the line joining these points represents the straight line trend.



For the trend line, the trend values 17.43 and 21.39 for the years 2002 and 2005 are plotted and then a straight line joining these two points is drawn and is extended on both the sides.

The estimate of trend for the year 2007 can also be obtained from the graph by drawing a perpendicular for the year 2007, from x-axis which meet the trend line at point P. From P, a perpendicular on y-axis gives the required ternd estimate as 24.

Now, to find straight line trend, when number of years is even, consider the following example.

Example 8:

Fit a straight line trend to the following time –series, representing sales in lakhs of Rs. of a company, for the year 1998 to 2005. Plot the given data well as the trend line on a graph paper. Hence or otherwise estimate trend for the year 2006.

Year	1998	1999	2000	2001	2002	2003	2004	2005
Sales (Lakhs of Rs.)	31	33	30	34	38	40	45	49

Solution:

Here the number of years = 8, an even number, so we define $x = \dfrac{year - 2001.5}{0.5}$, so that the values of x are -7, -3, -1, 1, 3, 5 and 7, to get $\sum x = 0$.

Prepare the following table to obtain the summations $\sum x^2$, $\sum y$, $\sum x y$.

Year	**Sales (in Lakhs of Rs.)**	**x**	**x^2**	**x y**	**Trend Values**
1998	31	-7	49	-217	28.33
1999	33	-5	25	-165	30.95
2000	30	-3	9	-90	33.57
2001	34	-1	1	-34	36.19
2002	38	1	1	38	38.81
2003	40	3	9	120	41.43
2004	45	5	25	225	44.05
2005	49	7	49	343	46.67
	300		168	220	

Here, n = 8, $\sum y = 300$, $\sum x^2 = 168$, $\sum xy = 220$

Now, a and b are calculated as follows:

$$a = \frac{\sum y}{n} = \frac{300}{8} = 37.5$$

$$b = \frac{\sum xy}{\sum x^2} = \frac{220}{168} = 1.31$$

So, the equation of the straight line trend is y = a + b x

i.e. y = 37.5 + 1.31 x

To obtain the trend values, first calculate y for x = -7, for the year 1998

y = 37.5 + 1.31 (-7)

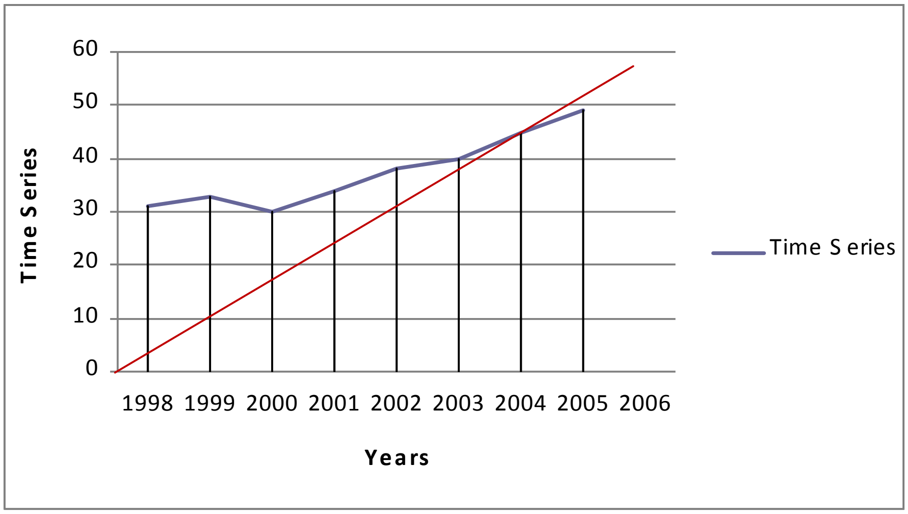
= 37.5 – 9.17 = 28.33

To find the successive trend values, go on addition 2b = 2 × 1.31 = 2.62, to the preceding values as in this case the different between x values is of 2 units.

So, the estimated values of trend for x = -5, -3, -1, 1, 3, 5, 7 and 7 are 30.95, 33. 57, 36.19, 38.81, 41.43, 44.05 and 46.67 respectively. Write down these values in the table.

Hence the estimated trend value for the year 2006 is 49.29 (in lakhs of Rs.).

Now, for the graph of trend line, note that only two trend values 30.95 and 46.67 w.r.t. years 1999 and 2005 are considered as point. The line joining these two points represents trend line.



To estimated the trend for the year 2006, drawn a perpendicular from x-axis at this point meeting the line in P. then from P, draw another perpendicular on y-axis which gives estimate of trend as 49.

Example 9:

Fit a straight line trend to the following data. Draw the graph of the actual time series and the trend line. Estimate the sales for the year 2007.

Year	1998	1999	2000	2001	2002	2003	2004	2005
Sales in'000Rs	120	124	126	130	128	132	138	137

Solution: let $y = a + bx$ be the straight line trend.

The number of years in the given time series is eight, which is an even number. The upper four years are assigned the values of x as 1, 2, 3, and 7. Note that ere the difference between the values of x is 2, but the sum is zero.

Now, the table of computation is completed as shown below:

Years	**Profit (y)**	**X**	**Xy**	**X^2**	**Trend Value: $Y_t = a + bx$**
1998	120	-7	-840	49	120.84
1999	124	-5	-620	25	123.28
2000	126	-3	-378	9	125.72
2001	130	-1	-130	1	128.16
2002	128	1	128	1	130.06
2003	132	3	396	9	133.04
2004	138	5	390	25	135.48
2005	137	7	359	49	137.92
Total	$\sum y = 1035$	$\sum x = 0$	$\sum xy = 205$	$\sum x^2 = 168$	

From the table : n = 8, $\sum xy = 205$, $\sum x^2 = 168$, $\sum y = 1035$

$$\therefore\ b = \frac{\sum xy}{\sum x^2} = \frac{205}{168} = 1.22 \qquad \text{and } a = \frac{\sum y}{n} = \frac{1035}{8} = 129.38$$

Thus, the straight line trend is y = 129.38 + 1.22x.
The trend values in the table for the respective years are calculated by substituting the corresponding value of x in the above trend line equation.

For the trend value for 1998: x = -7:
y_{1998} = 129.38 + 1.22 (-7) = 129.38- 8.54 = 120.84
Similarly, all the remaining trend values are calculated.

(A short-cut method in case of even number of years to find the remaining trend values once we calculate the first one, is to add twice the value of b to the first trend value to get the second trend value, then to the second trend value to get the third one and so on. This is because the difference in the values of x is 2. In this example we add 2 x 1.22 = 2.44)

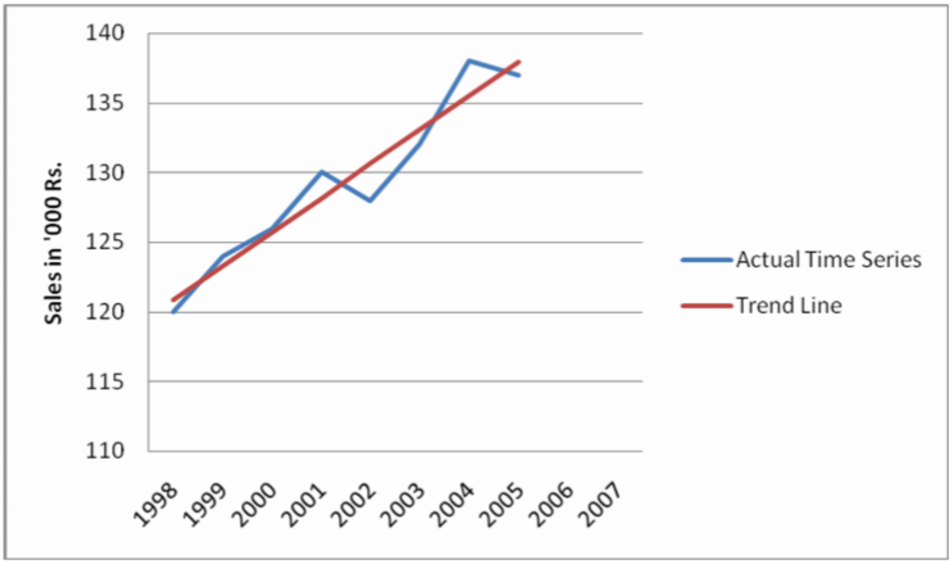
Estimation:
To estimate the profit for the years 2007 in the trend line equation, we substitute the prospective value of x, if the table was extended to 2007. i.e. we put x =11, the next value after x = 9 for the year 2006 and x = 7 for 2005.

y_{2007} = 129.38 + 1.22 (11) = 142.8

There fore the estimated profit for the year 2007 is Rs. 1,42,800.

Now we draw the graph of actual time series by plotting the sales against the corresponding year, the period is taken on the X-axis and the sales on the Y-axis. The points are joined by straight lines. To draw the trend line it is enough to plot any two point (usually we take the first and the last trend value) and join it by straight line.

To estimate the trend value for the year 2007, we draw a line parallel to Y-axis from the period 2007 till it meet the trend line at a point say A. From this point we draw a line parallel to the X-axis till it meet the Y-axis at point say B. This point is our estimate value of sales for the year 2007. The graph and its estimate value (graphically) is shown below:



From the graph, the estimated value of the sales for the year 2007 is 142 i.e. Rs 1,42,000 (approximately)

Example 10:

Fit a straight line trend to the following data. Draw the graph of the actual time series and the trend line. Estimate the import for the year 1998.

Year	1991	1992	1993	1994	1995	1996
Import in'000Rs	40	44	48	50	46	52

Solution: Here again the period of years is 6 i.e. even. Proceeding similarly as in the above problem, the table of calculation and the estimation is as follows:

Years	**Import (y)**	**x**	**xy**	**x^2**	**Trend Value: $Y_t = a + bx$**
1991	40	-5	-200	25	41.82
1992	44	-3	-132	9	43.76
1993	48	-1	-48	1	45.7
1994	50	1	50	1	47.64
1995	46	3	138	9	49.58
1996	52	5	260	25	51.52
Total	$\sum y = 280$	$\sum x = 0$	$\sum xy = 68$	$\sum x^2 = 70$	

From the table : n = 6, $\sum xy = 68$, $\sum x^2 = 70$, $\sum y = 280$

Their four b = $\frac{\sum xy}{\sum x^2} = \frac{68}{70} = 0.97$ and $\frac{\sum y}{n} = \frac{280}{8} = 46.67$

Thus, the straight line trend is y = 46.67 + 0.97x.

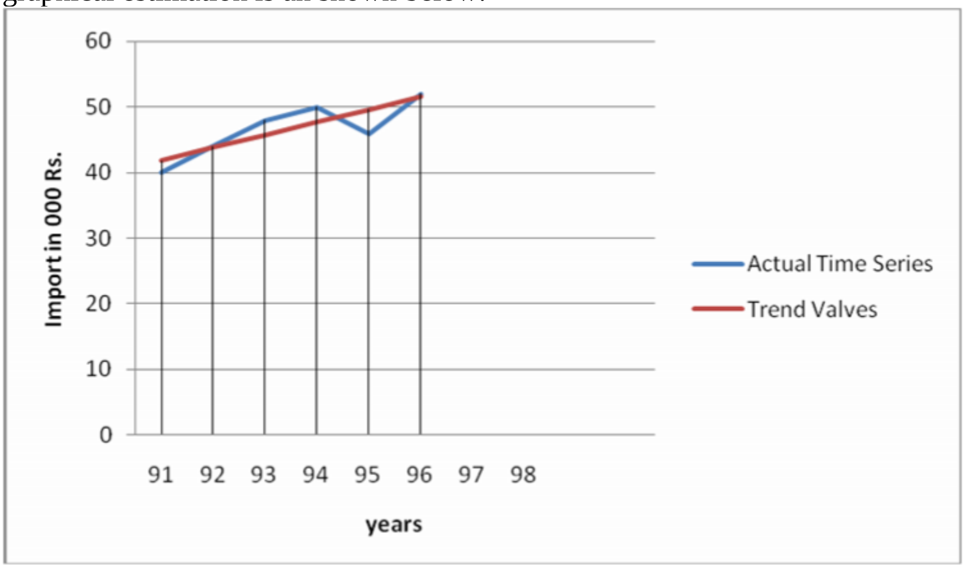
All the remaining trend values are calculated as described in the above problem.

Estimation:

To estimate the import for the year 1998, we put x = 9 in the tried line equation.There fore $y_{1997} = 46.67 + 0.97\ (9) = 55.4$

There fore the imports for the year 1997is Rs. 55,400.

The graph of the actual time series and the trend values along with the graphical estimation is an shown below:



From graph the estimated import are Rs. 55,000.

Example 11:

Fit a straight line trend to the following time –series, representing sales in lakhs of Rs. of a company, for the year 1998 to 2005. Plot the given data well as the trend line on a graph paper. Hence or otherwise estimate trend for the year 2006.

Year	1998	1999	2000	2001	2002	2003	2004	2005
Sales (Lakhs of Rs.)	31	33	30	34	38	40	45	49

Solution:

Here the number of years = 8, an even number, so we define $x = \frac{year - 2001.5}{0.5}$, so that the values of x are -7, -3, -1, 1, 3, 5 and 7, to get $\sum x = 0$.

Prepare the following table to obtain the summations $\sum x^2$, $\sum y$, $\sum x\, y$.

