69059

Duration 03 Hours. Total Marks assigned to the paper 80

Q. No. **Marks**

N.B. :

1. a) $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ (5)

$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \frac{(iy - ix)}{x^2+y^2}(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_C y\,dx - x\,dy \quad (\because x^2+y^2=1)$$

Let $x = \cos\theta$, $y = \sin\theta$, $dx = -\sin\theta\, d\theta$

$dy = \cos\theta\, d\theta$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \sin\theta(-\sin\theta) - \cos\theta(\cos\theta)\}\, d\theta$$

$$= -\int_0^{2\pi} d\theta = -(\theta)_0^{2\pi} = -2\pi \quad \#$$

b) $f(x) = \pi - x$, $0 < x < 2\pi$ (5)

$$f(x) = \frac{a_0}{2} + \sum_1^\infty a_n \cos nx + \sum_1^\infty b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = 0$$

$$b_n = \frac{2}{\pi}\int_0^{2\pi} (\pi - x)\sin nx\,dx = \frac{2}{n}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{n}\sin nx \quad \#.$$

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1\.

c) $f(x) = 1 - x^2, \quad |x| < 1$ (5)

$$F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (1 - |x|) e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2) e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{4}{s^3} \left[-s\cos s + \sin s \right].$$

(5)

d) $\text{curl } \vec{F} = \nabla \times \vec{F} = \nabla \times r^n \vec{r}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+y^2+z^2)^{n/2} x & (x^2+y^2+z^2)^{n/2} y & (x^2+y^2+z^2)^{n/2} z \end{vmatrix}$$

$= 0$

$$r^n \vec{r} = (x^2+y^2+z^2)^{n/2} (x\hat{i} + y\hat{j} + z\hat{k})$$

(6)

2\. a) fourier sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} e^{ax} \sin nx \, dx$$

$$= \frac{2}{\pi} n \left[\frac{1 - (-1)^n}{n^2+1} \right] e^{\pi}$$

$$f(x) = \frac{2}{\pi} \sum n \left[\frac{1 - (-1)^n e^{\pi}}{n^2+1} \right] \sin nx$$

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b) $f(x) = \frac{e^{-ax}}{x}$ (6)

$$F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx$$

$$\frac{d}{ds}\left[F(s)\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} (x \cos sx) \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$$

$$\therefore F(s) = \sqrt{\frac{2}{\pi}} \int \frac{a}{s^2+a^2} ds = \sqrt{\frac{2}{\pi}} \tan^{-1}\frac{s}{a} + c$$

for $s=0$, $F(s)=0$
$\therefore c=0$, $\therefore F(s) = \sqrt{\frac{2}{\pi}} \tan^{-1}\frac{s}{a}$.

c) Green's Theorem, (8)

$$\oint_c \phi \, dx + \psi \, dy = \iint_S \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx dy$$

$$= -\frac{1}{2}.$$

3. a) Work done $= \int_c \vec{F} \cdot d\vec{r}$ (6)

$$= \int_c 2x^2 y \, dx + 3xy \, dy$$

$y = 4x^2$, $dy = 8x \, dx$

$$= 104 \int_0^1 x^4 dx = \frac{104}{5}.$$

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3\. b) $y = (C_1 \cos c\sqrt{k}t + C_2 \sin c\sqrt{k}t)(C_3 \cos\sqrt{k}x + C_4 \sin\sqrt{k}x)$ (8)

Put $y=0$ when $x=0$

$C_3 = 0$

at $x=L$, $y = P_0 \cos pt$

$P_0 \cos pt = C_1 C_4 \cos c\sqrt{k}t \sin\sqrt{k}L$
$+ C_2 C_4 \sin c\sqrt{k}t \sin\sqrt{k}L$

Equating coefficients

$P_0 = C_1 C_4 \sin\sqrt{k}L \Rightarrow C_1 C_4 = \dfrac{P_0}{\sin\sqrt{k}L}$

& $C_2 = 0$ Let $c\sqrt{k} = p \Rightarrow \frac{p}{c} = \sqrt{k}$

$\therefore y = \dfrac{P_0}{\sin\sqrt{k}L} \cos pt \sin \frac{p}{c}x$

$$\boxed{y = \frac{P_0}{\sin \frac{p}{c}L} \cos pt \sin \frac{p}{c}x}$$

c) $f(x) = \dfrac{a_0}{2} + \sum a_n \cos \dfrac{n\pi x}{l} + \sum b_n \sin\dfrac{n\pi x}{L}$ (8)

$a_0 = \frac{1}{2}$, $a_n = \dfrac{2}{n^2\pi^2}$, n is odd

$= 0$, if n is even

$b_n = 0$

$\therefore f(x) = \dfrac{1}{4} + \dfrac{2}{\pi^2}\left[\dfrac{\cos 2\pi x}{1^2} + \dfrac{\cos 6\pi x}{3^2} + \cdots\right]$

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4. a) $f(x) = \sum b_n \sin nx$ (6)

