M.SC. (MATHEMATICS) PART-I
Algebra (R-2016)
(JUNE - 2019)

(3 Hours) **[Total Marks:80**

Instructions:
- Attempt **any two** questions from **each section**
- **All** questions carry **equal marks**.
- Answer to **section I** and**II** should be written on the **same answer book**

SECTION I (Attempt any two questions)

1. (a) If V is a finite dimensional vector space , then prove that any two Basis of V have the same number of elements.

   (b) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ and $S\colon \mathbb{R}^3 \to \mathbb{R}^3$ be two linear transformations defined by $T(x, y, z) = (x, 2y, 3z)$,$S(x, y, z) = (x + y, y + z, z + x)$ and $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis of $\mathbb{R}^3$.Verify that $[m(SoT)]_B^B = [m(S)]_B^B \cdot [m(T)]_B^B$

2. (a) Let $C_1, C_2, C_3, \cdots C_n$ be column vectors of dimension n . Then prove that they are linearly dependent if and only if $det(C_1, C_2, C_3, \cdots C_n) = 0$

   (b) (i) Find the rank of the matrix $\begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & -2 & 0 & 2 \\ 2 & -8 & 3 & -1 \end{pmatrix}$

   (ii) Solve using Cramer's Rule
   $$-2x - y - 3z = 3$$
   $$2x - 3y + z = -13$$
   $$2x - 3z = -11$$

3. (a) Define Minimal Polynomial of a square matrix. Show that the minimal polynomial of a real square matrix A divides every polynomial that annihilates $A_{n\times n}$. (Polynomial $f(x)$ with real coefficients annihilates matrix A if $f(A) = 0$)

   (b) If $(x - 1)(x + 2)^2$ is a characteristic polynomial of a matrix $A_{3\times 3}$ then find the characteristic polynomial of $(i) A^{-1} (ii) A^t (iii) A^2$

4. (a) Let W be a subspace of a finite dimensional inner product space V , then prove that $V = W \oplus W^\perp$

   (b) Let $S \subset \mathbb{R}^4$ be the set of vectors $X = (x_1, x_2, x_3, x_4)$ that satisfy $x_1 + x_2 - x_3 + x_4 = 0$
   What is the dimension of S. Find orthogonal complement of S

SECTION II (Attempt any two questions)

5. (a) Let G be a finite cyclic group of order n , $G =< a >$, then prove that G has a unique subgroup of order n for each divisor d of n

   (b) State and prove Cayley's Theorem

6. (a) Let G be a finite group and p be a prime that divides the order of group G. Then prove that G has an element of order p

   (b) Determine all groups of order 66 upto isomorphism

7. (a) Let R be a commutative ring . If I, J are ideals in R , show that $I \cap J, I + J$ and IJ are ideals of R , where

   $$I + J = \{x + y/x \in I, y \in J\}$$
   $$IJ = \{\sum_{i=1}^{n} x_i y_i / x_i \in I, y_i \in J\}$$

   (b) Determine all zero-divisors , units and idempotent elements in (i) $\mathbb{Z}_{18}$ (ii) $\mathbb{Z}_3 \times \mathbb{Z}_6$ (iii) $\mathbb{Z} \times \mathbb{Q}$

8. (a) Show that ring $\mathbb{Z}[\sqrt(2)]$ and H are isomorphic where $H = \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} /a, b \in \mathbb{Z} \right\}$ under addition and multiplication of 2×2 matrices

   (b) Prove that every Principal Ideal Domain(PID) is a Unique Factorization Domain(UFD)

M.SC. (MATHEMATICS) PART-I

Analysis & Topology (R-2016)

(JUNE - 2019)

Hours) **[Total Marks:80]**

Instructions:

- Attempt **any two** questions from **each section.**
- **All** questions carry **equal marks**.
- Answer to **section I** and **II** should be written on the **same answer book.**

SECTION I (Attempt any two questions)

1. (a) Let d_1 and d_2 be two metrics on X. If there exist $k > 0$, such that $\frac{1}{k}d_1(x,y) \leq d_2(x,y) \leq kd_1(x,y)$, for every $x, y \in X$, then show that d_1 and d_2 are equivalent metrics on X.

   (b) $i)$ In a metric space $(\mathbb{R}^2, \|\|)$, examine if the set $S = \{(x_1, x_2) \in \mathbb{R}^2 : 1 < x_1 + x_2 < 2\}$ is open set.
   $ii)$Let A be a non-empty set in a metric space (X, d). Show that $\mid d(x, A) - d(y, A) \mid \leq d(x, y)$, for all $x, y \in X$.