Year	**Sales (in Lakhs of Rs.)**	**x**	**x^2**	**x y**	**Trend Values**
1998	31	-7	49	-217	28.33
1999	33	-5	25	-165	30.95
2000	30	-3	9	-90	33.57
2001	34	-1	1	-34	36.19
2002	38	1	1	38	38.81
2003	40	3	9	120	41.43
2004	45	5	25	225	44.05
2005	49	7	49	343	46.67
	300		168	220	

Here, n = 8, $\sum y = 300$, $\sum x^2 = 168$, $\sum x y = 220$
Now, a and b are calculated as follows:

$$a = \frac{\sum y}{n} = \frac{300}{8} = 37.5$$

$$b = \frac{\sum xy}{\sum x^2} = \frac{220}{168} = 1.31$$

So, the equation of the straight line trend is y = a + b x

i.e. y = 37.5 + 1.31 x

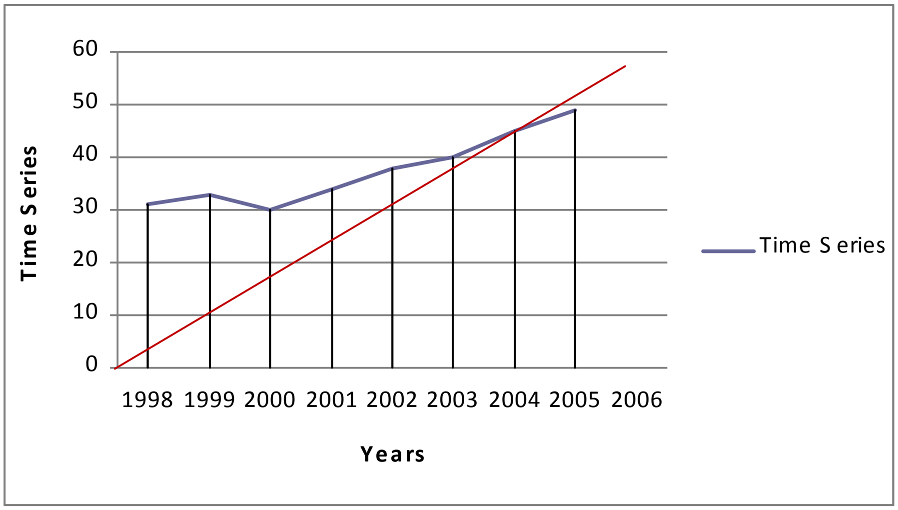
To obtain the trend values, first calculate y for x = -7, for the year 1998
y = 37.5 + 1.31 (-7)
= 37.5 – 9.17 = 28.33

To find the successive trend values, go on addition 2b = 2 × 1.31 = 2.62, to the preceding values as in this case the different between x values is of 2 units.

So, the estimated values of trend for x = -5, -3, -1, 1, 3, 5, 7 and 7 are 30.95, 33. 57, 36.19, 38.81, 41.43, 44.05 and 46.67 respectively. Write down these values in the table.

Hence the estimated trend value for the year 2006 is 49.29 (in lakhs of Rs.).

Now, for the graph of trend line, note that only two trend values 30.95 and 46.67 w.r.t. years 1999 and 2005 are considered as point. The line joining these two points represents trend line.



To estimate the trend for the year 2006, drawn a perpendicular from x-axis at this point meeting the line in P. then from P, draw another perpendicular on y-axis which gives estimate of trend as 49.

MEASURESMENT OF OTHER COMPONENTS

We have studied four method of estimation of Secular Trend. The following procedure is applied to separate the remaining components of the time series.

Using seasonal indices (s), the seasonal variations in a time series can be measured. By removing the trend and the seasonal factors, a combination of cyclical and irregular fluctuations is obtained.

If we assume, multiplicative model, represented by the equation

$$O = T \times S \times C \times I$$

Then, to depersonalize the data, the original time series (O) divided by the seasonal indices (S), which can be express as,

$$\frac{O}{S} = \frac{T \times S \times C \times I}{S} = T \times C \times I$$

If it is further divided by trend values (T), then we have

$$\frac{T \times C \times I}{T} = C \times I$$

Thus a combination of cyclical and irregular variation can be obtained. Irregular fluctuations, because of their nature, can not be eliminated completely, but these can be minimized by taking short term averages and then the estimate of cyclical variation can be obtained.

METHODS TO ESTIMATE SEASONAL FLUCTUATIONS

We have seen methods to separate the trend component of Time Series. Now, let us see, how to separate the seasonal component of it.
Methods of Seasonal Index

It is used to finds the effect of seasonal variations in a Time Series. The steps are as follows:

i. Find the totals for each season, as well as the grand total, say G.
ii. Find the arithmetic means of these total, and the grand total by dividing the values added.
iii. Find seasonal indices, representing the seasonal component for each season, using the formula

$$\text{Seasonal Index} = \frac{AverageforSeasonal \times 100}{GrandAverage}$$

$$\text{Where, Grand Average} = \frac{G}{TotalNo.ofValues}$$

Example 12:
Find the seasonal component of the time series, using method of seasonal indices.

Seasonal / Years	I	II	IV	Grand
2003	33	37	32	31
2004	35	40	36	35
2005	34	38	34	32
2006	36	41	35	36
2007	34	39	35	32

Solution:

	I	II	III	IV	Grand
Total	172	195	172	166	705 (G)
Average	34.4 (172 / 5)	39	34.4	33.2	35.25(G/20)
Seasonal Index	$\frac{34.4 \times 100}{35.25}$ = 97.59	$\frac{39 \times 100}{35.25}$ = 110.64	$\frac{34.4 \times 100}{35.25}$ = 97.59	$\frac{33.2 \times 100}{35.25}$ = 94.18	

The time series can be deseasonalised by removing the effect of seasonal component from it. It is done using the formula.

$$\text{Deseasonalised Value} = \frac{OriginalValue \times 100}{SeasonalIndex}$$

BUSINESS FORECASTING:

In this chapter, few methods of analyzing the past data and predicting the future values are already discussed. Analysis of time series an important role in Business Forecasting. One of the aspects of it estimating future trend values. Now-a-days, any business or industry is governed by factors like supply of raw material, distribution network, availability of land, labour and capital and facilitates like regular supply of power, coal, water, etc. a business has to sustain intricate government regulations, status, everchanging tastes and fashions, the latest technology, cut throat competition by other manufacturers and many other.

While making a forecast, combined effect of above factors should be considered. Scientific method are used to analyse the past business condition. The study reveals the pattern followed by the business in the past. It also bring out the relationship and interdependence of different industries which helps in interpretation of changes in the right perspective. The analysis gives an idea about the components of the time series and their movement in the past. Various indices such as index of production, prices, bank deposits, money rates, foreign exchange position etc. can provide information about short and long term variations, the general trend, the ups downs in a business.

The study of the past data and the comparison of the estimated and actual values helps in pinpointing the areas of shortcoming which can be overcome. For successful business forescasting co-ordination of all departments such as production, sales, marketing is sine-qua-nin, which result in achieving ultimate corporate goals.

There are different theories of Business Forecasting such as

i. Time lag or Sequence Theory
ii. Action and Reaction Theory
iii. Cross Cut Analysis Theory
iv. Specific Historical Analogy Theory

Of these, Time lag or Sequence Theory is most important. It is based on the fact that there is a time lag between the effect of changes at different stages but there is a sequence followed by these effect e.g. In 80’s, the invention of silicon ships brought fourth and fifth generation computers in use. The computers were introduced in various fields such as front-line and back house banking, airlines and railways reservation, new communication technique, home appliances like washing machine etc. this, in turn, increase the demand for qualified personnel in electronic filed to manufacture, handle and maintain these sophisticated machine. It has result in mad rush for admission to various branches of electronics and computer engineering in the recent past.

By applying any one of the these forecasting theories, business forecasting can be made. It should be noted that while collecting the data for analysis, utmost care has to be taken so as to increase the reliability of estimates. The information should be collected by export investigators, over a long period of time. Otherwise, it may lead to wrong conclusions.

EXERCISE

1. What is a time series ? Describe the various components of a time series with suitable example.
2. What are seasonal variation ? Explain briefly with example.
3. Describe the secular trend component of a time series,
4. What are the method of determining trend in a time series?
5. Compare method of moving average and least squares of estimating trend component.

6. Find the trend values using the method of semi-averages for the following data expressing production in thousand unit of a company for 7 years.
7. Explain the method to calculate 3 yearly and 4 yearly moving averages.
8. What are the merits and demerits of the method of moving average?
9. Explain the simple average method to find the seasonal indices of a time series
10. Calculate trend by considering three yearly moving average for the following time series of price indices for the years 2000-2007. Also plot on the graph the trend values.

Year	2000	2001	2002	2003	2004	2005	2006	2007
Price Index	111	115	116	118	119	120	122	124

11. Determine the trend for the following data using 3 yearly moving averages. Plot the graph of actual time series and the trend values.

Year	1989	1990	1991	1992	1993	1994	1995	1996	1997
Sales in'000Rs	24	28	30	33	34	36	35	40	44

12. Determine the trend for the following data using 3 yearly moving averages. Plot the graph of actual time series and the trend values.

Year	1977	1978	1979	1980	1981	1982	1983	1984
Sales in'000Rs	46	54	52	56	58	62	59	63

13. .Determine the trend for the following data using 3 yearly moving averages. Plot the graph of actual time series and the trend values.

Year	1979	1980	1981	1982	1983	1984	1985	1986
Profit in lakhs of Rs	98	100	97	101	107	110	102	105

14. Determine the trend for the following data using 5 yearly moving averages. Plot the graph of actual time series and the trend values.

Year	1980	1982	1984	1986	1988	1990	1992	1994	1996	1998	2000
Values	34	37	35	38	37	40	43	42	48	50	52

15. Determine the trend for the following data giving the production of steel in million tons, using 5 yearly moving averages. Plot the graph of actual time series and the trend values.

Year	1972	1973	1974	1975	1976	1977	1978	1979	1980	1981	1982
Production	28	30.5	32	36.8	38	36	39.4	40.6	42	45	43.5

16. Find five-yearly moving average for the following data which represents production in thousand unit of a small scale industry. Plot the given data as well as the moving average on a graph paper.

Year	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990
Production	110	104	78	105	109	120	115	110	115	122	130

Ans. The trend values are 101.2, 103.2, 105.4, 111.8, 113.8, 116.4 and 118.4 for the years 1982 to 1988.

17. Find the trend component of the following time series of production in thousand kilogram during 1971-1980. Plot the moving average and the original time on a graph paper.

Year	1971	1972	1973	1974	1975	1976	1977	1978	1979	1980
Production	12	15	18	17	16	20	23	22	24	25

Ans. The trend values are 16, 17.125, 18.375, 19.625, 21.25, 22.875 for the years 1973 to 1978.

18. Fit a straight line trend to the following data representing import in million Rs. of a certain company. Also find an estimate for the year 2008.

Year	2000	2001	2002	2003	2004	2005	2006
Import	48	50	58	52	45	41	49

Ans. The straight line trend is y = 49-x. the trend values are 52, 51, 50, 49, 48, 47 and 46 respectively and the estimate trend for the year 2006 is 44 million Rs.

19. The production of a certain brand of television sets in thousand unit is given below. Fit a straight line trend to the data. Plot the given data and the trend line on graph find an estimate for the year 2004.

Year	1997	1998	1999	2000	2001	2002	2003
production	865	882	910	925	965	1000	1080

Ans. The straight line trend is y = 947.71 + 33.43 x. the trend values are 846.42, 879.85, 913.28, 946.71, 1013.57 and 1047. The estimate for the year 2004 is 1080.43 thermal million.

20. The straight line trend by the method of least squares for the following data which represents the expenditure in lakhs od Rs. on advertisement of a certain company. Also find an estimate for the year 2005. Plot the given data and the trend line on a graph paper.

Year	1997	1998	1999	2000	2001	2002	2003	2004
Expenditure	21	24	32	40	38	49	57	60

Ans. The trend is y = 40.13 +2.9x. the trend values are 19.83, 25.62, 31.43, 37.23, 43.03, 48.83, 54.63 and 60.43, 2005 is 66.23.

21. Use the method of least squares to find straight line trend for the following time series of production in thousand units 1981 – 1988. Also estimate trend for the year 2003.

Year	1995	1996	1997	1998	1999	2000	2001	2002
Production	80	90	92	83	94	99	92	102

Ans. The straight line trend is y = 91.5 + 1.167 x. the trend values are 83.331, 85.665, 87.999, 90.333, 92.667, 95.001, 97.335 and 99.669. the estimate of trend, for the year 2003 is 102.003

22. Calculate seasonal indices for the following data:

Year	**I**	**II**	**III**	**IV**
2003	55	53	57	51
2004	56	55	60	53
2005	57	56	61	54

Ans. 100.59, 98.2, 106.57, 94.61

23. Determine the trend for the following data giving the production of wheat in thousand tons from the years 1980 to 1990, using the 5-yearly moving averages. Plot the graph of actual time series and the trend values.

Year	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990
Production	13.5	14.7	17	16.2	18.1	20.4	22	21.2	24	25	26.6

24. Determine the trend for the following data giving the income (in million dollars) from the export of a product from the year 1988 to 1999. Use the 4-yearly moving average method and plot the graph of actual time series and trend values.

Year	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999
Income	340	360	385	470	430	444	452	473	490	534	541	576

25. Using the 4-yearly moving average method find the trend for the following data.

Year	1967	1968	1969	1970	1971	1972	1973	1974	1975	1976	1977
Value	102	100	103	105	104	109	112	115	113	119	117

26. Determine the trend for the following data giving the sales (in ’00 Rs.) of a product per week for 20 weeks. Use appropriate moving average method.

Week	1	2	3	4	5	6	7	8	9	10
Sales	22	26	28	25	30	35	39	36	30	32
Week	11	12	13	14	15	16	17	18	19	20
Sales	29	34	36	35	35	39	43	48	52	49

27. An online marketing company works 5-days a week. The day-to-day total sales (in ‘000 Rs) of their product for 4 weeks are given below. Using a proper moving average method find the trend values.

Days	1	2	3	4	5	6	7	8	9	10
Sales	12	16	20	17	18	20	26	25	27	30
Days	11	12	13	14	15	16	17	18	19	20
Sales	35	32	32	38	36	35	34	38	40	41

28. Fit a straight line trend to the following data. Draw the graph of the actual time series and the trend line. Estimate the sales for the years 2000.

Year	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999
Sales in ‘000 Rs	45	47	49	48	54	58	53	59	62	60	64

29. Fit a straight line trend to the following data. Draw the graph of the actual time series and the trend line. Estimate the sales for the years 2001.

