Mathematics. : Algebra (R-2016)(Only for IDOL Students)

External] (3 Hours) [Total Marks:80

Instructions:
- Attempt **any two** questions from **each section**
- **All** questions carry **equal marks**.
- Answer to **section I** and **II** should be written on the **same answer book**

SECTION I (Attempt any two questions)

1. (a) State and prove Rank-Nullity theorem.

   (b) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the Linear Transformation defined by $T(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$.Find ImT and KerT.

2. (a) If A and B are two $n \times n$ matrices . Prove the following.

   (i) $|AB| = |A||B|$

   (ii) $|A^t| = |A|$

   (b) (i) Find the Rank of the matrix $\begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & -2 & 0 & 2 \\ 2 & -8 & 3 & 1 \end{pmatrix}$

   (ii) Find eigenvalues of a matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

3. (a) State and prove Cayley-Hamilton Theorem

   (b) Find the Minimal Polynomial of the matrix $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$

4. (a) Define Invariant Subspace of a Vector Space V. Let V be a finite dimensional inner product space and let T be any linear transformation on V. Suppose W is a subspace of V which is invariant under T. Then prove that the orthogonal complement of W is invariant under T

   (b) Let A be the following matrix. Show that the bilinear map $\mathbb{R}^3 \to \mathbb{R}^3 \to \mathbb{R}$ defined by $< x, y >= x^T Ay$ is a scalar product. $A = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

SECTION II (Attempt any two questions)

5. (a) State and prove fundamental theorem of Homomorphism of groups.

   (b) Let G be a group and $a \in G$. If $o(a) = n$. Prove that $o(a^r) = \frac{n}{gcd(o(a), r)}$

6. (a) Let G be a group and A be a non-empty subset of G. Define Centralizer $C_G(A)$ and Normalizer $N_G(A)$. Prove that both are subgroups of G. If G is an abelian group what can we say about its centralizer?. Justify your answer

   (b) Prove that a group of order p^n where p is prime and $n \geq 1$ has a non-trivial centre.

7. (a) Show that an ideal P in a commutative ring R is a prime ideal if and only if the quotient ring $R \backslash P$ is an integral domain

   (b) Let R be a commutative ring with prime characteristic p and $f\colon \mathbb{R} \to \mathbb{R}$ be defined as $f(a) = a^p$ for $a \in \mathbb{R}$.Show that f is a ring Homomorphism

8. (a) Prove that in UFD irreducibles are primes

   (b) Prove or disprove $\mathbb{Z}[x]$ is a PID

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M.SC. {MATHEMATIC} (PART-I)

ANALYSIS & TOPOLOGY (R-2016)

PAPER – II (JAN- 2020)

(3 Hours) **[Total Marks:80]**

Instructions:
- Attempt **any two** questions from **each section.**
- **All** questions carry **equal marks**.
- Answer to **section I** and **II** should be written on the **same answer book.**

SECTION I (Attempt any two questions)

1. (a) Define a metric space. Let d_1 and d_2 be two metrics on X. Then show that;
   $i))d(x,y) = \sqrt{d_1^2(x,y) + d_2^2(x,y)}$ is also metric space on X.
   $ii)d(x,y) = min\{1, d_1(x,y)\}$ is bounded metric on X.

   (b) $i)$ Let (X,d) be a metric space and A, B are any two subsets of X.
   Then show that $cl(A \cup B) = cl(A) \cup cl(B)$ and $cl(A \cap B) \subseteq cl(A) \cap cl(B)$. Also give an example to show that inclution can be proper.
   $ii)$ Show that if $A \subseteq B$, then $diam(A) \leq diam(B)$.

2. (a) Let (X,d) be a metric space, A and B are subsets of X with $A \subseteq B \subseteq \bar{A}$,and A is connected then B is also connected. Also give an example of sets A, B, C such that $A \subseteq B \subseteq C$, A, C are connected but B is not connected.

   (b) Show that continuous image of compact set is compact.

3. (a) If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then show that it's total derivative is unique.

   (b) State and prove Chain Rule.

4. (a) $i)$ Find the Taylor expansion of function, $f(x,y) = \sin(2x + 3y) at (a,b) = (0,0)$.
   $ii)$ Find the extreme value of, $f(x,y) = x^2 + y^3 + 3xy^2 - 2x$.

   (b) State and prove Inverse function theorem.

SECTION II (Attempt any two questions)

5. (a) Define a Topological Space and Base of a Topological Space. Let X be any infinite set and τ_1 consist of ϕ, X and all subsets A of X such that $X \backslash A$ is finite. Let τ_2 consist of ϕ, X and all subsets A of X such that $X \backslash A$ is countable. Show that τ_1 and τ_2 are topologies on X.