2. (a) $i)$ Show that close subset of compact set is compact.
   $ii)$ Show that two close sets are separated if and only if they are disjoint.

   (b) Show that a metric space is connected if and only if every continuous characteristic function is a constant function.

3. (a) Show that if $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable at p, then f is continuous at p.

   (b) $i)$ If $u(x, y) = x - 2y + 3, x = r + s + t, y = rs + t^2$ find u_r, u_s and u_t at (1,2,4).
   $ii)$ Write the matrix for f' and evaluate at the point $(2, 9)$ where
   $f(x, y) = (3x^2 + y, x^3 + y^2, x + y^3)$.

4. (a) $i)$ Locate and classify the stationary points of the function given by $f(x, y) = x^2+xy+2x+2y+1$.
   $ii)$ Find the distance of the point (10,1,-6) from the intersection of the plane $x + y - 2z = 5$ and $2x - 3y + z = 12$.

   (b) Determine whether the function $f(x, y) = x^3y + 3, y^2$ is locally invertible at $(1, 3)$.

SECTION II (Attempt any two questions)

5. (a) Define a Topological Space and Base of a Topological Space. Let X be any infinite set and τ_1 consist of ϕ, X and all subsets A of X such that $X \backslash A$ is finite. Let τ_2 consist of ϕ, X and all subsets A of X such that $X \backslash A$ is countable. Show that τ_1 and τ_2 are topologies on X.

   (b) Define open and closed sets in a topological space. Prove that a set G in a topological space (X, τ) is closed if and only if $X \backslash G$ is open in (X, τ).

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6. (a) Define T_1. Show that a topological space (X, τ) is T_1 if and only if every one point subset of X is a closed subset.

   (b) Let (X, τ) be a topological space. When is X said to be separable? Show that a topological space being separable is a topological property. Is being separable a hereditary property? Justify your answer.

7. (a) Show that closed subsets of a compact space is compact. Also prove that compact subset of a Hausdorff space is closed.

   (b) State Tube Lemma and hence show that cartesian product of two compact topological spaces is compact.

8. (a) Define limit point compact space . Show that if a topological space X is compact then X is limit point compact.

   (b) Prove that every closed and bounded interval in $\mathbb{R}$ is compact.

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M.SC. (MATHEMATICS) PART-I

Complex Analysis (R-2016)

(JUNE - 2019)

rs) **[Total marks: 80]**

1) Attempt **any two** questions from **each section.**
2) All **questions** carry **equal marks**.
3) Answer to Section I and section II should be written in the same answer book.

SECTION-I (Attempt any two questions)

Q1 a) If $a_n \neq 0$ for all but finitely many values of n then prove that the radius of convergence R of $\sum_{n=0}^{\infty} a_n z^n$ is related by $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \frac{1}{R} \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$. In particular, if $\lim \left|\frac{a_{n+1}}{a_n}\right|$ exist, then $\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |a_n|^{1/n}$. **10**

b) Find the domain of region of convergence of the following power series $\sum_{n=1}^{\infty} \left(\frac{(iz-1)}{3+4i}\right)^n$. **10**

Q2 a) Prove that the Mobius Transformation takes circles onto circles. **10**

b) Find the bilinear transformation which maps the points $z = \infty, i, 0$ onto the points $0, i, \infty$. **10**

Q3 a) Prove that $\log z$ is not continuous on negative real axis. **10**

b) Prove that $u(x, y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + 2$ is a harmonic function. Also find its harmonic conjugate and the corresponding analytic function. **10**

Q4 a) Let γ be such that $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$ be a smooth curve and suppose that f is a continuous function on an open set containing $\{\gamma\}$. Then prove that **10**

(i) $\int_{-\gamma} f(z)dz = -\int_{\gamma} f(z)dz$

(ii) $\left|\int_{\gamma} f(z)dz\right| \leq \int_{\gamma} |f(z)||dz|$

(iii) If $M = \max_{t\in[a,b]} |f(\gamma(t))|$ and $L = L(\gamma)$ (length of γ) then $\left|\int_{\gamma} f(z)dz\right| \leq ML$

b) Evaluate $\int_{1-i}^{2+i} (2x + iy + 1)\,dz$ along **10**

(i) The straight line joining (1 - *i*) to (2 + *i*).

(ii) Along the curve whose parametric equation is $x = t + 1, y = 2t^2 - 1$.