Year	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998
Profit in ‘000 Rs	76	79	82	84	81	84	89	92	88	90

30. Fit a straight line trend to the following data. Draw the graph of the actual time series and the trend line. Estimate the sales for the years 2007.

Year	1998	1999	2001	2002	2003	2004	2005	2006
Profit in ‘000 Rs	116	124	143	135	138	146	142	152

31. Fit a straight line trend to the following data giving the number of casualties (in hundred) of motorcyclists without helmet. Estimate the number for the year 1999.

Year	1992	1993	1994	1995	1996	1997	1998
No of casualties	12	14.2	15.2	16	18.8	19.6	22.1

32. Fit a straight line trend to the following data. Draw the graph of the actual time series and the trend line. Estimate the import for the years 2002

Year	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
Import in'000 Rs.	55	52	50	53	54	56	58	60	57	59

33. Fit a straight line trend to the following data giving the price of crude oil per barrel in USD. Draw the graph of the actual time series and the trend line. Estimate the sales for the year 2003

.

Year	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001
Price per barrel	98	102	104.5	108	105	109	112	118	115	120

34. Apply the method of least squares to find the number of student attending the library in the month of May of the academic year 2005 – 2006 from the following data.

Month	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Jan	Feb	Mar	Apr
Students	105	120	160	225	180	115	124	138	176	230	180

35. Assuming that the trend is absent, find the seasonal indices for the following data and also find the deseasonalized values.

Quarters	**I**	**II**	**III**	**IV**
1977	10	12	14	16
1978	12	15	18	22
1979	16	18	20	24
1980	24	26	28	34

36. Calculate seasonal indices for the following data:

Year	**I**	**II**	**III**	**IV**
2003	55	53	57	51
2004	56	55	60	53
2005	57	56	61	54

Ans. 100.59, 98.2, 106.57, 94.61

❖❖❖

14

INDEX NUMBERS

OBJECTIVES

- To understand about the importance of Index Numbers.
- To understand different types of Index Numbers and their computations.
- To understand about *Real Income* and *Cost of Living Index Numbers*.
- To understand the problems in constructing Index Numbers.
- To know the merits and demerits of Index Numbers.

Every variable undergoes some changes over a period of time or in different regions or due to some factors affecting it. These changes are needed to be measured. In the last chapter we have seen how a time series helps in estimating the value of a variable in future. But the magnitude of the changes or variations of a variable, if known, are useful for many more reasons. For example, if the changes in prices of various household commodities are known, one can plan for a proper budget for them in advance. If a share broker is aware of the magnitude of fluctuations in the price of a particular share or about the trend of the market he can plan his course of action of buying or selling his shares. Thus, we can feel that there is a need of such a measure to describe the changes in prices, sales, profits, imports, exports etc, which are useful from a common man to a business organization.

Index number is an important statistical relative tool to measure the changes in a variable or group of variables with respect to time, geographical conditions and other characteristics of the variable(s). Index number is a relative measure, as it is independent of the units of the variable(s) taken in to consideration. This is the advantage of index numbers over normal averages. All the averages which we studied before are absolute measures, *i.e.* they are expressed in units, while index numbers are percentage values which are independent of the units of the variable(s). In calculating an index number, a base period is considered for comparison and the changes in a variable are measured using various methods.

Though index numbers were initially used for measuring the changes in prices of certain variables, now it is used in almost every field of physical sciences, social sciences, government departments, economic bodies and business organizations. The gross national product (GNP), per capita income, cost of living index, production index, consumption, profit/loss etc every variable in economics uses this as a tool to measure the variations. Thus, the fluctuations, small or big, in the economy are measured by index numbers. Hence it is called as a barometer of economics.

14.2 IMPORTANCE OF INDEX NUMBERS

The important characteristics of Index numbers are as follows:

(1) ***It is a relative measure***: As discussed earlier index numbers are independent of the units of the variable(s), hence it a special kind of average which can be used to compare different types of data expressed in different units at different points of time.

(2) ***Economical Barometer***: A barometer is an instrument which measures the atmospheric pressure. As the index numbers measure all the ups and downs in the economy they are hence called as the economic barometers.

(3) ***To generalize the characteristics of a group***: Many a time it is difficult to measure the changes in a variable in complete sense. For example, it is not possible to directly measure the changes in a business activity in a country. But instead if we measure the changes in the factors affecting the business activity, we can generalize it to the complete activity. Similarly the industrial production or the agricultural output cannot be measured directly.

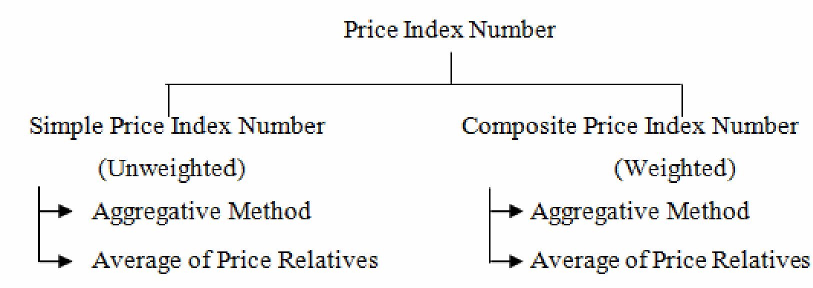
(4) ***To forecast trends***: Index numbers prove to be very useful in identifying trends in a variable over a period of time and hence are used to forecast the future trends.

(5) ***To facilitate decision making***: Future estimations are always used for long term and short term planning and formulating a policy for the future by government and private organizations. Price Index numbers provide the requisite for such policy decisions in economics.

(6) ***To measure the purchasing power of money and useful in deflating***: Index numbers help in deciding the actual purchasing power of money. We often hear from our elders saying that " In our times the salary was just Rs. 100 a month and you are paid Rs. 10,000, still you are not happy!" The answer is simple (because of index numbers!) that the money value of Rs. 100, 30 years before and now is drastically different. Calculation of *real income* using index numbers is an important tool to measure the actual income of an individual. This is called as deflation.

There are different types of index numbers based on their requirement like, price index, quantity index, value index etc. The price index is again classified as single price index and composite price index.

14.3 PRICE INDEX NUMBERS



The price index numbers are classified as shown in the following diagram:

Notations:

P_0: Price in Base Year **Q_0: Quantity in Base Year**

P_1: Price in Current Year **Q_1: Quantity in Current Year**

The *suffix* **'0'** stands for the ***base year*** and the *suffix* **'1'** stands for the ***current year***.

14.3.1 Simple (Unweighted) Price Index Number By Aggregative Method

In this method we define the price index number as the ratio of sum of prices in current year to sum of prices in base year and express it in percentage. *i.e.* multiply the quotient by 100.

Symbolically,

$$I = \frac{\Sigma P_1}{\Sigma P_0} \text{ x } 100 \qquad \ldots (1)$$

Steps for computation:

1. The total of all base year prices is calculated and denoted by ΣP_0.
2. The total of all current year prices is calculated and denoted by ΣP_1.
3. Using the above formula, simple price index number is computed.

***Example* 1**

For the following data, construct the price index number by simple aggregative method:

Commodity	Unit	Price in	
		1985	**1986**
A	Kg	10	12
B	Kg	4	7
C	Litre	6	7
D	Litre	8	10

Solution: Following the steps for computing the index number, we find the totals of the 3rd and 4th columns as shown below:

Commodity	Unit	Price in	
		1985(P_0)	1986(P1)
A	Kg	10	12
B	Kg	4	7
C	Litre	6	7
D	Litre	8	10

$\therefore I = \frac{\Sigma P_1}{\Sigma P_0} \text{ x } 100 = \frac{36}{28} \text{ x } 100 = 128.57$

Meaning of the value of I:

I = 128.57 means that the prices in 1986, as compared with that in 1985 have increased by 28.57 %.

14.3.2 Simple (Unweighted) Price Index Number by Average of Price Relatives Method

In this method the price index is calculated for every commodity and its arithmetic mean is taken. *i.e.* the sum of all price relative is divided by the total number of commodities.

Symbolically, if there are *n* commodities in to consideration, then the simple price index number of the group is calculated by the formula:

$$I = \frac{1}{n}\Sigma\left(\frac{P_1}{P_0} \text{ x } 100\right) \qquad \dots (2)$$

Steps for computation

1. The price relatives for each commodity are calculated by the formula: $\frac{P_1}{P_0}$ x 100 .

2. The total of these price relatives is calculated and denoted as: $\Sigma\left(\frac{P_1}{P_0} \text{ x } 100\right)$.

3. The arithmetic mean of the price realtives using the above formula no. (2) gives the required price index number.

Example 2

Construct the simple price index number for the following data using average of price relatives method:

Commodity	Unit	Price in	
		1997	1998
Rice	Kg	10	13
Wheat	Kg	6	8
Milk	Litre	8	10
Oil	Litre	15	18

Solution: In this method we have to find price relatives for every commodity and then total these price relatives. Following the steps for computing as mentioned above, we introduce first, the column of price relatives. The table of computation is as follows:

Commodity	Unit	Price in		$\frac{P_1}{P_0}$ x 100
		1997(P_0)	1998(P_1)	
Rice	Kg	10	13	130
Wheat	Kg	6	8	133.33
Milk	Litre	8	10	125
Oil	Litre	15	18	120
			Total:	**508.33**

Now, n = 4 and the total of price relatives is 508.33

$$\therefore I = \frac{1}{n}\Sigma\left(\frac{P_1}{P_0} \text{ x } 100\right) = \frac{508.33}{4} = 127.08$$

The prices in 1998 have increased by 27 % as compared with in 1997.

<u>Remark</u>:

1. The simple aggregative method is calculated without taking into consideration the units of individual items in the group. This may give a misleading index number.

2. This problem is overcome in the average of price relatives method, as the individual price relatives are computed first and then their average is taken.

3. Both the methods are unreliable as they give equal weightage to all items in consideration which is not true practically.

14.3.3 Weighted Index Numbers by Aggregative Method

In this method weights assigned to various items are considered in the calculations. The products of the prices with the corresponding weights are computed; their totals are divided and expressed in percentages.

Symbolically, if W denotes the weights assigned and P_0, P_1 have their usual meaning, then the weighted index number using aggregative method is given by the formula:

$$I = \frac{\Sigma P_1 W}{\Sigma P_0 W} \text{ x } 100 \qquad \dots (3)$$

Steps to find weighted index number using aggregative method

1. The columns of P_1W and P_0W are introduced.

2. The totals of these columns are computed.

3. The formula no. (3) is used for computing the required index number.

***Example* 3**

From the following data, construct the weighted price index number:

Commodity	**A**	**B**	**C**	**D**
Price in 1982	6	10	4	18
Price in 1983	9	18	6	26
Weight	35	30	20	15

Solution: Following the steps mentioned above, the table of computations is as follows:

Commodity	Weight (W)	Price in 1982 (P_0)	P_0W	Price in 1983 (P_1)	P_1W
A	35	6	210	9	315
B	30	10	300	18	540
C	20	4	80	6	120
D	15	18	270	26	390
Total	-	-	ΣP_0W = 860	-	ΣP_1W = 1365

Using the totals from the table, we have

Weighted Index Number $I = \frac{\Sigma P_1W}{\Sigma P_0W} \text{x} 100 = \frac{1365}{860} \text{ x } 100 = 158.72$

<u>*Remark*</u>**:**

There are different formulae based on what to be taken as the weight while calculating the weighted index numbers. Based on the choice of the weight we are going to study here three types of weighted index numbers: (1) Laspeyre's Index Number, (2) Paasche's Index Number and (3) Fisher's Index Number.

(1) Laspeyre's Index Number:

In this method, Laspeyre assumed the *base quantity (Q_0) as the weight* in constructing the index number. Symbolically, P_0, P_1 and Q_0 having their usual meaning, the Laspeyre's index number denoted by I_L is given by the formula:

$$I_L = \frac{\Sigma P_1Q_0}{\Sigma P_0Q_0} \text{ x } 100 \qquad \ldots (4)$$

Steps to compute I_L:

1. The columns of the products P_0Q_0 and P_1Q_0 are introduced.
2. The totals of these columns are computed.
3. Using the above formula no. (4), I_L is computed.

***Example* 4**

From the data given below, construct the Laspeyre's index number:

Commodity	1965		1966
	Price	**Quantity**	**Price**
A	5	12	7
B	7	12	9
C	10	15	15
D	18	5	20

Solution: Introducing the columns of the products P_0Q_0 and P_1Q_0, the table of computation is completed as shown below:

Commodity	1965		1966	P_0Q_0	P_1Q_0
	Price (P_0)	Quantity (Q_0)	Price (P_1)		
A	5	12	7	60	84
B	7	12	9	84	108
C	10	15	15	150	225
D	18	5	20	90	100
Total	-	-	-	$\Sigma P_0Q_0 = 384$	$\Sigma P_1Q_0 = 517$

Using the totals from the table and substituting in the formula no. (4), we have

$$I_L = \frac{\Sigma P_1Q_0}{\Sigma P_0Q_0} \text{ x } 100 = \frac{517}{384} \text{ x } 100 = 134.64$$

(2) Paasche's Index Number:

In this method, Paasch assumed the current year quantity (Q_1) as the weight for constructing the index number. Symbolically, P_0, P_1 and Q_1 having their usual meaning, the Paasche's index number denoted by I_P is given by the formula:

$$I_P = \frac{\Sigma P_1Q_1}{\Sigma P_0Q_1} \text{ x } 100 \qquad \ldots (5)$$

The steps for computing I_P are similar to that of I_L.

