(3 Hours) **[Total Marks: 100]**

Note: (i) All questions are compulsory.
(ii)Figures to the right indicate marks for respective parts.

Q.1 Choose correct alternative in each of the following (20)

$i.$ Let T: V →V' be a linear transformation. ker(T) is a subspace of

(a) V (b) V'
(c) V and V' (d) None of the above.

$ii.$ Let $T:\mathbb{R}^2\to \mathbb{R}^2$ such that T(x ,y)=(x,0) be the linear transformation. Then, ker(T) is

(a) X –axis (b) Y-axis
(c) {(0,0)} (d) None of the above

$iii.$ If $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$ then E^{-1} is

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$ (d) None of the above.

$iv.$ Which one of the following is **NOT TRUE**

(a) $Det(A^t) = Det\ A$ (b) $Det(A+B) = DetA + DetB$
(c) $Det(AB) = DetA\ DetB$ (d) $Det(A^{-1}) = (Det\ A)^{-1}$, when A is invertible

$v.$ Let A be a matrix then which of the following is **NOT TRUE**

(a) If any of the row of a matrix A is zero then its determinant is zero

(b) If i^{th}row is multiplied with non-zero α then its determinant does not change.

(c) If $i^{th}$and j^{th}rows of A are interchanged then determinant changes by a sign, for any $i \neq j$

(d) If the rows are linearly dependent then its determinant is zero.

vi. $Det(2e_{2,}, e_1+5e_2, -e_3)$ wheree_1,e_2, e_3 are standard basis elements of $\mathbb{R}^3$ is

(a) -10 -2

(c) $+2$ $+10$

vii. If $I_{31} \in M_3(\mathbb{R})$ then I_{31} is

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

viii. Let V be a finite dimensional inner product space and W be a subspace of V and $W^{\perp}$ be the orthogonal complement of W in V. If dim $V = n$, dim $W = r$, then dim $W^{\perp}$ is

(a) r (b) $n - r$

(c) n (d) None of these

ix. Let $e_1 = (1,0)$and $e_2 = (0,1)$. Consider $S = \left\{\frac{e_1+e_2}{\sqrt{2}}, \frac{e_1-e_2}{\sqrt{2}}\right\}$ in $\mathbb{R}^2$ with dot product. Then

(a) S is not a basis of $\mathbb{R}^2$

(b) S is a basis of $\mathbb{R}^2$but not orthogonal.

(c) S is an orthogonal basis of $\mathbb{R}^2$ but not orthonormal

(d) S is an orthonormal basis of $\mathbb{R}^2$

x. Consider the following sets of vectors in $\mathbb{R}^2$ under dot product.

(i) $S_1 = \{(0,1), (2,0)\}$ (ii) $S_2 = \left\{\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}$

(iii) $S_3 = \{(0,-1), (0,1)\}$ (iv) $S_4 = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\}$

(a) All the sets are orthogonal sets (b) S_2 and S_4 are orthogonal sets

(c) S_1 and S_2 are orthogonal sets (d) S_1 and S_3 are orthogonal sets

Q2. Attempt any **ONE** question from the following: (08)

a) i. State and prove Rank-Nullity theorem.

ii Prove that Elementary matrices are invertible and their inverses are also elementary matrices.

Q.2 Attempt any **TWO** questions from the following: (12)

b) i. Let T :V→V' be a linear transformation. Prove that T is one-one if and only if kerT={0_v}

ii If T:V→W is linear transformation and B={ v_1, v_2, ……..v_n} is linearly independent subset of V and kerT ={0} then show that { T(v_1), T(v_2) ,………T(v_n)} is linearly independent subset of W.

iii Find the rank of a matrix A where A=$\begin{bmatrix} 1 & 3 & 4 & 1 \\ 2 & 6 & 8 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$.

iv Let T:$\mathbb{R}^2$→ $\mathbb{R}^2$ T(x,y,z)=(x,2y+z) be a linear transformation. Find the matrix of this linear transformation w.r.t standard basis.