SECTION-II (Attempt any two questions)

Q5 a) State and prove Cauchy's Integral Formula. **10**

b) Evaluate $\int_C \frac{\sin \pi z + \cos \pi z}{(z-1)(z-2)} dz$, using Cauchy's Integral Formula where C is the circle $|z| = 3/2$. **10**

Q6 a) State and prove Identity theorem. **10**

b) Let $f(z) = e^z$ and $T = \overline{B(2+3i,1)}$. Find the point in T at which $|f|$ attains its maximum value. **10**

Q7 a) State and prove Casorti Weiestrass theorem. **10**

b) Find all the possible Laurent Series expansions of $f(z) = \frac{z^2-1}{z^2+5z+6}$. **10**

Q8 a) State and prove Rouche's theorem. **10**

b) Use the residue theorem to evaluate $\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2+4} dx$. **10**

M.SC. (MATHEMATICS) PART-I
Discrete Mathematics & Differential Equations (R-2016)
(JUNE - 2019)

urs) [Total marks: 80]

Instructions:

1) Attempt any two questions from each section.
2) All questions carry equal marks.
3) Answer to Section I and section II should be written in the same answer book.

SECTION-I (Attempt any two questions)

Q.1] A] Prove that ' if $(a,m) = d$, then a Diophantine equation $ax + my = b$ is solvable if and only if $d|b$'. [10]

B] Use Cardanos method to find the roots of the cubic equation $64x^3 - 48x^2 + 12x - 1$. [10]

Q.2] A] Define Derangement on $'n'$. Prove that $D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \ldots\ldots + (-1)^n \frac{1}{n!}\right]$ by using the principle of inclusion and Excusion. [10]

B] If $S(n,k)$ denotes the Striling number of second kind, then prove that

$$S(n,k) = \sum_{i=0}^{k} (-1)^i \times \binom{k}{i} (k-i)^n, \quad k \le n.$$ [10]

Q.3] A] Given 5 points in the plane with integer coordinates, show that there exists a pair of points whose midpoint also has integer coordinates. [10]

B] Let m and n be relative prime positive integer then prove that the system $x \equiv a(mod m)$ and $x \equiv (mod n)$ has a solution. [10]

Q.4] A] Let a simple graph $G(v,e)$ is connected then prove that $v \le e + 1$. [10]

B] Find the disjunctive normal to the function $F(x,y,z) = (x \vee y) \wedge \bar{z}$. [10]

SECTION-II (Attempt any two questions)

Q.5] A] Show that solution matrix $\emptyset$ of $y' = A(x)y$ is fundamental matrix if and only if $\det(\emptyset) \neq 0$. [10]

B] Solve $\frac{dx}{dt} = 2x + 4y,\ \ \frac{dy}{dt} = x - y,\ x(0) = 4,\ y(0) = 5.$ [10]

Q.6] A] Show that $\emptyset_{1,} u_2\emptyset_1, \dots\dots, u_n\emptyset_1$ is a basis for the solutions of $L_n(y) = 0$ on I, where

$\emptyset_1(x) \neq 0$ on I, and $v_k = u'_k$ $(k = 1,2, \dots\dots, n)$ are the linearly independent solutions

of $\emptyset_1 v^{(n-1)} + \cdots \dots + \left(n\emptyset_1{}^{(n-1)} + a_1(n-1)\emptyset_1{}^{(n-2)} + \cdots \dots + a_{n-1}\emptyset_1\right)v = 0.$ [10]

B] Verify that $\emptyset_1(x) = x,\ (x > 0)$ is a solution of $x^2y'' - xy' + y = 0$, hence find other Linearly independent solution. [10]

Q.7] A] Let $u(x)$ be any non-trivial solution of $u'' + q(x)y = 0$ where $q(x) > 0\ \forall x > 0$

if $\int_1^\infty q(x)dx = \infty$, than show that $u(x)$ has infinitely many zero on positive x-axis. [10]

B] Solve Bessel's equation $x^2y'' + xy' + (x^2 - p^2)y = 0,\ p \geq 0.$ [10]

Q.8] A] Solve the Cauchy problem $(x^2 + 1)u_x + \frac{2xy}{x^2+1}u_y = 2xu,\ y > 0,\ u(0, y) = \log y.$ [10]

B] For the PDE $u_x^2 + u_y^2 = u^2$. i) Find the characteristic strips and ii) The integral surface $z = u(x, y)$ passing through the circle $x = \cos S,\ y = SinS,\ z = 1.$ [10]

M.SC. (MATHEMATICS) PART-I
Set theory, Logic & Elementary Probability Theory (R-2016) (JUNE - 2019)

Duration: 3 Hours **Marks: 80**

N.B. 1) Attempt any two questions from section - I and any two questions from section - II.
2) All questions carry equal marks.