***Example* 5**

From the data given below, construct the Paasche's index number:

Commodity	**1985**		**1986**
	Price	**Price**	**Quantity**
A	5	8	10
B	10	14	20
C	6	9	25
D	8	10	10

Solution: Introducing the columns of the products P_0Q_1 and P_1Q_1, the table of computations is completed as shown below:

Commodity	**1985**		**1986**	P_0Q_1	P_1Q_1
	Price (P_0)	**Price (P_1)**	**Quantity (Q_1)**		
A	5	8	10	50	80
B	10	14	20	200	280
C	6	9	25	150	225
D	8	10	10	80	100
Total	-	-	-	$\Sigma P_0Q_1 = 480$	$\Sigma P_1Q_1 = 685$

Using the totals from the table and substituting in the formula no. (5), we have: $I_P = \frac{\Sigma P_1 Q_1}{\Sigma P_0 Q_1}$ x 100 = $\frac{685}{480}$ x 100 = 142.71

(3) Fisher's Index Number:
Fisher developed his own method by using the formulae of Laspeyre and Paasche. He defined the index number as the geometric mean of I_L and I_P. Symbolically, the Fisher's Index number denoted as I_F is given by the formula: $I_F = \sqrt{I_L \text{ x } I_P} = \sqrt{\frac{\Sigma P_1 Q_0}{\Sigma P_0 Q_0} \text{ x } \frac{\Sigma P_1 Q_1}{\Sigma P_0 Q_1}}$ x 100 .. (6)

Note:
1. *The multiple 100 is outside the square root sign*.
2. *While computing products of the terms, care should be taken to multiply corresponding numbers properly.*

***Example* 6**
From the following data given below, construct the (i) Laspeyre's index number, (ii) Paasche's index number and hence (iii) Fisher's index number.

Item	**1975**		**1976**	
	Price	**Quantity**	**Price**	**Quantity**
A	4	12	6	16
B	2	16	3	20
C	8	9	11	14

Solution: Introducing four columns of the products of P_0Q_0, P_0Q_1, P_1Q_0 and P_1Q_1, the table of computations is completes as shown below:

Item	P_0	Q_0	P_1	Q_1	P_0Q_0	P_0Q_1	P_1Q_0	P_1Q_1
A	4	12	6	16	48	64	72	96
B	2	16	3	20	32	40	48	60
C	8	9	11	14	72	112	99	154
				Total	152	216	219	310

From the table, we have $\Sigma P_0Q_0 = 152$, $\Sigma P_0Q_1 = 216$, $\Sigma P_1Q_0 = 219$ and $\Sigma P_1Q_1 = 310$

$$\therefore I_L = \frac{\Sigma P_1 Q_0}{\Sigma P_0 Q_0} \text{ x } 100 = \frac{219}{152} \text{ x } 100 = 144.08$$

$$\therefore I_P = \frac{\Sigma P_1 Q_1}{\Sigma P_0 Q_1} \text{ x } 100 = \frac{310}{216} \text{ x } 100 = 143.52$$

$$\therefore I_F = \sqrt{I_L \text{ x } I_P} = \sqrt{144.08 \text{ x } 143.52} = 143.8$$

Remark:
1. **Laspeyre's** index number though popular has a drawback that it does not consider the change in consumption over a period. (as it does not take into account the current quantity).
2. Paasche's index number overcomes this by assigning the current year quantity as weight.

3. Fisher's index number being the geometric mean of both these index numbers, it considers both the quantities. Hence it is called as the ideal index number.

***Example* 7**
From the following data given below, construct the Kelly's index number:

Item	**Base Year**		**Current Year**	
	Price	**Quantity**	**Price**	**Quantity**
A	18	20	24	22
B	9	10	13	16
C	10	15	12	19
D	6	13	8	15
E	32	14	38	18

Solution: Introducing the columns of $Q = \frac{Q_0 + Q_1}{2}$, P_0Q and P_1Q, the table of computations is completed as shown blow:

Item	Q_0	Q_1	Q	P_0	P_0Q	P_1	P_1Q
A	20	22	**21**	18	**378**	24	**504**
B	10	16	**13**	9	**117**	13	**169**
C	15	19	**17**	10	**170**	12	**204**
D	13	15	**14**	6	**84**	8	**112**
E	14	18	**16**	32	**512**	38	**608**
				Total	**1261**	--	**1597**

From the table, we have $\Sigma P_0Q = 1261$ and $\Sigma P_1Q = 1597$

$$\therefore I_K = \frac{\Sigma P_1 Q}{\Sigma P_0 Q} \text{ x } 100 = \frac{1597}{1261} \text{ x } 100 = 126.65$$

14.3.4 Weighted Index Numbers using average of price relatives method

This is similar to what we have seen in subsection 7.3.2. Here the individual price relatives are computed first. These are multiplied with the corresponding weights. The ratio of the sum of the products and the total value of the weight is defined to be the weighted index number.
Symbolically, if W denotes the weights and I denote the price relatives then the weighted index number is given by the formula: $\frac{\Sigma IW}{\Sigma W}$ … (8)
One of the important weighted index number is the *cost of living index number*, also known as the *consumer price index (CPI) number*.

14.4 Cost Of Living Index Number Or Consumer Price Index Number

There are two methods for constructing this index number:
(1) Aggregative expenditure method and (2) Family Budget Method

(1) In aggregative expenditure method we construct the index number by taking the base year quantity as the weight. In fact this index number is nothing but the Laspeyre's index number.

(2) In family budget method, value weights are computed for each item in the group and the index number is computed using the formula:

$$\frac{\Sigma IW}{\Sigma W}, \text{ where } I = \frac{P_1}{P_0} \text{ x } 100 \text{ and } W = P_0Q_0 \qquad \ldots (9)$$

***Example* 8**

A survey of families in a city revealed the following information:

Item	**Food**	**Clothing**	**Fuel**	**House Rent**	**Misc.**
% Expenditure	30	20	15	20	15
Price in 1987	320	140	100	250	300
Price in 1988	400	150	125	250	320

What is the cost of living index number for 1988 as compared to that of 1987?

Solution: Here % expenditure is taken as the weight (W). The table of computations is as shown below:

Item	P_0	P_1	$I = \frac{P_1}{P_0}$ **x 100**	**% Expenditure (*W*)**	***IW***
Food	320	400	125	30	3750
Clothing	140	150	107.14	20	2142.8
Fuel	100	125	125	15	1875
House Rent	250	250	100	20	2000
Miscellaneous	300	320	106.67	15	1600.05
			Total	ΣW =100	11367.85

From the table, we have $\Sigma W = 100$ and $\Sigma IW = 11376.85$

$\therefore$ cost of living index number = $\frac{\Sigma IW}{\Sigma W} = \frac{11367.85}{100} = 113.68$

14.5 Use Of Cost of Living Index Numbers

1. These index numbers reflect the effect of rise and fall in the economy or change in prices over the standard of living of the people.
2. These index numbers help in determining the purchasing power of money which is the reciprocal of the cost of living index number.
3. It is used in deflation. *i.e.* determining the actual income of an individual. Hence it also used by the management of government or private organizations to formulate their policies regarding the wages, allowance to their employees.

14.6 REAL INCOME

As discussed earlier in this chapter, index numbers are very useful in finding the real income of an individual or a group of them, which facilitates the different managements to decide their wage policies. The process of measuring the actual income vis-a-vis the changes in prices is called as *deflation*.

The formula for computing the real income is as follows:

$$\text{Real Income of a year} = \frac{\text{Money Income for the year}}{\text{Price Index of that year}} \text{ x } 100$$

***Example* 12**

Calculate the real income for the following data:

Year	**1990**	**1991**	**1992**	**1993**	**1994**	**1995**
Income in Rs.	800	1050	1200	1600	2500	2800
Price Index	100	105	115	125	130	140

Solution: The real income is calculated by the formula:

$$\textbf{real income} = \frac{\textbf{Money Income for the year}}{\textbf{Price Index of that year}} \textbf{ x 100}$$

The table of computation of real income's is completed as shown below:

Year	**Income in Rs.**	**Price Index**	**Real Income**
1990	800	100	800
1991	1050	105	$\frac{1050}{105}$ x100 = 1000
1992	1200	115	$\frac{1200}{115}$ x 100 = 1043
1993	1600	125	$\frac{1600}{125}$ x 100 = 1280
1994	2500	130	$\frac{2500}{130}$ x 100 = 1923
1995	2800	140	$\frac{2800}{140}$ x 100= 2000

14.7 Demerits Of Index Numbers

(1) There are numerous types and methods of constructing index numbers. If an appropriate method is not applied it may lead to wrong conclusions.
(2) The sample selection may not be representative of the complete series of items.
(3) The base period selection also is personalized and hence may be biased.
(4) Index number is a quantitative measure and does not take into account the qualitative aspect of the items.
(5) Index numbers are approximations of the changes, they may not accurate.

Check Your Progress

1. Define Index Numbers.
2. Write a short note on the importance of Index Numbers.
3. "Index Numbers are the Economical barometers". Discuss this statement with examples.
4. Discuss the steps to construct Index Numbers.
5. What are the problems in constructing an Index Number?
6. Define Cost of Living Index Number and explain its importance.
7. What do you mean by (i) Chain Based Index Number and (ii) Fixed Base Index Number? Distinguish between the two.
8. Define (i) Laspeyre's Index Number, (ii) Paasche's Index Number and (iii) Fisher's Index Number. What is the difference between the three? Which amongst them is called as the ideal Index Number? Why?
9. What are the demerits of Index Numbers?
10. From the following data, construct the price index number by simple aggregative method:

Commodity	Unit	Price in	
		1990	1991
A	Kg	14	18
B	Kg	6	9
C	Litre	5	8
D	Litre	12	20

Ans: 148.65

11. From the following data, construct the price index number for 1995, by simple aggregative method, with 1994 as the base:

Commodity	Unit	Price in	
		1994	1995
Rice	Kg	8	10
Wheat	Kg	5	6.5
Oil	Litre	10	13
Eggs	Dozen	4	6

Ans: 131.48

12. From the following data, construct the price index number for 1986, by average of price relatives method:

Commodity	Unit	Price in	
		1985	1986
Banana	Dozen	4	5
Rice	Kg	5	6
Milk	Litre	3	4.5
Slice Bread	One Packet	3	4

Ans: 132.08

13. From the following data, construct the price index number, by method of average of price relatives:

Commodity	Unit	Price in	
		1988	1990
A	Kg	6	7.5
B	Kg	4	7
C	Kg	10	14
D	Litre	8	12
E	Litre	12	18

Ans: 148

14. From the following data, construct the price index number for 1998, by (i) simple aggregative method and (ii) simple average of price relatives method, with 1995 as the base:

Commodity	Unit	Price in	
		1995	1998
Rice	Kg	12	14
Wheat	Kg	8	10
Jowar	Kg	7	9
Pulses	Kg	10	13

Ans: (i) 124.32, (ii) 125.06

15. From the following data, construct the weighted price index number:

Commodity	A	B	C	D
Price in 1985	10	18	36	8
Price in 1986	12	24	40	10
Weight	40	25	15	20

Ans: 121.29

16. From the following data, construct the index number using (i) simple average of price relatives and (ii) weighted average of price relatives:

Commodity	Weight	Price in	
		1988	1990
Rice	4	8	10
Wheat	2	6	8
Pulses	3	8	11
Oil	5	12	15

Ans: (i) 130.13, (ii) 128.82

17. From the data given below, construct the Laspeyre's index number:

Commodity	1975		1976
	Price	Quantity	Price
A	5	10	8
B	6	15	7.5
C	2	20	3
D	10	14	12

Ans: 131.40

18. From the data given below, construct the Paasche's index number:

Commodity	**1980**		**1985**
	Price	**Price**	**Quantity**
A	4	7	10
B	14	22	16
C	5	7	30
D	8	10	21

Ans: 144.67

19. From the following data given below, construct the (i) Laspeyre's index number, (ii) Paasche's index number and hence (iii) Fisher's index number.

Commodity	1980		1990	
	Price	Quantity	Price	Quantity
A	6	15	9	21
B	4	18	7.5	25
C	2	32	8	45
D	7	20	11	29

Ans: (i) 203.83, (ii) 203.37, (iii) 203.60

20. From the following data given below, construct the (i) Laspeyre's index number, (ii) Paasche's index number and (iii) Fisher's index number.

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
Cement	140	200	167	254
Steel	60	150	95	200
Coal	74	118	86	110
Limestone	35	50	46	60

Ans: (i) 103.98, (ii) 127.4, (iii) 115.09

21. From the following data given below, construct the Fisher's index number:

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	2	8	4	14
B	6	14	7	20
C	8.5	10	12	15
D	14	8	19	12
E	22	60	38	85

Ans: 131.37, 120.15

22. From the following data, construct the aggregative price index numbers by taking the average price of the three years as base.

Commodity	Price in 1980	Price in 1981	Price in 1982
A	10	12	16
B	16	19	25
C	5	7	10

Ans: 81.58, 100, 134.21

23. From the following data, construct the price index number by taking the price in 1978 as the base price using aggregative method:

Commodity	Price in 1978	Price in1979	Price in 1980
A	16	18	24
B	4	6	7.5
C	11	15	19
D	20	28	30

Ans: 131.37, 120.15

24. From the following data, construct the price index number by taking the price in 1998 as the base price:

Commodity	Weight	Price in 1998	Price in 1999	Price in 2000
A	3	12	15	20
B	1	8	9	11
C	4	16	20	25
D	2	15	18	22

25. From the following data, construct (i) I_L, (ii) I_P, (iii) I_F

Commodity	1969		1970	
	Price	Quantity	Price	Quantity
Rice	2	10	3	12
Wheat	1.5	8	1.9	10
Jowar	1	6	1.2	10
Bajra	1.2	5	1.6	8
Pulses	4	14	6	20

Ans: 144.4, 144.56, 144.28

26. Construct the cost of living index number for 1980 using the Family Budget Method:

Item	Quantity	Price in	
		1975	1980
A	10	5	7
B	5	8	11
C	7	12	14.5
D	4	6	10
E	1	250	600

Ans: 192.95

27. Construct the cost of living index number for the following data with base year as 1989.

Item	Weight	Price in		
		1989	1990	1991
Food	4	45	50	60
Clothing	2	30	33	38
Fuel	1	10	12	13
House Rent	3	40	42	45
Miscellaneous	1	5	8	10

Ans: for 1990: 114.49, **for 1991**: 132.20

28. A survey of families in a city revealed the following information:

Item	Food	Clothing	Fuel	House Rent	Misc.
% Expenditure	30	20	15	20	15
Price in 1987	320	140	100	250	300
Price in 1988	400	150	125	250	320

What is the cost of living index number for 1988 as compared to that of 1987? **Ans:** 113.65

29. Construct the consumer price index number for the following industrial data:

Item	Weight	Price Index
Industrial Production	30	180
Exports	15	145
Imports	10	150
Transportation	5	170
Other activity	5	190

Ans: 167.30

30. Calculate the real income for the following data:

Year	1988	1989	1990	1991	1992	1993
Income in Rs.	500	550	700	780	900	1150
Price Index	100	110	115	130	140	155

31. The employees of Australian Steel ltd. have presented the following data in support of their contention that they are entitled to a wage adjustment. Dollar amounts shown represent the average weekly take home pay of the group:

Year	1973	1974	1975	1976
Pay in $	260.50	263.80	274	282.50
Index	126.8	129.5	136.2	141.1

Compute the real wages based on the take home pay and the price indices given. Also compute the amount of pay needed in 1976 to provide buying power equal to that enjoyed in 1973.