   (b) Let $f : (X, \tau) \to (Y, \tau')$ be any map. Show that the following conditions are equivalent:

   i) f is continuous on X.
   ii) If $H \in \tau'$, then $f^{-1}(H) \in \tau$.
   iii) If C is a closed subset of (Y, τ'), then $f^{-1}(C)$ is a closed subset of (X, τ).

[TURN OVER]

iv) For any subset A of X, $f(c(A))$ is a subset of $c(f(A))$, where $c(A)$ denotes the closure of A and $c(f(A))$ denotes the closure of $f(A)$.

6. (a) Define T_0, T_1 and T_2 space. Show that a topological space being T_0 is a topological and hereditary property.

   (b) Let (X, τ) be a topological space. When is X said to be separable? Show that a topological space being separable is a topological property. Is being separable a hereditary property? Justify your answer.

7. (a) Show that closed subsets of a compact space is compact. Also prove that compact subset of a Hausdorff space is closed.

   (b) Define local compactness and one point compactification. Show that the one point compactification of $(0, 2\pi)$ is homeomorphic with S^1, where S^1 denotes the unit circle.

8. (a) Define limit point compact and sequentially compact space. Show that if a topological space X is limit point compact then X is sequentially compact.

   (b) Prove that every closed and bounded interval in $\mathbb{R}$ is compact.

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M.SC. {MATHEMATIC} (PART-I)

COMPLEX ANALYSIS (R-2016)

PAPER – III (JAN- 2020)

Q.P. Code: 37371

(3 Hours) [Total marks: 80]

Instructions:

1) Attempt any two questions from each section.
2) All questions carry equal marks.
3) Answer to Section I and section II should be written in the same answer book.

SECTION-I (Attempt any two questions)

1) (a) Suppose $z_n = x_n + iy_n$ and $z_0 = x_0 + iy_0$ then prove that $\lim_{n\to\infty} z_n = z_0$ if and only if $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} y_n = y_0$. (10)

(b) Given a series $\sum_{n=1}^{\infty} z^n(1-z)$. Prove that

(i) The series converges for $|z| < 1$ and find its sum.

(ii) The series uniformly converges to the sum z for $|z| \le 1/2$.

(iii) Does the series converge uniformly for $|z| \le 1$? Explain. (10)

2) (a) Prove that a Mobius Transformation is a composition of translation, rotation, inversion, and magnification. (10)

(b) Prove that the circle $|z-2| = 3$ is mapped onto a circle $\left|w + \frac{2}{5}\right| = \frac{9}{25}$ under the transformation $w = \frac{1}{z}$. (10)

3) (a) Define logarithmic function of a complex variable. Hence or otherwise prove that $\log z$ is not continuous on negative real axis. (10)

(b) Determine whether the following functions are analytic. (i) $\cos z$ (ii) $z^2 - \bar{z}$ (iii) $x^2 - y^2 + 2ixy$. (10)

4) (a) Let γ be such that $\gamma(t)$ be a smooth curve defined on $[a,b]$ and suppose that f is a continuous function on an open set containing $\gamma[a,b]$. Then, prove that

(i) $\int_{-\gamma} f(z)dz = -\int_{\gamma} f(z)dz$

(ii) $\left|\int_{\gamma} f(z)dz\right| \le \int_{\gamma} |f(z)||dz|$

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(iii) If $M = \underset{t\in[a,b]}{Max}\left|f\left(\gamma(t)\right)\right|$ and $L = L(\gamma)$ (length of γ) then, $\left|\int_{\gamma} f(z)dz\right| \le ML$. (10)

(b) Evaluate $\int_{0}^{1+i} x^2 + iy\,dz$, along

(i) The line y = x.

(ii) Along the parabola y = x^2. Is the integral independent of path? (10)

SECTION-II (Attempt any two questions)

5) (a) State and prove Morera's theorem. (10)

(b) Evaluate $\int_C \frac{z+1}{z^3 - 2z^2} dz$ where C is

(i) The circle $|z| = 1$.

(ii) The circle $|z-2-i| = 2$.