SECTION-I (Attempt any two questions)

1. a) Prove or disprove : 10
i) (A ∩ B) × C = (A × C) ∩ (B × C)
ii) A – (B ∩ C) = (A–B) ∩ (A–C)

b) Show that the relation of congruence modulo m, $a \equiv b(mod\ m)$ in the Set of integers $\mathbb{Z}$ is an equivalence relation. 7

c) Let p and q be propositions : 3
i) Determine whether p ∨ q and ¬ p→ q are logically equivalent.

2. a) If B is non-empty set then show that following statements are equivalent. 10
1) B is countable
2) There is surjective function f : Z_+ → B
3) There is injective function g : B → Z_+.

b) Show that the mapping $f: \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by $f(x) = x^2, x \in \mathbb{Z}^+$ where $\mathbb{Z}^+$ is the set of positive integers is one-one and into. 5

c) Prove or disprove if f and g are injective then g ∘ f is injective. 5

3. a) By using mathematical induction prove that every integer n ≥ 2 is a product of prime numbers.(Here a prime number is itself to be viewed as a one – factor product of prime numbers.) 10

b) Define the following terms in partially ordered set : 5
i) Chain (totally ordered set)
ii) Maximal element
iii) Upper bound

c) Using mathematical induction prove that if n≥ 3 then 2n-1 > 1. 5

4. a) Show that every permutation of a finite set can be written as a cycle or as a product of disjoint cycles. 10

b) What is the order of each of the following permutations? 5
i) (124)(35)
ii) (124)(356)
iii) (124)(3578)

c) If β = (1 2 5 7 3 6) then compute β^{122}. 5

SECTION-II (Attempt any two questions)

5. a) If $\{A_j\}$ is increasing sequence of events of space (Ω, C) then show that $\lim_{n\to\infty} P(A_n) = P(\cup_{n=1}^{\infty} A_n)$ 5

b) In a random arrangement of alphabets in word CHILDREN, find probability
that i) All vowels are together. ii) No two vowels are together. 5

c) Let $\{F_i : i\epsilon\ I\}$ be a collection of sigma-fields of subsets of Ω. Prove or disprove 6
i) $\cap_{i=1}^{\infty} F_i$ is a sigma-field.

ii) $\bigcup_{i=1}^{\infty} F_i$ is a sigma-field.

d) A class C of subsets of $\sum$, such that either A or $\bar{A}$ are countable. Is C a sigma field? 4

6. a) If A, B, C are independent events then show that 6
i) $A, B \cap C$ are independent.
ii) $A, B \cup C$ are independent.
iii) $A, B - C$ are independent.

b) Find $P(A)$ if $f(x) = \frac{1}{4}\mathbb{I}_{[-1,4]}(x)$ is a density function and P is absolutely continuous probability measure with respect to it. Where $A = (-1, 0.5]$. 4

c) Let $(\mathbb{R}, \mathcal{B}, m)$ be a measure space and $P(A) = \int_A f\, dm$ f is a density function. Show that P is a probability measure on Borel subset A of $\mathbb{R}$. 6

d) Define conditional probability of A given B, where A and B are events in a probability space. Also show that: 4
$P(A \cup B/C) = P(A/C) + P(B/C) - P(A \cap B/C)$

7. a) Let X and Y be two independent random variables with binomial distribution $B(m,p)$ and $B(n,p)$ respectively. What is the distribution of $X + Y$. 6

b) Let X be a simple random variable and $E(X)$be the expectation of X .Show that if $X \geq 0$ and $E(X) = 0$ then $P(\{X = 0\}) = 1$. 4

c) The joint p.d.f of X,Y is f(x, y) = 2 for 0 < x < y <1 ; find conditional p.d.f of X given Y. 5

d) X has Binomial with (n = 8, p=0.4). Find mean and variance if X. 5

8. a) For any r.v.s X, Y show that (a) $E^2[XY] \leq E[X^2]E[Y^2]$ (b)EyE[X/Y = y] = E[X]. 6

b) Show that if non-negative random variable X has finite expectation $E(X)$, then $P(\{X \geq C\}) \leq \frac{E(X)}{C}$ for any $C \geq 0$. 4

c) State the Chebyshev inequality. A fair coin is tossed independently n times. Let S_n be the number of heads obtained. Find a lower bound of the probability that $\frac{S_n}{n}$ differs from ½ by less than 0.1 when $n = 1000$. 6

d) Examine whether the Strong law of large numbers holds for sequence of independent r.v.'s 4

$$\{X_k\}: X_k = \begin{cases} \pm k & \text{with probability } \frac{1}{2\sqrt{k}} \\ 0 & \text{with probability } 1 - \frac{1}{\sqrt{k}} \end{cases}.$$