32. Calculate the real income for the following data:

Year	1977	1978	1979	1980	1981	1982
Income in Rs.	250	300	350	500	750	1000
Price Index	100	105	110	120	125	140

33. The per capita income and the corresponding cost of living index numbers are given below. Find the per capita real income:

Year	1962	1963	1964	1965	1966	1967
per capita income	220	240	280	315	335	390
cost of living I.N.	100	110	115	135	150	160

34. The following data gives the salaries (in '00 Rs.) of the employees of Hindusthan Constructions Ltd with the cost of living index number. Find the real income and suggest how much allowance should be paid to them to maintain the same standard of living.

Year	1990	1991	1992	1993	1994	1995
Income	12	14	17	20	24	28
Price Index	100	120	135	155	180	225

35. The income of Mr. Bhushan Damle in 1999 was Rs. 8,000 per month. If he gets an increment of Rs. 1,200 in 2000 and the price index being 115 with base as 1999, can you conclude that Mr. Damle has got an increment which will maintain his standard of living as compared with the previous year?

❖❖❖

UNIT IX

15

STATISTICAL DECISION THEORY

OBJECTIVES

After going through this chapter you will be able to understand:

- The decision making situation.
- The decision making criteria to arrive at optimum decision.
- The decision tree technique for multi-stage decision making.

15.1 INTRODUCTION

The decision theory may be applied to problems whether it involves financial management or a plant assembly line, whether the time span is ten years, five years or one day and whether it is in public or private sector. In each of such decision-making problems, there are certain common elements which are called ingredients of decision problems. These ingredients are:

1. Alternative Courses of Action: The process of decision-making involves the selection of a single act from among some set of alternative acts. Decision is needed in a problem situation where two or more alternative courses of action are available, and where only one of these actions can be taken. Obviously, if there is only one course of action available, no decision is required since that action must be taken in order to solve the problem. For the sake of simplicity the possible actions are symbolised by a_1 a_2, a_3, ... etc. The totality of all possible actions is called action space denoted by *A*. If there are only three possible actions, we write *A* = action space = (a_1 a_2, a_3,). The decision procedure involves selecting among the alternatives a single course of action that can be actually carried out. If such a course of action is selected that cannot be carried out in the existing situation and circumstance, it will then amount to waste of time and resources. Quite often the objective of precision is to select an act which will accomplish some predesignated purpose. The decision taken may be regarded as satisfactory or not depending upon whether it has helped in the attainment of that objective.

2. Uncertainty: In all decision problems "uncertainty" is found to be a common element. When the outcome of some action is not known in advance. the outcome is said to be uncertain. When there are many possible outcomes of an event (also called *states of nature*) one cannot predict with certainty what will happen—it is only in terms of probabilities we may be able to talk. The term "state of nature" does not mean nature in the ordinary sense of the word. it is a general term which is used to encompass all those factors beyond the control of the decision-maker that affect the outcome of his decision. The various states of nature (outcomes) are symbolised by $\theta_{,1}, \theta_2$ θ_3 ... etc. Totality of all outcomes is called nature space or state space symbolised by Ω. If an action leads to three outcomes θ_1, θ_2 and θ_3 then we write : $\mathbf{\Omega = (\theta_1, \theta_2, \theta_3)}$.

For example, if a product is marketed it may be highly appreciated (outcome θ_1), it may not appeal to the customers (outcome θ_2) or it may be liked by a certain fraction of the customers say, 25% (outcome θ_3).

It must be pointed out that sometimes a distinction is made between decision-making under "risk" and decision-making under "uncertainty". When the state of nature is unknown, but objective or empirical data is available so that the decision-maker can use these data to assign probabilities to the various states of nature the procedure is generally referred to as decision-making under 'risk.' When the state of nature is unknown and there is no objective information on which probabilities can be based, the procedure is referred to as decision-making under 'uncertainty'. It may, however, be noted that even when no objective information is available, the decision-maker may, in Bayesian decision theory, assign subjective probability to the states of nature to help in taking a decision. Once probabilities are assigned, regardless of the manner in which they were obtained, the decision procedure that follows is exactly the same. Hence for practical purposes risk and uncertainty are essentially the same and decision-making under both circumstances will be referred to as decision-making under uncertainty.

3. Pay Offs: In order to evaluate each possible course of action, the result of each event with each course of action has a *value (or pay off)* placed upon it . A number of consequences result from each action under different conditions , the conditions being various states of nature, the consequences will be more in number. In practical situations, particularly in business and economic problems, consequences can be expressed .in terms of money and utility .
The consequences may be evaluated in several ways such as:
(i) in terms of profit,
(ii) in terms of cost.
(iii) in terms of opportunity loss: (The opportunity loss is defined as the difference between the highest possible profit for an event and the actual profit obtained for the actual action taken.)
(iv) units of satisfaction or utility.

4. Decision criteria: The decision-maker must-determine how to select the best course of action. In most decision problems the expected monetary value is used as a decision criterion. When consequences are evaluated in terms of profit, they are called payoffs. A payoff table is prepared and it shows the relation between all possible states of nature, all possible actions and the values associated with the consequences. A specimen of payoff table is given below;

GENERAL FORMAT OF A PAYOFF TABLE

Acts / States *of nature*	a_1	a_2	a_3	……	a_k	……	a_m
θ_1	p_{11}	p_{12}	p_{13}		p_{1k}		p_{1m}
θ_2	p_{21}	p_{22}	p_{23}		p_{2k}		p_{2m}
θ_3	p_{31}	p_{32}	p_{33}		p_{3k}		p_{3m}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
θ_i	$p_{/1}$	$p_{/2}$	$p_{/3}$				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
θ_n	p_{n1}	p_{n2}	p_{n3}		p_{nk}		p_{nm}

In the above table, the column heading designate the various actions out of which the decision-maker may choose while the row heading shows the admissible states of nature under which the decision-maker has to take decision .

A payoff may be thought of as a conditional value or conditional profit (loss). It is conditional value in the sense that is associated with each course of action there is a certain profit (or loss), given that a specific state of nature has occurred. A payoff table thus contains all conditional values of all possible combinations and states of nature. The payoff table indicates that there is no single act which is best for all the states of natue. Therefore, in order to make an optimal decision some criterion and additional information are necessary.

The calculation of payoff depends on the problem. Very often it is a relatively easy matter and sometimes a bit algebraic reasoning is required.

With the payoff table, the decision-maker may be able to reach the optimal solution of a problem, if he has a knowledge of what event is going to occur. Since there is uncertainty about occurrence of events, a decision-maker must make some prediction or forecast usually in terms of probability of occurrence of events. With the probabilities of occurrence assigned, the last step of statistical decision theory is to analyse these probabilities by calculating expected payoff *(EP*) of *expected monetary value* for each course of action. The decision criterion here is to choose the Optimal Act (OA), the act that yields the highest EP.

An alternative decision criterion of statistical decision theory is what is called *expected opportunity loss, EOL*. This criterion also leads to the same result as obtained from expected profits (EP).

Calculations of EOL are the same as those for EP except for the fact that we have to use conditional opportunity loss (COL) instead of payoffs. It may be pointed out that COL of the optimal acts is zero, COL of any act other than *OA* is positive and is the difference between the payoff of OA and the act taken.

If we replace the payoffs by their corresponding opportunity losses we get a new table called loss table.

Example 1:
A baker produces a certain type of special pastry at a total average cost of Rs. 3 and sells it at a price of Rs. 5. This pastry is produced over the weekend and is sold during the following week; such pastry being produced but not sold during a week's time is totally spoiled and has to be thrown away. According to past experience the weekly demand for these pastries is never less than 78 or greater than 80. You are required to formulate action space, payoff table and opportunity loss table.

Solution: It is clear from the problem given that the manufacturer will not produce less than 78 or more than 80 pastries. Thus, there are three courses of action open to him:
a_1 = produce 78 pastries
a_2 = produce 79 pastries
a_3 = produce 80 pastries
thus the action space or A = (a_1, a_2, a_3)

The state of nature is the weekly demand for pastries. There are three possible states of nature, *i.e.,*
θ_1 = demand is 78 pastries
θ_2 = demand is 79 pastries
θ_3= demand is 80 pastries
Hence the state space Ω= (θ_1, θ_2, θ_3)

The uncertainty element in the problem is the weekly demand. The bakery profit conditioned by the weekly demand. Cell values of payoff table are computed as follows:'
Demand x Price - Production x Cost
P_{11} = payoff when action a_1 is taken but the state of nature is θ_1
= Rs. [5 x 78 - 3 x 78] = Rs. 156.

P_{12} = payoff when action a_2 is taken but the state of nature is θ_1
= Rs. [5 x 78 - 3 x 79] = Rs. 153.

P_{13} = payoff when action a_3 is taken but the state of nature is θ_1
= Rs. [5 x 78 - 3 x 80] = *Rs.* 150.

Similarly P_{21} = payoff when action a_1 is taken and the state of nature is θ_2
= Rs. [5 x 78 - 3 x 78] = Rs. 156.
P_{22} = Rs. [5x79-3x79]= Rs. 158.
P_{23} = Rs. [5x79 -3x80] = Rs. 155.

Also, P_{31} = payoff when action a_1 is taken and the state of nature is θ_3
P_{31} = Rs. [5x78-3x78]= Rs. 156.
P_{32} = Rs. [5x79-3x79] = Rs. 158.
P_{33} = Rs. [5 x 80-3x80] = Rs. 160

An EMV indicates the average profit that would be gained If a particular alternative were selected in many similar decision-making situations. Since decisions arc often made on a one time basis, the decision criterion is to choose the alternative course of action that maximizes the expected monetary value.

These values are tabulated below:

PAYOFF TABLE

Action *State of nature*	a_1 =78	a_2 =79	a_3 =80
θ_1 = 78	156	153	150
θ_2 = 79	156	158	155
θ_3 = 80	156	158	160

To calculate opportunity loss, we first determine maximum payoff in each state of nature.
In first state of nature θ_1, maximum payoff = Rs. 156
In second state of nature θ_2, maximum payoff = Rs. 158
In third state of nature θ_3, maximum payoff = Rs. 160
L_{11} 156-156 = 0, L_{12} = 156-153 = 3, L_{13} = 156-150 = 6
L_{21} = 158 - 156 = 2, L_{22} = 158 - 158 = 0, L_{23}= 158 - 155 = 3
L_{31} = 160 - 156 = 4, L_{32} = 160 - 158 = 2, L_{33} = 158 - 158 = 0.
The loss table corresponding to payoff table is given below.

Action **State of nature**	a_1	a_2	a_3
θ_1	0	3	6
θ_2	2	0	3
θ_3	4	2	0

Example 2: Suppose in a process of producing bulbs, the number of producing defective bulbs is constant but unknown. Consider a lot of 100 bulbs which is either to be sold at Rs. 5 each giving a double money back guarantee for each defective item or to be junked at a cost of Rs. 100 for the lot. Construct action space, state space, and payoff table.

Solution: Two possible actions are open to the manufacturer, namely,
a_1 = Junk the lot at a loss of Rs. 100 for the lot,
a_2 = Sell the lot at Rs. 5 each giving double money back guarantee for each defective item.

Thus A = (a_1 a_2) = action space
The number of defective bulbs in the lot designates the status of nature. The number of defective bulbs can be 0, 1, 2, 3. .... 100. *i.e.* there are 101 possible states of nature. Let 0_i denote that there are "*i*" defective bulbs in the lot.

Thus state space or $\Omega = \{0_0, 0_1, 0_2, \ldots, 0_{100}\}$
Since there are 101 states of nature, it is very difficult to work out all possible payoff. Suppose the manufacturer takes action a_1, then he will lose Rs. 100.

$P_{i1} = -100$ for all $i = 0,1,2.....100$
Suppose the manufacturer takes action a_2 and the state of nature is that, there are defective bulbs in the lot then the payoff is given by
$P_{i2} = 100x5 - i\, x10 = 500 - 10\, i$ For all i = 0, 1, 2. .... 100.
Thus his payoffs are

$$P_{i1} = -100 \quad \} \quad \text{for all } i = 0,1,2.....100$$
$$P_{i2} = 500 - 10i \quad \} \quad \text{for all I} = 0, 1, 2 \ldots 100$$

I (i) Maximax or Hurwicz Decision criterion: The maximax or Hurwicz decision criterion is a criterion of super optimism. It is based upon the idea that we do get some favourable or lucky breaks. Since, nature can be good to us, the decision maker should select that state of nature which will yield him the highest payoff for the selected strategy. Thus, the maximax criterion attempts to maximise the maximum gain i.e. maximax chooses the act that is the "best of the best". The procedure involved in the criterion is to look at the various payoffs for each strategy and select the highest amount. Therefore, the maximum of these maximum payoffs is selected. This payoff is referred to as a maximax (maximum of maximums).

i) Example 3 : Consider the following payoff table :

Strategies / States of Nature	S_1	S_2	S_3
N_1	24	17	10
N_2	24	27	20
N_3	24	27	30
Maximum gains	24	27	(30)
Minimum gains	(24)	17	10

For maximax criterion, first of all maximum payoff of each strategy is identified. Maximum payoff for strategy S_1 is 24, for S_2 is 27 and for S_3, it is 30. Next, we must determine the largest payoff of these maximum payoffs which is 30. Hence, based on maximax criterion the best strategy would be S_3.

