M.SC. {MATHEMATIC} (PART-II)
ALGEBRA – II & FIELD THEORY (REV)
PAPER – I (JAN- 2020)

(3 Hours) **[Total Marks:80**

Instructions:
- Attempt **any two** questions from **each section**
- **All** questions carry **equal marks**.
- Answer to **section I** and **II** should be written on the **same answer book**

SECTION I (Attempt any two questions)

1. (a) Prove that subgroup of solvable group is solvable
   (b) Prove that a group of order 42 cannot be simple
2. (a) Prove that homomorphic image of a nilpotent group is nilpotent
   (b) State and prove Jordan-Holder theorem
3. (a) (i) State (without proof) the Hilbert basis theorem. Define the terms: Noetherian ring, Noetherian module.
   (ii) Prove that any Artinian ring has finitely many maximal ideals.
   (b) Show that a submodule of a free module over a PID is free
4. (a) Define Module and submodule. Show that (i) $\mathbb{Z}$-modules are same as abelian groups and (ii) $\mathbb{Z}$-submodules are same as subgroups.
   (b) (i) Prove that the matrix $\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$ is nilpotent. Reduce it to the triangular form
   (ii) Find Jordan canonical form of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{pmatrix}$

SECTION II (Attempt any two questions)

5. (a) Define algebraic extension. Prove that if F is a field of characteristic 0 and a, b are algebraic over F then there is an element c in $F(a, b)$ such that $F(a, b) = F(c)$
   (b) Define a splitting field. Find splitting field for $f(x) = x^4 - x^2 - 2$ over $\mathbb{Q}$
6. (a) If K/F and L/K be a finite separable extensions , prove that L/F is separable
   (b) Prove that the multiplicative group of a finite field is cyclic.
7. (a) Prove that the field of complex number is algebraically closed
   (b) Find the Galois group of $x^3 - 2$ over a field $\mathbb{Q}$

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8. (a) Prove that it is impossible by ruler and compass to trisect the angle ${\frac{\pi}{3}}^c$

   (b) Define constructible number. If a and b are constructible numbers then prove that ab is constructible.

[Total marks: 80]

Instructions:

1) Attempt any two questions from each section.
2) All questions carry equal marks.
3) Answer to Section I and section II should be written in the same answer book.

SECTION-I (Attempt any two questions)

Q.1]

A) $f: A \to \mathbb{R}$ is non-negative and $\int_A f = 0$, where A is rectangle, then show that

$\{x \in A, f(x) \neq 0\}$ has measure zero. [10]

B) State and prove first mean value theorem for Riemann integral. [10]

Q.2]

A) Let $\{A_i\,; i \in I\}$ be collection of σ –Algebra. Show that the $\bigcap_{i\in I} A_i$ is a σ –Algebra, but $\bigcup_{i\in I} A_i$ is not in general. [10]

B) Show that f is measurable if and only if $\{x \in E: f(x) \leq a\}$ is measurable for any $a \in \mathbb{R}$. [10]

Q.3]

A)Show that lebesgue integration of simple function is independent of points representation.[10]

B) Define $f(x) = \frac{1}{x}, \;\; if \;\; 0 < x \leq 1$

$= 9 \;\; if \;\;\;\; x = 0$

Show that f is not Lebesgue integrable on [0, 1] . [10]

Q.4]

A) State and prove Fatou's lemma. [10]

B) Verify Bounded convergence theorem for the sequence of functions

$f_n(x) = \frac{1}{(1+\frac{x}{n})^n}\,, 0 \leq x \leq 1, n \in N.$ [10]

SECTION-II (Attempt any two questions)

Q.5]

A) Find the Fourier series expansion of the function $f(x) = x^2$ where $-\pi \leq x \leq \pi$. Evaluate series at $x = \pi$ and find $\sum_{n=1}^{\infty} \frac{1}{n^2}$. [06]

B) Show that $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(\theta)d\theta = 1$ where D_N is N^{th} Dirichlet's kernel. [08]

C) Find complex from of Fourier series of $g(\theta) = \theta$; $-\pi \leq \theta \leq \pi$. [06]