(iii) The circle $|z-1-2i| = 2$. (10)

6) (a) State and prove Liouville's Theorem. (10)

(b) Evaluate $\int_{\gamma} \frac{f'(z)}{f(z)} dz$, where

i) $f(z) = \frac{z(z-1)^2}{z^3+5}$, where is γ circle $|z| = 1.2$

ii) $f(z) = \frac{z^2 + (z+3)(z-1)}{z^3+2}$, where γ is circle $|z| = 1.2$ (10)

7) (a) Prove that if f has an isolated singular point at z_0, then $z = z_0$ is a removable singularity of f if and only if $\lim_{z\to z_0} (z-z_0) f(z) = 0$. (10)

(b) Find all the possible Laurent Series expansions of $f(z) = \frac{2-z^2}{z(1-z)(2-z)}$. (10)

8) (a) State and prove Argument theorem. (10)

(b) Using contour integration evaluate $\int_0^{2\pi} \frac{d\theta}{25 - 16\cos^2\theta}$. (10)

M.SC. {MATHEMATIC} (PART-I)
DISCRETE MATHEMATICS & DIFFERENTIAL EQUATIONS (R-2016)
PAPER – IV (JAN- 2020)

(3 Hours) **[Total marks: 80]**

1) Attempt any two questions from each section.
2) All questions carry equal marks.
3) Answer to Section I and section II should be written in the same answer book.

SECTION-I (Attempt any two questions)

Q.1] A] If $ca \equiv cb(modn)$ then show that$a \equiv b\ (mod\frac{n}{d})$, where $d = \gcd(c, n)$. [10]

B] If x is an odd integer, not divisible by 3, prove that $x^2 \equiv 1(mod24)$. [10]

Q.2] A] In a survey of 60 people, It was found that 25 read magazine. 26 read Times of India and 26 read DNA. Also 9 read both magazine and DNA, 11 read both magazine and times of India , 8 read times of India and DNA and 8 are not reading anything. [10]

i). Determine the number of people who read at least one of them.
ii). Determine the number of people who read magazine and times of India only.
iii). Determine the number of people who read exactly one magazine.

B] i) Find the integer solution of $x + y + z = 12, where\ 0 \leq x, y, z$. [05]

ii) S is a set of mn objects. Prove that S can be split up into n sets of m elements in $\frac{(mn)!}{(m!)^n\ n!}$ different ways. [05]

Q.3] A] During a month with 30 days a baseball team plays at least a game a day but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games. [10]

B] State and prove Erdos-Szekers theorem on monotonic subsequence by using pigeon-hole principle. [10]

Q.4] A] Let $G(v, e)$ be a graph then show that G is a tree if and only if it is connected and $e = v - 1$. [10]

B] i) Let L be a bounded distributive lattice then prove that compliments are unique if they exits. [05]

ii) State and prove De-Morgan's law for Boolean expressions in two variables. [05]

SECTION-II (Attempt any two questions)

Q.5] A] Show that k^{th} successive approximation $\emptyset_k$ to the solution $\emptyset$ of the IVP satisfies

$$|\emptyset(x) - \emptyset_k(x)| \leq \frac{M}{k} \frac{(k\alpha)^{k+1}}{(k+1)!} e^{k\alpha}.$$ [10]

B] Find the fundamental matrix of the system $y' = Ay$, where $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$. Hence find the unique solution of $y' = Ay$ with $y(0) = (2,4)$. [10]

Q.6] A] Consider the n^{th} order linear differential equation

$$L(y) = y^n + a_1(x)y^{n-1} + a_2(x)y^{n-2} + \cdots \ldots. + a_n(x)y = 0$$

where $a_1(x), a_2(x), \ldots \ldots \ldots, a_n(x)$ are continuous functions on an interval *I*.

If $\emptyset_1, \emptyset_2, \ldots \ldots .., \emptyset_n$ are *n* solutions of *L(y)* on an interval *I* then show that

$$W(\emptyset_1, \emptyset_2, \ldots \ldots .., \emptyset_n) = \exp\left[-\int_{x_0}^{x} a_1(t)dt\right] W(\emptyset_1, \emptyset_2, \ldots \ldots .., \emptyset_n), \ x, x_0 \in I.$$ [10]

B] Verify that $\emptyset_1(x) = x^3, \ (x > 0)$ is a solution of $x^2y'' - 7xy' + 15y = 0$, hence find other Linearly independent solution . [10]

Q.7] A] Find the normal form of the Strum-Liouville equation $\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + \lambda q(x)y = 0.$ [10]

B] Let $u(x)$ be any non-trivial solution of $u'' + q(x)u = 0$ where $q(x) > 0, \forall x > 0$ if $\int_1^{\infty} q(x)dx = \infty$ then show that $u(x)$ has infinitely many zeros on the positive x-axis. [10]

Q.8] A] Find the integral surface generated by the partial differential equation

$$x(y^2 + z)z_x - y(x^2 + z)z_y = (x^2 - y^2)z$$

Which contains the straight line $x + y = 0, \ z = 1$. [10]

B] Solve the Cauchy problem $(x^2 + 1)u_x + \frac{2xy}{x^2+1}u_y = 2xu, \ y > 0, \ u(0, y) = \log y$. [10]

M.SC. {MATHEMATIC} (PART-I)
SET THEORY, LOGIC & ELEMENTARY PROBABILITY THEORY (R-2016)
PAPER – V (JAN- 2020)

Duration: 3 Hours **Marks: 80**

N.B. 1) Attempt any two questions from section - I and any two questions from section - II.
2) All questions carry equal marks.