(ii) Maximin or Wald decision criterion: The maximin criterion is a criterion of pessimism. Wald suggested that the decision maker should always be pessimistic or conservative, resulting in a maximin criterion. This maximin payoff approach means that the decision maker, under continually adverse circumstances, should select the strategy that will give him as large a minimum payoff as possible i.e. the criterion attempts to maximise the minimum gains or it tries to "pick the best of the worst".
Under maximin criterion, minimum payoff of each strategy is selected. In the example given in (i) part, minimum payoffs for S_1 is 24, for S_2 it is 17, and for S_3, the minimum payoff is 10. Once the minimum payoff corresponding to each strategy has been selected, the next step is to select the maximum of these minimum payoffs (maximum of minimums) which is 24 in .our above example. Hence, strategy S_1 is the maximin gain strategy; In essence, the worst state of nature that could happen would give a payoff of 24.

(iii) Minimax Regret Criterion: Suppose that the manager is concerned about how the decision he makes might be viewed in the future after the state of nature is known with certainty. In such situation, the minimax regret criterion can be used. To employ this criterion, one must transform the payoff matrix into a regret matrix by replacing every payoff in a row of the payoff matrix with the difference obtained by subtracting the payoff from the row's maximum payoff. This regret or opportunity loss represents the amount of profit that a person lost because he did hot select the most profitable act. *When the minimax criterion is used, the decision maker expects the worst event to materialise and so he selects the act that will give the minimum of the maximum opportunity losses.*

Example 4:
Given the following pay-off function for each act a_1 and a_2 $Qa_1 = -30 + 50x$
$Qa_2 = -90 + 20x$
(i) What is the break-even value of x?
(ii) If $x = 10$, which is the better act?

(iii) If x = 10, what is the regret of the poor strategy?
(iv) If x =-5 which is the better act?
(v) If x = -5 what is the regret of the poorer strategy?

Solution: Equating Qa_1 and Qa_2

$$-30 + 50x = -90 + 20x$$

or $x = -2$

∴ -2 is the break even point.

(ii) Substituting x = 10 in Qa_1 and Qa_2

Qa_1 =470 Qa_2 =110

Hence a_1 is a better strategy.

(iii) Regret with x = 10 is equal to 470 -110 = 360
This happens when strategy a_2 is adopted.

(iv) for x = -5 Qa_1 =-280
Qa_2 = -190
a_2 is a better strategy.

(v) Regret in (iv) = - 190 – (- 280)
= - 190 + 280
= 90

Example 5:

A food products company is contemplating the introduction of a revolutionary new product with new packaging to replace the existing product at much higher price (S_1) or a moderate change in the composition of the existing product with a new packaging at a small increase in price (S_2) or a small change in the composition of the existing except the word 'New' with a negligible increase in price (S_2). The three possible states of nature of events are (*i*) high increase in sales (*N*), (*ii*) no change in sales (N_2), and (*iii*) decrease in sales(N_3). The marketing department of the company worked out the payoffs in terms of yearly net profits for each of the strategies for these events (expected sales) This is represented in the following table:

State of nature	Pay offs (in Rs.)		
Strategies	N_1	N_2	N_3
S_1	7,00.000	3,00,000	1 ,50,000
S_2	5,00,000	4,50,000	0
S_3	3,00,000 ,	3,00.000	3,00,000

Which strategy should the executive concerned choose on the basis of
(a) Maximin Criterion,
(b) Maximax Criterion,
(c) Minimax Regret Criterion,
(d) Laplace Criterion ?

Solution. a) *Maximin Criterion.* When this criterion is adopted that course of action is selected which maximises the minimum payoffs.

Strategy	**Minimum Payoffs Rs**
S_1	1,50,000
S_2	0
S_3	3.00.000

The executive would choose strategy S_2

(b) Maximax Criterion. In this criterion we select that strategy which gives the maximum payoffs.

Strategy	Maximum Payoffs
S_1	7,00,000
S_2	5.00,000
S_3	3,00,000

The executive should choose S_1

(c) *Minimax Regret Criterion.* When this criterion is adopted three steps are necessary :

(i) to determine the opportunity loss for each strategy by subtracting from maximum pay -off of a state of nature, the payoffs of all the strategies.

(ii) to determine the maximum opportunity loss for each strategy.

(iii) to select the strategy which minimises the maximum of the loss.

OPPORTUNITY LOSS TABLE

	N_1	N_2	N_3
S_1	7,00,000 - 7,00.000 = 0	4,50,000 - 3,00,000 = 1,50,000	3.00,000 - 1,50,000 = 1.50,000
S_2	7,00,000 - 5,00,000 = 2,00,000	4,50,000 - 4,50,000 = 0	3,00,000 - 0 = 3,00,000
S_3	7,00,000 - 3.00,000 = 4,00,000	4,50,000 - 3,00.000 = 1,50,000	3,00.000 - 3.00.000 = 0

Maximum opportunity loss

S_1	= 1,50,000
S_2	= 3,00,000
S_3	= 4,00,000

The executive should choose strategy $S_1$for it minimises the maximum opportunity loss.

Laplace criterion. This involves three things:

(i) assigning equal opportunity to each state of nature.

(ii) calculating the expected monetary value E_1MV.

(iii) selecting that strategy whose E_1MV is maximum.

E_1MV TABLE

$S_1 : \frac{1}{3}(7,00,000 + 3,00,000 + 1,50,000) = \frac{11,50,000}{3} = 3,83,333.33$

$S_2 : \frac{1}{3}(5,00.000 + 4,50,000 + 0) = \frac{9,50,000}{3} = 3,16,666.67$

$S_3 : \frac{1}{3}(3,00.000 + 3,00,000 + 3,00,000) = \frac{9,00,000}{3} = 3,00,000$

Since the E_2MV is highest for strategy 1, hence the executive should select strategy S_1.

Example 6:

A management is faced with the problem of choosing one of three products for manufacturing. The potential demand for each product may turn out to be good, moderate or poor. The probabilities for each of the state of nature were estimated as follows :

Nature of Demand

Product	Good	Moderate	poor
X	0.70	0.20	0.10
Y	0.50	0.30	0.20
Z	0.40	0.50	0.10

The estimated profit or loss under the three states may be taken as

X	30,000	20,000	10,000
Y	60,000	30,000	20,000
Z	40,000	10,000	-15,000(loss)

Prepare the expected value table and advise the management about the choice of product.

Computation of EV for various Acts

States of Nature	Expected payoff (in Rs. '0000) for various acts								
	X			Y			Z		
	x_{1j}	p_{1j}	$x_{1i}p_{1j}$	x_{2j}	p_{2j}	$x_{2i}p_{2j}$	x_{3j}	p_{3j}	$x_{3i}p_{3j}$
Good	30	0.7	21	60	0.5	30	40	0.4	16
Moderate	20	0.2	04	30	0.3	09	10	0.5	05
Poor	10	0.1	01	20	0.2	04	-15	0.1	-1.5
Expected Monetary value			26			43			19.5

Since the expected value is highest for second course of action, the management is advised to produce Y.

Example 7:

A TV dealer finds that the procurement cost of a TV is Rs. 20 and the cost of a unit shortage is Rs. 50. For one particular model of TV the probability distribution of weekly sales is as follows:

Weekly Sales	0	1	2	3	4	5	6
Probability	0.10	0.10	0.20	0.20	020	0.15	0.05

How many units per week should the dealer buy? Also find E.V.P.I.
Solution : The cost matrix is constructed below. Also derived from it are the expected costs for the various strategies.

Demand	*Prob.*	*Strategies (buy so many)* 0	1	2	3	4	5	6
0	0.10	0	20	40	60	80	100	120
1	0.10	50	20	40	60	80	100	120
2	0.20	100	70	40	60	80	100	120
3	0.20	150	120	90	60	80	100	120
4	0.20	200	170	140	110	80	100	120
5	0.15	250	220	190	160	130	100	120
6	0.05	300	270	240	210	180	150	120
Expected		147.5	122.5	102.5	92.5	92.5	102.5	120

Thus either buy 3 or 4.

Example 8: Following are ihe records of demand of an item for the past 300 days.

θ = Demand in units	*No. of days*	*Prob (θ)*
10,000	18	0.06
11,000	90	0.30
12,000	120	0.40
13,000 .	60	0.20
14,000	12	0.04
	300	1.00

(*i*) What is the expected demand?
(*ii*) It costs Rs. 15 to make an item which sells for Rs. 20 normally but at the end of the day surplus has to be disposed at Rs. 10 per item. What is the, optimum output?

Solution: Expected demand = 10 x 0.06 = 0.60 .
= 11 x 0.30 = 3.30
= 12 x 0.40 = 4.80
= 13 x 0.20 = 2.60
= 14 x 0.04 = 0.56
11.86

Expected demand = 11,860 (Answer)
The daily pay-off matrix is constructed below:

		Strategies				
P(*θ*)	states of nature	10	11	12	13	14
0.06	10	50	45	40	35	30
0.30	11	50	55	50	45	40
0.40	12	50	55	60	55	50
0.20	13	50	55	60	65	60
0.04	14	50	55	60	65	70
Expected Pay off		50	54.4	55.8	53.2	48.6

Demand of 12,000 is the optimum wth an expected payoff of 55.8x 1000 = 55,800 rupees.;

Example 9: The sales manager of Beta Co. is highly experienced in the fad market. He is sure that the sales of JUMBO (during the period it has special appeal) will not be less than 25,000 units. Plant capacity limits total production to a maximum of 80000 units during JUMBO's, brief life. According to the sales manager, there are 2 chances in 5 for a sales volume of 50,000 units. The probability that it will be more than 50,000 is four times the probability that it will be less than 50.000. If sales exceed 50,000 units volumes of 60,000 and 80,000 are equally likely. A 70,000 unit volume is 4 times as likely as either. It costs Rs. 30 to produce a unit of JUMBO whereas its selling price is estimated at Rs. 50 per unit. Initial investment is estimated at Rs. 8,00,000. Should the venture of production be undertaken?

Solution:

Prob. (50,000) = $\frac{2}{5}$ = 0.40

Prob. (less than or more than 50,000) = 0.60

Prob. (less than 50,000): Prob. (More than 50,000): 1:4

Thus Prob. (less than 50,000) = 0.60 x $\frac{1}{1+4}$ = 0.12

Prob. (More than 50,000) = 0.60 x $\frac{4}{1+4}$ = 0.48

Prob. (60,000): Prob. (70,000): Prob. (80,000):: 1:4:1

Thus Prob (60,000) = 0.48 x $\frac{1}{1+4+1}$ = 0.08

∴ Prob (80,000) = 0.08

and Prob 70,000 = $\frac{4}{1+4+1}$ x 0.48 = 0.32

Now we have complete probability distribution of sales except that for sales less than 50,000 we have a summarised probability of 0.12. The payoff matrix is compiled below. We used 25,000 in place of less than 50,000 as the worst contingency.

Pay-off Table

(Rs.in'000)

Demand	Prob	< 50,000 (take it 25,000)	50,000	60,000	70,000	80,000
< 50,000 (worst 25,000)	0.12	500	-250	-550	-850	-1150
50,000	0.40	500	1,000	700	400	100
60,000	0.08	500	1,000	1200	900	600
70,000	0.32	500	1,000	1200	1400	1100
80,000	0.08	500	1,000	1200	1400	1600
Expected Payoff		500	850	790		

Expected Payoff is, thus, Rs. 850,000. This exceeds 800,000 the initial investment; therefore, the venture ought to be undertaken.

2 Decision Trees

A Decision Tree diagram is a graphical representation of various alternatives and the sequence of events in a decision problem. In constructing a decision tree, there are certain conventions to be followed. The tree is constructed starting from left and moving towards right. The square box denotes a decision point at which the available strategies are considered. The circle O represents the chance node or event, the various states of nature or outcomes emanate from this chance event. At each decision or chance node, there can be one or more branches represented by a straight line. In figure below, there are two branches at the decision node, each branch represents a strategy. Similarly, at each of the two chance nodes, there are three branches. Any branch that is not followed by another decision or chance node is called a *terminal branch.*

Let us illustrate the decision tree with an example.

Example 10: A manufacturer of toys is interested to know whether he should launch a deluxe model or a popular model of a toy. If the deluxe model is launched, the probabilities that the market will be good, fair or poor are given by 0.3,0.4 and 0.3 respectively with payoffs Rs. 1,40,000, Rs. 70,000 and Rs. (-10,000). If the popular model is introduced, the corresponding probabilities are given by 0.4, 0.3 and 0.3 with respective payoffs Rs 1,50,000, Rs. 80,000 and Rs. (-15,000). The problem is to decide which model should be launched. The decision tree for the given problem is drawn below:

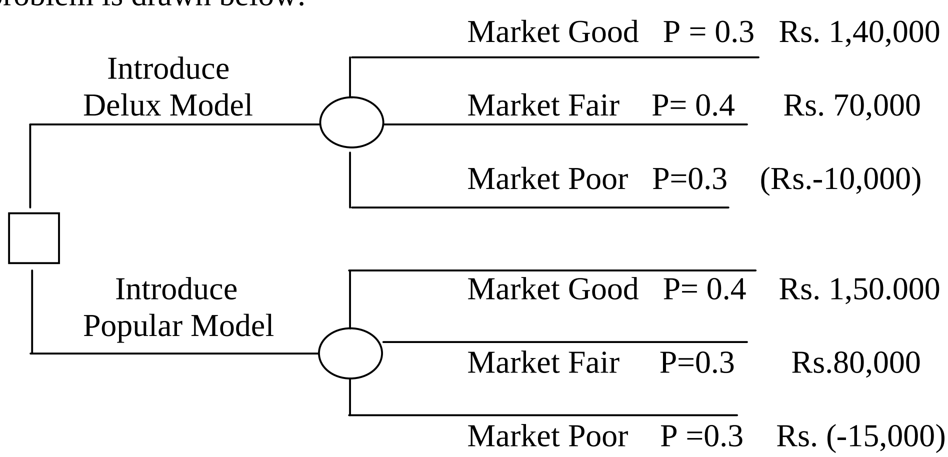


Fig. 1

Thus the decision tree shows that structure of the decision problem. To carry out the decision tree analysis, conditional payoffs are estimated for every combination of actions and events (i.e., for every path through the tree). These payoffs can be either positive or negative. Also the probabilities for each event must be assessed by the decision maker. To analyse the decision tree begin at the end of the tree and work backward. For each set of events branches, the *Expected Monetary Value (EMV)* is calculated and for each set of decision branches, the one with the highest EMV is selected. This EMV now represents the EMV for the decision point from which this set of event branches emanates. Following the same procedure we move back on the decision tree to the next decision point. This technique of analysing tree is called the roll-back technique. Thus, there are two rules concerning roll-back technique:

(i) If branches emanate from a circle, the total expected pay off may be calculated by summing the expected value of all the branches.