Q.6]

A) Show that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm. [08]

B) Let H be Hilbert space. Show that for any $x, y \in H$,

$4\langle x,y\rangle\ ||x + y||^2 - ||x - y||^2 + i||x + iy||^2 - i||x - iy||^2$. [06]

C) Let $H = L^2[-\pi,\pi]$ and $f_n(t) = e^{int}$ for $n = 0, \pm1, \pm2, ...$ and $t \in [-\pi,\pi]$ then prove that $\{ f_n(t) \mid n = 0, \pm1, \pm2, ... \}$ is an orthonormal basis for $L^2[-\pi,\pi]$. [06]

Q.7]

A) Prove that if $\sum_{n=0}^{\infty} c_n$ is Cesaro summable to σ, then it is Able summable to σ. [10]

B) Solve Dirichlet problem on unit disc defined by $D = \{(r,\theta)/\ 0 \leq r \leq 1,\ 0 \leq \theta < 2\pi\}$. whose boundary is unit circle $C = \{(r,\theta)/r = 1, 0 \leq \theta \leq 2\pi\}$ subject to boundary condition $u = sin\theta$ on C. [10]

Q.8]

A) Prove that the space $L^2(\mathbb{R}^d)$ is complete in its metric. [10]

B) State and prove Riez-Fischer theorm. [10]

Paper III- Differential Geometry and Functional Analysis (Rev)

Q. P. Code : 40988

[Marks: 80]

N.B. 1) All questions carry equal marks.
2) Solve any **Two** questions from section A.
3) Solve any **Two** questions from section B.

Section A

1. (a) (i) Find the equation of line passing through point (x_1, x_2, x_3) and normal to the plane $ax + by + cz = d$. (5)

(ii) Show that the 3×3 rotation matrices are the elements of special orthogonal group SO_3. (5)

(b) (i) Let A be $n \times n$ real matrix then show that matrix A is orthogonal if and only if $(AX \cdot AY) = (X \cdot Y)$ for all column vectors X and Y. (5)

(ii) Let m be an isometry of the plane. Show that $m = t_v \rho_\theta$ for a uniquely determined vectors v and angle θ where t_v denotes translation by v and ρ_θ denotes rotation by an angle θ. (5)

2. (a) (i) Show that any reparametrization of a regular curve is regular. (5)

(ii) Show that if a curve $\gamma : (a, b) \to \mathbb{R}^2$ has constant curvature $\kappa(s)$ then its signed curvature is constant. (5)

(b) (i) Let $\gamma : I \to I\!R^3$ be a parametrized regular curve. Show that $|\gamma(t)|$ is nonzero constant if and only if $\gamma(t)$ is orthogonal to $\dfrac{d\gamma}{dt}$ for all $t \in I$. (5)

(ii) The parametric equation of Cissoid of Diocles is given by $\gamma(t) = (\dfrac{2at^2}{1+t^2}, \dfrac{2at^3}{1+t^2})$, $t \in I\!R$. Show that $\lim_{t\to\infty} \dot{\gamma} = (0, 2a)$. (5)

3. (a) (i) If $f : U \longrightarrow I\!R$ is a differentiable function in an open set U of $I\!R^2$ then show that the subset of $I\!R^3$ given by $(x, y, f(x, y))$ for $(x, y) \in U$ is a regular surface. (5)

(ii) Let U and $\tilde{U}$ be open subsets of $I\!R^2$ and $\sigma : U \to I\!R^3$ be a regular surface patch. Also let $\Phi : \tilde{U} \to U$ be a bijective smooth map with smooth inverse map $\Phi^{-1} : U \to \tilde{U}$. Then show that $\tilde{\sigma} = \sigma o \Phi : \tilde{U} \to I\!R^3$ be a regular surface patch. (5)

(b) (i) Define orientable surface. The surface S be defined by a smooth function $f(x, y, z) = 0$ such that f_x, f_y and f_z do not vanish simultanuously at any point of S. Show that the vector $\nabla f = (f_x, f_y, f_z)$ is perpendicular to the tangent plane at every point of S. Is S is orientable? Justify. (5)