SECTION-I (Attempt any two questions)

1. a) For sets A and B, Prove or Disprove: 10
   a) $A \subseteq B$ if and only if $P(A) \subseteq P(B)$
   b) $P(A) \cup P(B) = P(A \cup B)$
   c) $P(A) \cap P(B) = P(A \cap B)$

   b) Let p and q be propositions : 6
   i) Determine whether $p \vee q$ and $\neg p \rightarrow q$ are logically equivalent.
   ii) Determine whether $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$ is a tautology.

   c) Let n denote an integer. Prove the implication "If $n^3 + 5$ is odd then n is even " using contrapostive method. 4

2. a) Show that the product $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countably infinite. 10
   b) Show that countable union of countable sets is countable. 6
   c) Show that if a set A is finite then there is no bijection of A with a proper subset of itself. 4

3. a) Using mathematical induction prove the following : 10
   i) Let S_n denote the sum of first n natural numbers then derive the formula $S_n = \frac{n(n+1)}{2}$
   ii) If $n \geq 3$ then prove that $2n-1 > 1$.

   b) Define the terms partially ordered set, totally ordered set and give the statement of Zorn's lemma. 6
   c) Explain Russell's paradox. 4

4. a) For $n > 1$, show that A_n has order $n!/2$. 10
   b) If $\beta = (1\ 2\ 5\ 7\ 3\ 6)$ then compute β^{122}. 5
   c) If A and B are two sets and $|A|=|B|=n$ then show that there are exactly n! bijections from A to B. 5

SECTION-II (Attempt any two questions)

5. a) Let $\{F_i : i \in I\}$ be a collection of sigma-fields of subsets of Ω. Prove or disprove 6
   i) $\cap_{i=1}^{\infty} F_i$ is a sigma-field.
   ii) $\cup_{i=1}^{\infty} F_i$ is a sigma-field.

   b) A class C of subsets of $\sum$, such that either A or $\bar{A}$ are countable. Is C a sigma field? 4

   c) A boy is throwing stones at a target, if the probability of hitting the target is 3/5. 6
   (i) Write the probability distribution of X = no. of attempts required before hitting the target.
   (ii) What is the probability that target it hit on the 5th attempt ?

(iii) Find the mean and variance of X.

d) Let F be a field. If $A, B \in F$ then show that $A - B$ and $A \Delta B$ are also events of F. 4

6. a) Let $(\mathbb{R},\ \mathcal{B},\ m)$ be a measure space and $P(A) = \int_A f\, dm$ f is a density function. Show that P is a probability measure on Borel subset A of $\mathbb{R}$. 6

b) Let $\Omega = \{1,2,3,4\}$ with uniform probability and $A = \{1,2\}$. List all $B \subset \Omega$ such that A, B are independent. 4

c) State and prove Borel Cantelli lemma. 6

d) Define conditional probability of A given B, where A and B are events in a probability space. Also show that: 4

$P(A \cup B/C) = P(A/C) + P(B/C) - P(A \cap B/C)$

7. a) Let X and Y be two independent random variables with binomial distribution $B(m, p)$ and $B(n, p)$ respectively. What is the distribution of $X + Y$. 6

b) Let X be a simple random variable and $E(X)$be the expectation of X .Show that if $X \geq 0$ and $E(X) = 0$ then $P(\{X = 0\}) = 1$. 4

c) Let X and Y be independent discrete random variables show that $E(XY) = E(X)E(Y)$ and $V(X + Y) = V(X) + V(Y)$. 6

d) The joint p.d.f of X, Y is f(x, y) = 8xy for 0 < x < y < 1; find conditional p.d.f of X given y. 4

8. a) Define the characteristic function of a random variable. If X is a random variable such that, $P(\{X \in \mathbb{Z}\}) = 1$ then show that characteristic function of S is a periodic function with period 2π. 6

b) State the Chebyshev inequality. A fair coin is tossed independently n times. Let S_n be the number of heads obtained. Find a lower bound of the probability that $\frac{S_n}{n}$ differs from ½ by less than 0.1 when $n = 1000$. 6

c) Find the characteristic function of X such that $P(\{X = k\}) = 2^{-k}$, $k = 1,2, \ldots$ 3

d) State the Strong law of large numbers and Examine whether the Strong law of large numbers holds for sequence of independent r.v.'s 5

$$\{X_k\}: X_k = \begin{cases} \pm k & \text{with probability } \frac{1}{2\sqrt{k}} \\ 0 & \text{with probability } 1 - \frac{1}{\sqrt{k}} \end{cases}.$$