(ii) If branches emanate from a square, we calculate the total expected benefit for each branch emanating from that square and let the total expected pay-off be equal to the value of the branch with the highest expected benefit.

Let us analyse the tree given above by the roll back technique. Here point 1 is decision point and C & D are the chance nodes.

EMV (at C) = .3 x Rs. 1,40,000 + .4 x Rs. 70,000 + .3 x (- Rs.10,000)
= Rs. 42,000+ Rs. 28,000 - Rs. 3,000 = Rs. 67,000

EMV (at D) = .4 x Rs. 1,50,000 +. 3 x Rs. 80,000 + .3 x (- Rs. 15,000)
= Rs. 60,000 + Rs.24,000-Rs. 4,500 = Rs. 79,500

The decision at point 1 is to introduce the popular model since it results in the highest EMV

Example 11: Mr. X is trying to decide whether to travel to Sri Lanka from Delhi to negotiate the sale of a shipment of china novelties. He holds the novelties stock and is fairly confident, but by no means sure that if he makes the trip, hewill sell the novelties at price that will give him profit of Rs. 30,000. He puts the probability of obtaining the order at 0.6. If he does not make the trip, he will certainly not get the order.

If the novelties are not sold in Sri Lanka there is an Indian customer who will certainly buy them at a price that leaves him a profit of Rs. 15,000 and his offer will be open at least till Mr.X returns from Sri Lanka. Mr. X estimates the expenses of trip to Sri Lanka at Rs. 2,500. He is however, concerned that his absence, even for only three days, may lead to production inefficiencies in the factory. These could cause him to miss the deadline on another contract, with the consequence that a late penalty of Rs. 10,000 will be invoked. Mr. X assesses the probability missing the deadline under these circumstances at 0.4. Further, he believes that in his absence there will be a lower standard of house-keeping in the factory, and the raw material and labour costs on the other contract will rise by about Rs. 2,000 above the budgeted figure.

Draw an appropriate decision tree for Mr. X's problem and using EMV as the appropriate criterion for decision, find the appropriate initial decision.

Solution:

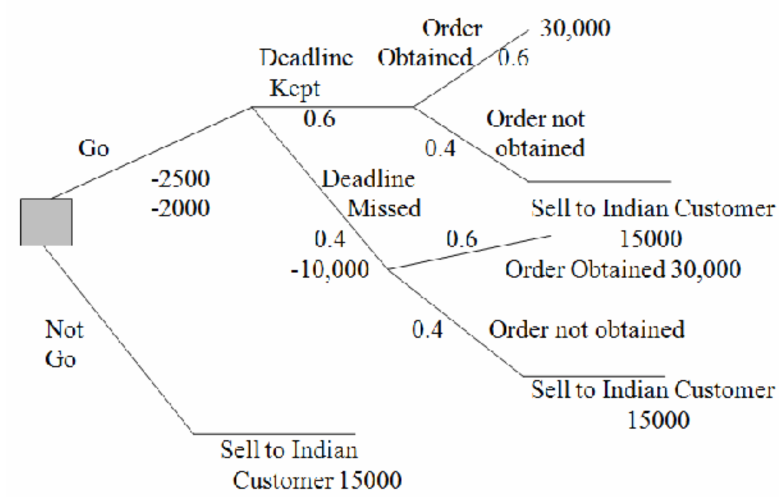


Fig. 2

The decision tree is drawn in the above figure. Calculations of EMV proceed from right to left.

Go	*Not Go*
30,000 x 0.6 = 18,000	15,000 X 1 = 15,000

15,000 x 0.4 = 6000 (where 1 is the probability)
24000
24,000 x 0.6 = 14,400

plus
24,000 x 0.4 = 9600
minus
10,000 x 0.4 = 4000 Hence Mr. X should proceed to Sri Lanka
20,000
minus 4,500
15,500

Steps in Decision Tree Analysis

In a decision analysis, the decision maker has usually to proceed through the following six steps:

1. Define the problem in structured terms. First of all. the factors relevant to the solution should be determined. Then probability distribiutions that are appropriate to describe future behaviour of those factors estimated. The financial data concerning conditional outcomes is collected.

2. Model the decision process. A decision tree that illustrates all alternatives in the problem is constructed. The entire decision process is presented schematically and in an organised step-by-step fashion.

3. Apply the appropriate probability values and financial data. Toeach of the branches and sub-branches of the decision tree the appropriate probability values and financial data are applied. This would enable to distinguish between the probability value and conditional monetary value associated with
each outcome.

4. "Solve" the decision tree. Using the methodology mentioned above proceed to locate that particular branch of the tree that has the largest expected value or that maximises the decision criterion.

5. Perform sensitivity analysis. Determine how the solution reacts in changes in inputs. Changing probability values and conditional financial values allows the decision maker to test both the magnitude and the direction of the reaction.

6. List the underlying assumptions. The accounting cost finding and other assumptions used to arrive at a solution should be explained. This would also enable others to know what risks they are taking when they use the results of your decision tree analysis. The limits under which the results obtained will be valid and not valid should be clearly specified.

Advantages of Decision Tree Approach

The decision tree analysis as a tool of decision-making is important because of the following:

1. Decision trees are of great help in complicated kinds of decision problems. However, in a simple problem there is no advantage of constructing a decision tree.
2. It structures the decision process making managers, approach decision making in an orderly sequential fashion.
3. It requires the decision-maker to examine all possible outcomes desirable and undesirable.
4. It communicates clearly the decision-making process to others.
5. It allows a group to discuss alternatives by focusing on each financial figure, probability value and underlying assumption one at a time. Thus, group can move in orderly steps towards a consensus decision instead of debating a decision in its entirety.
6. It can be used with a computer so that many different sets of consumptions can be simulated and their effects on the final outcomes observed.

Example 12: M/s J. Bloggs & Co. is currently working with a process, which, after paying for materials, labour etc. brings a profit of Rs. 10,000. The following alternatives are made available to the company.

(i) The company can conduct research(R_1) which is expected to cost Rs. 10,000 and having 90% probability of success, the company gets a gross income of Rs. 25,000. (ii) The company can conduct research

(R_2) expected to cost Rs. 5,000 and having a probability *of* 60% success. If successful, the gross income will be Rs. 25,000.

(iii) The company can pay Rs. 6,000 as royalty of a new process which will bring a gross incomeof 20,000.

(iv) The company continues the current process.

Because of limited resources, it is assumed, that only one of the two types of research can be carried out at a lime. Which alternative should be accepted by the company?

Solution: The decision tree which represents the possible courses of action is depicted in figure 4. Point 1 is a 'decision box' located 'now' on the time scale. The four possibilities arising here are shown. Upon failure of a particular research, say, R_1 , there are again 3 original alternatives to be sorted out, that of R_1 being excluded. lf R_2 fails after failure of R_1 , the company is left with only two choices, *i.e.*, either to pay royalty or continue the existing process.

Branch Current

Net Return = Rs. 10,000

Branch Licence

Net Return = 20,000-6,000 =Rs. 14,000.

Branch R_1 First

Value at Point 3

Current Rs. 10,000

Licence Rs. 14,000

Value at point C: Expected gross profit

= 25,000 x 0.6 + 14,000 x 0.4

= 20,600.

Value at Point 2

Licence Rs. 14,000

Current Rs. 10,000

R_2 Rs. 20,600- Rs. 5,000 = Rs.. 15,600

Value at Point A Rs. 25,000 x 0.9 + 0.1 x 15,600 = Rs. 24,060

Value at Branch R_1 Rs. 24,060- Rs. 10,000 = Rs. 14,060

Branch R_2First

Value at point 5

Current Rs. 10,000

Licence Rs. 14,000

Value at Point at D: Expected gross profit

= 0.9 x 25,000 + 0.1 X 14,000 = Rs. 23.900

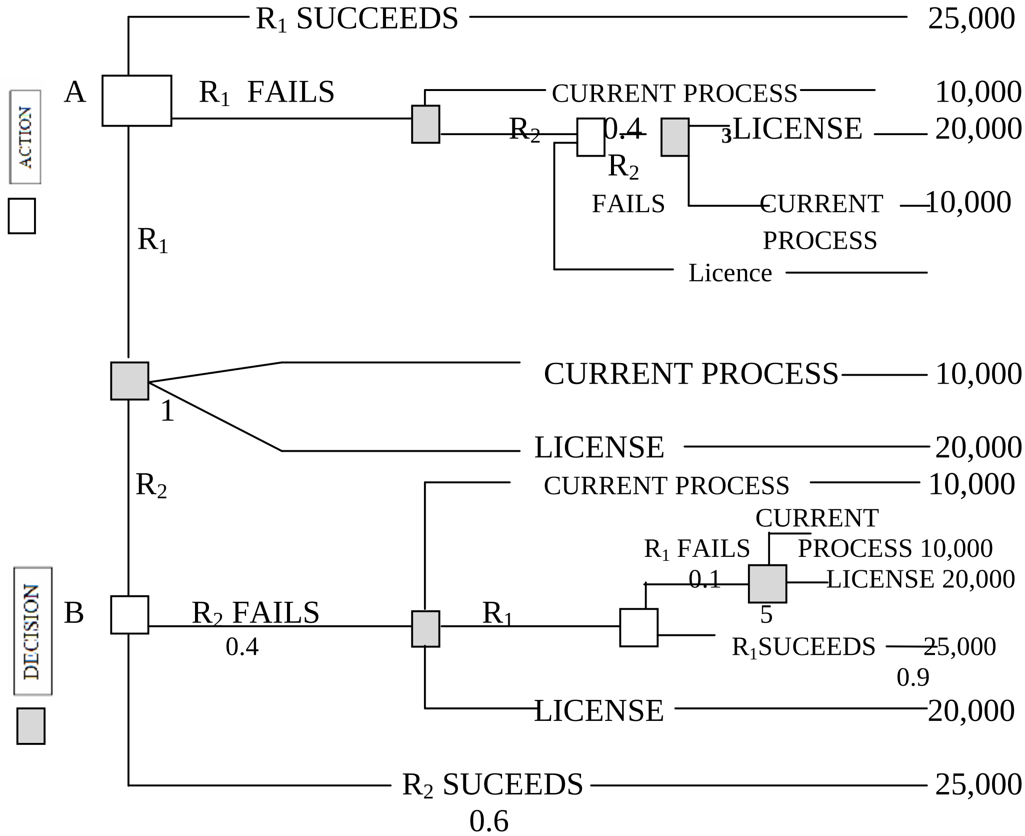


Fig 3

Value at point 4

Current	Rs. 10,000
Licence	Rs. 14,000
R_1	= Rs. 23,900 - Rs. 10,00= Rs. 13,900
Value at Point B	= 0.4 x 14,000 +0.6x25,000=Rs.20,600
Value of Branch R_2	= Rs. 20,600 - Rs. 5,000 - Rs. 15,600

Thus R_2. followed by licence upon former's failure is the best course of action.

EXERCISES

1. The research department of consumer products division has recommended the marketing department to launch a soap with 3 different perfumes. The marketing manager has to decide the type of perfume to launch under the following estimated payoff for various levels of sales.

Type of perfume	Estimated sales (units)		
	25,000	15.000	10,000
I	200	25	20
II	30	20	10
III	50	30	5

Find the best decision using (i)Maximax , (ii) Maximin , (iii) Minimax Regret and (iv) Laplace criteria.

Ans: (i) I, (ii) I, (iii) I, (iv) I

2. A consumer product company is examining the introduction of a new product with new packaging or replace the existing product at much higher price (S_1) or moderate change in the composition of the existing product with new packaging at a small increase in price (S_2) or a small change in the composition of the existing product except the word 'new' with the negligible increase in price (S_3). The three possible states of nature are E_1 : increase in sales, E_2 :no change in sales, E_3 : decrease in sales. The marketing department has worked out the following payoffs in terms of profits. Which strategy should be considered under (i) Minimax Regret (ii) optimistic (iii) equi-probability conditions?

States of Nature	Strategies		
	S_1	S_2	S_3
E_1	30.000	40,000	25,000
E_2	50,000	45,000	10,000
E_3	40.000	40,000	40,000

Ans: (i) S_2, (ii) S_1, (iii) S_2

3. Construct a payoff matrix for the following situations and find the best decision using (i) Maximin, (ii) Maximax, (iii) Laplace criteria.

Product		
	Fixed Cost Rs.	Variable cost unit in Rs.
X	250	12
Y	350	10
Z	500	5

The likely demand (units) of products
Poor demand 300
Moderate demand 700
High demand 1000
Selling price of each product is Rs. 25.
Ans: (i) Z, (ii) Z, (iii) Z

4. The management of a company is faced with the problem of choosing one of the three products A, B, C for manufacturing. The demand for these products can be good, moderate or poor. The probabilities of each state of nature are estimated as follows :

Product	Nature of demand		
	Good	Moderate	Poor
A	0.7	0.2	0.1
B	0.4	0.5	0.1
C	0.5	0.3	0.2

The estimated profit/loss under the three states of nature are given as follows:

Product	Profit in 10,000 Rs.		
	Good	Moderate	Poor
A	30	20	10
B	40	10	-15
C	60	30	20

Using EMV criterion, advice the management the optimum decision.