(ii) Let $f(x, y, z) = (x + y + z - 1)^2$. Locate the critical points and critical values of f. (5)

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4. (a) (i) Show that $\|\sigma_u \times \sigma_v\| = (EG - F^2)^{\frac{1}{2}}$ where E,F and G are notations as in first fundamental form. (5)
(ii) Show that a curve on a surface is a geodesic if and only if its geodesic curvature is zero everywhere. (5)
(b) Consider parametrized equation of Mobius strip

$$\sigma(u, v) = ((2 - v\sin\frac{u}{2})\sin u, (2 - v\sin\frac{u}{2})\cos u, v\cos\frac{u}{2})$$

Calculate
(i) The coefficients of the first fundamental form (2)
(ii) The coefficients of the Second fundamental form (2)
(iii) The Gaussian curvature (2)
(iv) The Principal curvatures (2)
(v) The Mean curvature (2)

Section B

5. (a) (i) Show that a mapping $T : X \to Y$ of a metric space (X, d) into a metric space $(Y, \bar{d})$ is continuous at a point $x_0 \in X$ if and only if $x_n \to x_0 \Longrightarrow T(x_n) \to T(x_0)$. (5)
(ii) Is the function space $C[a, b]$ complete? Justify your answer. (5)
(b) (i) Let X be a compact metric space and F a subset of $\mathcal{C}(X)$. Show that F is a compact subspace of $\mathcal{C}(X)$ if and only if F is closed, uniformly bounded and equicontinuous. (5)
(ii) Show that $\mathcal{C}[0, 1]$ and $\mathcal{C}[a, b]$ are isometric. (5)

6. (a) (i) State and prove the Cauchy-Schwarz inequality for l^p space. (5)
(ii) Show that every finite dimensional subspace Y of a normed space X is closed in X. (5)
(b) Let $\mathcal{C}[a, b]$ be the vector space of all continuous real valued functions on $[a, b]$ with norm defined by $\|x\| = (\int_a^b [x(t)]^2 dt)^{\frac{1}{2}}$. Is $(\mathcal{C}[a, b], \|.\|)$ complete normed space? Justify your answer. If not, what is the completion of $(\mathcal{C}[a, b], \|.\|)$? (10)

7. (a) (i) A real matrix A with r rows and n columns defines an operator $T : \mathbb{R}^n \to \mathbb{R}^r$ by $y = Ax$ where x and y are column vector with n and r components respectively. Is the operator T a bounded linear operator? Justify your answer. (5)
(ii) Show that a linear operator T is continuous if and only if T is bounded. (5)
(b) Define dual space of a normed space. Show that the dual space of l^1 is l^∞. (10)

8. (a) State and prove closed graph theorem. (10)
(b) State and prove generalized Hahn Banach theorem. (10)

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M.SC. {MATHEMATIC} (PART-II)

NUMERICAL ANALYSIS (REV)

PAPER – IV (JAN- 2020)

[Total Marks:80

(1) Attempt any two questions from each section.

(2) All questions carry equal marks. Scientific calculator can be used.

(3) Answer to Section-I and Section-II should be written in the same answer book

Section-I

Que. 1 (a) Define the following terms with one example of each:

(i) Absolute error.
(ii) Relative error.
(iii) Percentage error.
(iv) Round-off error.
(v) Truncation error.

(b) Convert the decimal fraction $(205.453125)_{10}$ to the binary form and then convert to the hexadecimal form.

Que. 2 (a) Define the term rate of convergence of an iterative method. Find the rate of convergence of the Newton-Raphson method.

(b) Use the method of iteration to solve the following system of non-linear equations:

$$x^2 + y = 11, \quad x + y^2 = 7.$$

Use initial approximation $x_0 = 2.5$ and $y_0 = 1.1$. Perform four iterations.

Que. 3 (a) Describe the Jacobi's method to obtain eigenvalues and eigenvectors of a real symmetric matrix $A = [a_{ij}]$ of order $n \times n$.

(b) Find the inverse of the following matrix by using Gauss-Jordan method:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Que. 4 (a) Derive Newton's backward difference interpolation formula.