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin nx\,dx$$

$$= -\frac{2}{n\pi}\left[(-1)^n - 1\right] = \frac{4}{n\pi}, \text{ if } n \text{ is odd}$$

$$= 0, \text{ if } n \text{ is even}$$

$$\therefore 1 = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \cdots$$

Now $\int_0^{\pi}[f(x)]^2 dx = \frac{c}{2}\left[b_1^2 + b_2^2 + \cdots\right]$

$$\int_0^{\pi}(1)^2 dx = \frac{\pi}{2}\left[\left(\frac{4}{\pi}\right)^2 + \left(\frac{4}{3\pi}\right)^2 + \cdots\right]$$

$$[x]_0^{\pi} = \left(\frac{\pi}{2}\right)\left(\frac{16}{\pi^2}\right)\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\pi = \frac{\pi}{2}\left(\frac{16}{\pi^2}\right)\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\Rightarrow \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \quad \#.$$

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4. b) $\iint_S f_1\,dy\,dz + f_2\,dz\,dx + f_3\,dx\,dy$ (6)

$$= \iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\,dy\,dz$$

$$= \iiint_V (z^2+x^2+y^2)\,dx\,dy\,dz$$

$\because f_1 = xz^2$, $f_2 = x^2y - z^3$, $f_3 = 2xy + y^2z$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$.

$$\therefore = \iiint_V r^2(r^2\sin\theta)\,dr\,d\theta\,d\phi$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta\,d\theta \int_0^a r^4\,dr = \frac{2\pi a^5}{5}$$

c) $\frac{\partial^2 u}{\partial x^2} = h^2 \frac{\partial u}{\partial t}$, $h^2 = \frac{1}{c^2}$ (8)

sol^n is $u = c_1 e^{-\frac{p^2 t}{h^2}} (c_2 \cos px + c_3 \sin px)$

$x=0, u=0$ $\quad c_1 \neq 0, c_2 = 0$

$\therefore u = c_3 \sin px\, c_1 e^{-p^2 t/h^2}$

$x = l, u = 0$ $\quad c_3 \neq 0$, $\sin pl = 0 = \sin n\pi$

$\Rightarrow p = \frac{n\pi}{l}$

$$u = c_1 c_3 \sin\frac{n\pi x}{l} e^{-\frac{n^2\pi^2 t}{h^2 l^2}}$$

$$= \sum b_n \sin\frac{n\pi x}{l} e^{-\frac{n^2\pi^2 t}{h^2 l^2}}$$

at $t=0$, $u = 100\frac{x}{l}$

$\therefore b_n = \frac{2}{l}\int_0^l 100\frac{x}{l} \sin\frac{n\pi x}{l}\,dx = \frac{200}{\pi}\frac{(-1)^{n+1}}{n}$

$$\therefore u = \frac{200}{\pi}\sum \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{l} e^{-\frac{c^2 n^2\pi^2}{l^2}\cdot t}.$$

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5. a) $\int_C \vec{F}\cdot d\vec{r}$ (6)

The integral is independent of path of integration if $\vec{F} = \nabla\phi \Rightarrow \nabla\times\vec{F} = \vec{0}$.

$\nabla\times\vec{F} = \vec{0}$ (prove it)

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= d(x^2yz^2) + d(\sin yz)$$

$$\therefore \phi = x^2yz^2 + \sin yz$$

$$\phi\Big]_{(0,\pi/2,1)}^{(1,0,1)} = 1.$$

b) $$C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-\frac{in\pi x}{L}}\,dx$$ (6)

$$= \frac{1}{2}\int_{-1}^{1} e^{-x} e^{-in\pi x}\,dx$$

$$= \frac{(-1)^n(1-in\pi)}{1^2+n^2\pi^2}\sinh 1$$

$$\therefore f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}$$

$$= \sum \frac{(-1)^n(1-in\pi)}{1^2+n^2\pi^2}\sinh 1\, e^{in\pi x}$$ #

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5. c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $u(0,y) = 0 \ \forall y$ (8)

$u(a,y) = 0 \ \forall y$, $u(x,\infty) = 0 \ \forall x$

$u(x,0) = \begin{cases} x, & 0 \le x \le a/2 \\ a-x, & a/2 \le x \le a. \end{cases}$

Sol^n is $u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$

using boundary cond^n

$u(x,y) = c_2 c_4 \sin px \, e^{-py}$

$= \sum b_n \sin px \, e^{-py}$, $p = \frac{n\pi}{a}$

$u(x,0) = \sum b_n \sin px$

$b_n = \frac{2}{a}\int_0^a f(x) \sin px \, dx$

$u(x,y) = \frac{2}{\pi^2} \sum \frac{(-1)^{n+1}}{(2n-1)^2} a \sin\frac{(2n-1)\pi x}{a} e^{\frac{(2n-1)\pi y}{a}}$ #

6. a) $\int_C (2x-y)\,dx - yz^2\,dy - y^2 z\,dz$ (6)

$= \int_C ((2x-y)\hat{i} - yz^2\hat{j} - y^2 z\hat{k}) \cdot (i\,dx + \hat{j}\,dy + \hat{k}\,dz)$

By Stoke's Theorem,

$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \vec{F} \cdot \eta \, ds$

$\text{curl} \vec{F} = \nabla \times \vec{F} = k$

$\therefore \iint k \cdot \hat{\eta} \, ds = \iint \hat{k} \cdot \hat{\eta} \frac{dx\,dy}{\hat{\eta} k} = \iint dx\,dy = \pi.$