Ans: Select product C

5. You are given, the following payoff table for three acts A_1, A_2, A_3 and the states of nature S_1 , S_2 , S_3.

States of nature	A_1	A_2	A_3
E_1	25	-10	-125
E_2	400	440	400
E_3	650	740	750

The probabilities of the states of nature are 0.1, 0.7, 0.2 respectively. Find the optimum decision using EMV criterion.

Ans: Optimum decision at A_2

6. An investor is given the following investment alternatives and percentage rate of return.

State of Nature	Course	of	Action
	Regular Share	Risky Share	Property
Good	7%	-10%	-12%
Better	10%	12%	18%
Best	15%	18%	30%

Over past 100 days, 50 days have market in better condition and for 20 days market conditions are best. Using this information state optimum investment strategy for the investment. Use EMV criterion.

Ans: Invest in property.

7. A retailer purchases grapes every morning for Rs. 50/- a case nd sells for Rs. 80/- a case. Unsold cases at the end of the day are donated to old age homes. Past sales have ranged from 15 to 18 cases per day. The following are the details of demand and probability.

Cases demanded :	15	16	17	18
Probability :	0.1	0.2	0.4	0.3

The retailer wants to know how many cases should he buy to maximize his profit. Use EMV criterion.

Ans: 17 cases

8. Following is the pay-off matrix corresponding to four states of nature S_1, S_2, S_3, S_4 and four courses of action A_1, A_2 , A_3, A_4.

State of Nature	Course of Action				Probability of State
	A_1	A_2	A_3	A_4	
s_1	50	400	-50	0	0.15
S_2	300	0	200	300	0.45
S_3	-150	100	0	300	0.25
s_4	50	0	100	0	0.15

(i) Calculate expected pay off and find best course of action using EMV.

(ii) Calculate EOL for each course of action hence find best action using EOL.

Ans: (i) A_4, (ii) A_4

9. Following is pay-off table corresponding to four acts A_1, A_2 ,A_3 ,$A_4$and four states of nature E_1, E_2, E_3, E_4 with the probability of the events of this table $P(E_1) = 0.20$, $P(E_2) = 0.15$, $P(E_3) = 0.40$, $P(E_4) = 0.25$. Calculate the expected pay off and expected opportunity loss and suggest best course of action.

Acts	Events			
	E_1	E_2	E_3	E_4
A_1	40	50	200	0
A_2	300	200	0	100
A_3	50	100	40	200
A_4	300	0	100	50

Ans: A_2

10. The florist shop promises its customers delivery within three hours on all orders. All flowers are purchased the previous day and delivered to florist by 8.00 a.m. the next morning. Demand distribution of roses is as follows.

Dozens of roses	7	8	9	10
Probability	0.1	0.2	0.4	0.3

Florist purchases roses for Rs. 10.00 per dozen and sells them at Rs. 30 per dozen. All unsold roses are distributed in a local hospital free of cost. How many dozens of roses should the florist order each evening?

Ans: 9 dozens of roses

11. The probability of the demand for tourist cars for hiring on any day at ABC travels and tours are as under:

No. of cars demanded	0	1	2	3	4
Probability	0.1	0.2	0.3	0.2	0.2

The cars have a fixed cost of Rs. 900 each day to keep and the daily hire charges are Rs. 2000. (i) If the tour owner company owns 4 cars what is its daily expectation? (ii) Use EMV and EOL criterion to suggest how many cars the company should keep.

Ans: (i) Rs. 800/-, (ii) 2 cars.

12. Unique home appliances finds that the cost of holding a cooking ware in stock for a month is Rs. 200. Customer who cannot obtain a cooking ware immediately tends to go to other dealers and he estimates that for every customer who cannot get immediate delivery he loses an average of Rs. 500. The probabilities of a demand of 0, 1, 2, 3, 4, 5 cooking ware in a month are 0.05, 0.1, 0.2, 0.3, 0.2, 0.15 respectively. Determine the optimum stock level of cooking wares. Using EMV criterion.

Ans: 4 Cooking wares

13. A person has the choice of running a hot snack stall or an ice cream stall at a certain holiday hotel during coming season. If weather is cool and rainy he can expect to make a profit of Rs, 1,50,000 and if it is warm he can expect to make profit of Rs. 40,000 by running hot snack stall. If he runs ice-cream and cold drink shop he can make a profit of Rs. 1,60,000 if the weather is warm and only Rs. 30,000 if whether is cool and rainy. The odds in favour of warm weather are 2 : 3 and that of having cool and rainy are 3 : 2. Use EMV to find Best f Action:

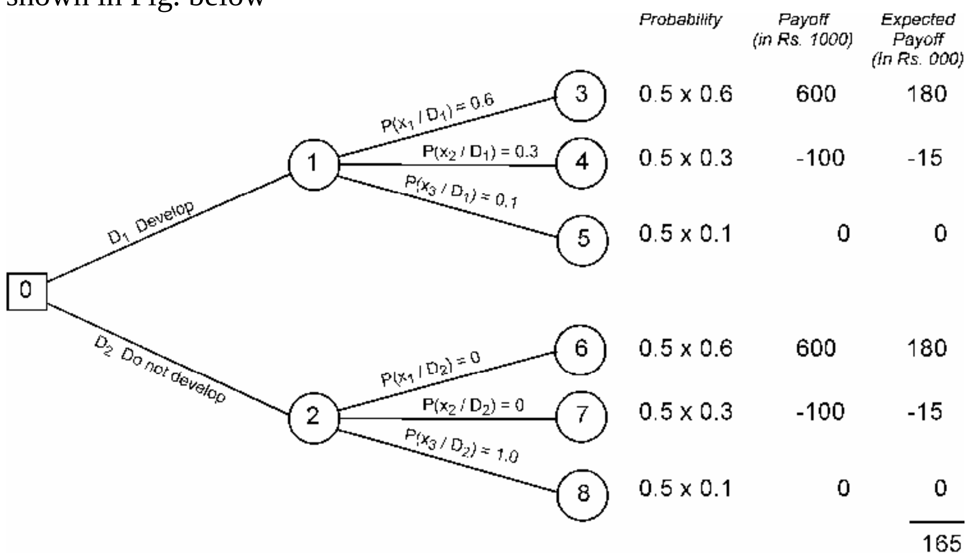
Ans: Hot snack stall.

14. You are given the following estimates concerning a Research and Development programme:

Decision Di	Probability of decision Di given research R P(Di \ R)	Outcome number	Probability of outcome Xi given Di P(Xi \ Di)	Payoff value of outcome, Xi (Rs '000)
Develop	0.5	1	0.6	600
		2	0.3	-100
		3	0.1	0
Do not develop	0.5	1	0.0	600
		2	0.0	-100
		3	1.0	0

Construct and evaluate the decision tree diagram for the above data. Show your working for evaluation.

The decision tree of the given problem along with necessary calculation is shown in Fig. below



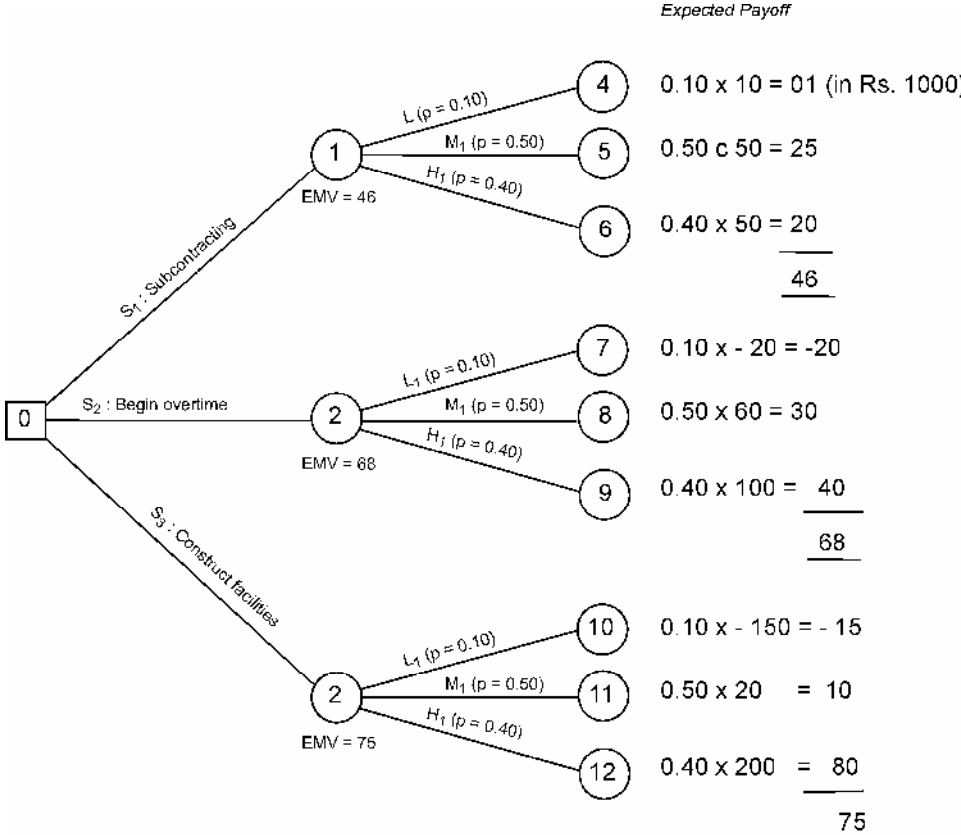
Decision tree

15. A glass factory specializing in crystal is developing a substantial backlog and the firm' management is considering tree courses of sub-contracting (S1), being overtime production (S2), and construct new facilities (S3). The correct choice depends largely upon future demand which may be low, medium, or high. By consensus, management ranks the respective probability as 0.10, 0.50 and 0.40. A cost analysis reveals effect upon the profits that is shown in the table below:

Show this decision situation in the form of a decision tree and indicate the most preferred decision and corresponding expected value.

A decision tree which represents possible courses of action and nature are shown in the Fig. In order to analyses the tree we start working backward from the end branches.

He most preferred decision at the decision node 0 is found by calculating expected value of each decision branch and selecting the path (course of action) with high value.



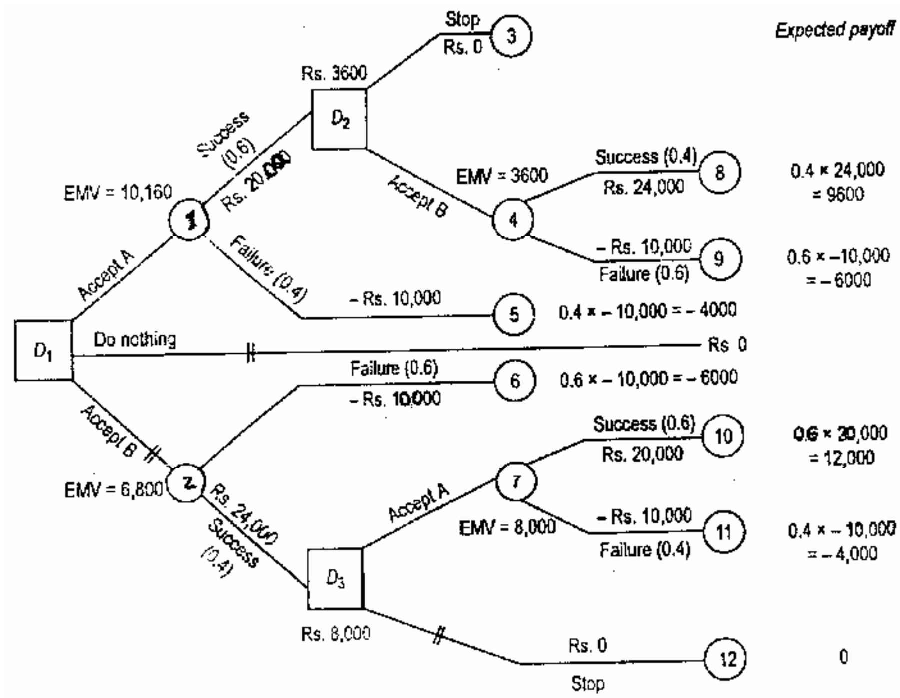
Decision tree

	Decision Point	Outcome	Probability	Conditional value (Rs.)	Expected value
D_3	(i) Accept A	Success Failure	0.6	20,000 -10,000	12,000 -4000 Rs. 8,000
	(ii) Stop	-	-	-	0
D_2	(i) Accept B	Success Failure	0.4 0.6	24,000 -10,000	9,600 - 6,000 Rs. 3,600
	(ii) Stop	-	-	-	0
D_1	(i) Accept A	Success Failure	0.6 0.4	20,000+3,600 -10,000	14,160 - 4,000 Rs. 10,160
	(ii) Accept B	Success Failure	0.4 0.6	24,000+8,000 -10000	12,800 - 6,000 Rs. 6,8000
	(ii) Do nothing	-	-	-	Rs. 0

Since node 3 has the highest EMV, therefore, the decision at node 0 will be to choose the course of action S3, i.e., construct new facilities.

16. A person wants to invest in two independent investment schemes: A and B, but he can undertake only at a time due to certain constraints. He can choose A first and then stop, or if A is not successful then B or vice-versa. The probability of success of A is 0.6, while for B it is 0.4. The investment in both the schemes requires an initial capital outlay of Rs. 10,000 and both return nothing if the venture is unsuccessful. Successful completion of A will return Rs. 20,000 (over cost) and successful completion of B will return Rs. 24,000 (over cost). Draw decision tree and determine the best strategy.

The decision tree corresponding to the given information is depicted in the Fig.



Decision tree

Since EMV = Rs 10,160 at node 1 is the highest the best strategy at node D1 is to accept course of Action A first and if A is successful then Accept B.

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