(b) Using Newton's forward difference formula, find the sum

$$S_n = 1^3 + 2^3 + 3^3 + \cdots + n^3.$$

Section-II

Que. 5 (a) Estimate the error in the Simpson's $3/8^{\text{th}}$ rule.

(b) Evaluate the integral $\int_0^2 \frac{1}{x^3+2x+1}\,dx$ using two-point Gaussian quadrature formula and three-point Gaussian quadrature formula.

Que. 6 (a) Let $y = f(x)$ be continuous function on $[a, b]$. Let the function $y(x)$ be approximated by $Y(x) = a_0 f_0(x) + a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x)$, where $f_j(x),\ j = 0, 1, 2, 3$ are orthogonal polynomials on $[a, b]$ of degree j with respect to the weight function $W(x)$. Find the unknown parameters a_0, a_1, a_2 and a_3 by least squares method.

(b) Explain the term: Discrete Fourier Transform (DFT). Find the DFT of the sequence $\{1, -1, i, -i\}$.

Que. 7 (a) Derive the Adams-Bashforth predictor formula to solve the differential equation $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.

(b) Given the differential equation

$$y'' - xy' - y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Use Taylor's series method to determine the value of $y(0.1)$ correct to seven decimal places.

Que. 8 (a) Derive a numerical method (Bender-Schmidt method) to obtain the numerical solution of one dimensional heat equation with initial and boundary conditions.

(b) The Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ satisfies the conditions $u(0, y) = 0$, $u(4, y) = 8 + 2y$, $u(x, 0) = \frac{1}{2}x^2$ and $u(x, 4) = x^2$. Using Liebmann's method find the values of $u(i, j)$, $i = 1, 2, 3;\ j = 1, 2, 3$, correct to two places of decimals.

M.SC. {MATHEMATIC} (PART-II)
GRAPH THEORY (REV.)
PAPER – V (JAN- 2020)

(3 Hours) [Total Marks: 80]

N.B 1) Both the Sections are **Compulsory**.
2) Attempt **ANY TWO** questions from each Section.
3) Figures to the right indicate full marks.
4) Answers to section I and section II should be written in same answer book.

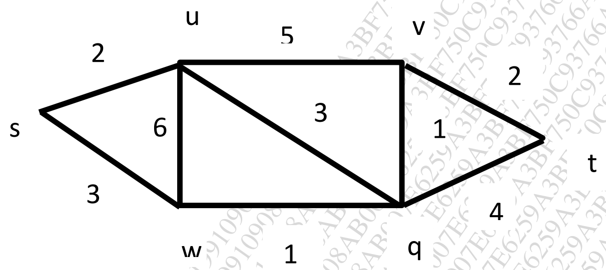
Section - I

Q.1 20M

a) Prove that the block graph of a connected graph is a tree. 5M

b) Let G be a graph in which all vertices have degree atleast 2. Then prove that G contains a cycle. 5M

c) Use Dijkstra Algorithm to find the shortest path from vertex s to t in the following graph. 5M

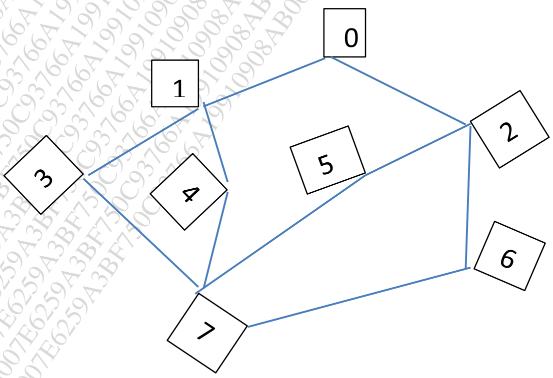


d) Draw all non – isomorphic graph on four vertices. 5M

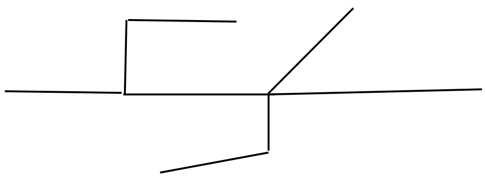
Q.2 20M

a) If T is a tree with k edges and G is a simple graph with $\delta(G) \ge k$, then prove that T is subgraph of G. {where $\delta(G)$: minimum dgree of graph G. } 5M

b) Draw Depth first search tree for the following graph. 5M



c) Find Prüfer sequence for the given tree . 5M

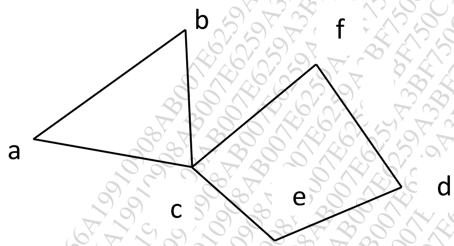


d) Prove that the minimal cost spanning tree generated by Kruskal's algorithm is optimal. 5M

Q.3 20M

a) Let G be a simple graph on n vertices . let u and v be two non adjacent vertices in G such that $d(u)+d(v) \geq n$. Then prove that G is Hamiltonian if and only if $G+uv$ is Hamiltonian. 5M

b) Using Fleury's algorithm find an Eulerian circuit for given connected even graph G. 5M



c) If G is Hamiltonian, then prove that for every non empty subset $S \subseteq V(G)$, $c(G \setminus S) \leq |S|$. 5M
Where $c(G \setminus S)$: the number of distinct component obtained after remaining S from G.

d) Prove that every 5- vertex path in the dodecahedron lies in Hamiltonian cycle. 5M

Q.4 20M

a) If T is an m – vertex tree then prove that $R(T, K_n) = (m-1)(n-1)+1$. 5M

b) Prove that $R(p,q) \leq \binom{p+q-2}{p-1}$. 5M

c) State and prove Hall's theorem. 5M

d) If G is a graph without isolated vertices then prove that $\alpha'(G)+\beta'(G)=n(G)$. 5M
Where $\alpha'(G)$: maximum size of matching. , $\beta'(G)$: minimum size of edge cover.

Section -II

Q.5 20M

a) If G is a simple graph and $e \in E(G)$ then prove that $\chi(G:K) = \chi(G \setminus e; K) - \chi(G.e; K)$. 5M

b) If G is a graph on p- vertices then prove that $2\sqrt{p} \le \chi(G) + \chi(\overline{G}) \le p+1$. 5M

c) Prove that for any graph G $\chi'(G) \le 2\Delta(G) - 1$.
Where $\chi'(G)$: edge chromatic number. 5M

d) Prove that for any graph G, $\chi(G) \le \Delta(G) + 1$. 5M

Q.6 20M

a) If G (p , q) is a plane graph in which every face is bounded by a cycle of length atleast n then prove that $q \le \frac{n(p-2)}{n-2}$. 5M

b) If G is a plane graph with $p \ge 3$ vertices and q edges then prove that $q \le 3p - 6$. Hence show that K_5 is non-planar. 5M

c) Prove that for a plane graph G, G is bipartite if and only if every face of G has even length. 5M

d) Prove that the boundary of outer face of 2 connected outer plane graph is spanning cycle. 5M

Q.7 20M

a) If P is an f-augmenting path with tolerance Z then prove that changing flow by +Z on edges followed forward by P and by -Z an edge followed backward by P produces a feasible flow f' with $val(f') = val(f) + Z$. 5M

b) Prove that a diagraph D contains a directed path of length $\chi - 1$. 5M

c) Prove that every strong tournament D on p- vertices ($p \ge 3$) contains a cycle of length k $\forall k, 3 \le k \le p$. 5M

d) Prove that a diagraph D is unilaterally connected if and only if there is a directed walk not necessary closed containing all its vertices. 5M

Q.8 20M

a) Define spectrum of graph and find the spectra of path K_3. 10M

b) For every graph G, Prove that $\delta(G) \le \lambda_{\max}(G) \le \Delta(G)$. Where $\delta(G)$ is minimum degree of G, $\Delta(G)$ is maximum degree of G and $\lambda_{\max}(G)$ is maximum eigen value of G. 10M