Q3. Attempt any **ONE** question from the following: (08)

a) i. Let $\phi: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a bilinear function such that $\phi(A^1, A^1) = 0, \forall A^1 \in \mathbb{R}^2and\phi(E^1, E^2) = 1$ whereE^1, E^2 are the standard unit vectors of $\mathbb{R}^2$. Prove that $\phi(A^1, A^2) = \det(A^1, A^2)$ for any column vectors $A^1, A^2 \in \mathbb{R}^2$. Further prove that determinant of a $n \times n$ lower triangular matrix is product of its diagonal elements.

ii. Let $A \in M_n(\mathbb{R})$ then prove that $AX = 0$ has a non zero solution $\Leftrightarrow$ det A = 0

Q3. Attempt any **TWO** questions from the following: (12)

b) i. Let A be an $n \times n$ invertible matrix. Prove that $\det(adj\ A) = (\det A)^{n-1}$. Further using adjoint, find inverse of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $ad - bc \neq 0$.

ii. Find the determinant of the following matrices using definition and its properties

$$\begin{pmatrix} 1 & t & t^2 & t^3 \\ 1 & a & a^2 & a^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{pmatrix}$$

iii. State the Laplace expansion formula for determinant.
Use Laplace expansion to find the determinant of the following matrix

$$\begin{pmatrix} 1 & -1 & 3 & 1 \\ -1 & 2 & -1 & 0 \\ 2 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

iv. State Cramer's Rule for a $n \times n$ linear system. Use it to find solution to

$$\begin{pmatrix} 1 & 4 & 2 \\ 3 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 15 \\ 10 \\ 5 \end{pmatrix}$$

Q4. Attempt any **ONE** question from the following: (08)

$a)$ i. Let V be an inner product space. Prove that $\|x+y\| = \|x\| + \|y\|$ if and only if $x = \alpha y$ or $y = \alpha x$ for $\alpha \geq 0$.

ii. Let W be a subspace of a finite dimensional inner product space Vover $\mathbb{R}$. Show that
$(p) W^{\perp}$ is a subspace of V
$(q) (W^{\perp})^{\perp} = W$

Q4. Attempt any **TWO** questions from the following: (12)

$b)$ i. Show that $(C[a,b], \langle , \rangle)$ the space of continuous real valued functions on $[a,b]$ is an inner product space where $\langle f,g \rangle = \int_a^b f(t)g(t)dt$ for $f,g \in C\,[a,b]$.

ii. Let V be a real inner product space and u be an unit vector in V. If $P_u(x)$ denotes the projection of x along u, show that

$$\|x - P_u(x)\| \leq \|x - \alpha u\| \ \forall\, \alpha \in \mathbb{R}.$$

iii. Define angle between two vectors in an inner product space. Find the angle between the vectors $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and B $= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ using $\langle A,B \rangle = ae + 2bf + 3cg + 4dh$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$.

iv. Using Gram-Schmidt Process, find orthonormal set corresponding to $S = \{(-1,2,0), (1,5,7)\}$ in $\mathbb{R}^3$ with dot product.

Q5. Attempt any **FOUR** questions from the following: (20)

a) Let T:U→V and S:V→ W be linear transformations. Prove that $S \circ T$ is also a linear transformation.

b) Verify Rank-Nullity theorem for the following linear transformation
T: $\mathbb{R}^2$→:$\mathbb{R}^2$ T(x,y) =(2x+y,3x-y)

c) Prove that det(AB) = detAdetB for A,B∈ $M_n(\mathbb{R})$

d) Find the inverse of the following matrix using adjoint
$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

e) Show that $\left\{\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\left(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)\right\}$ is an orthonormal set in $\mathbb{R}^3$ with respect to dot product and extend it to an orthonormal basis of $\mathbb{R}^3$.

f) Verify Cauchy-Schwarz inequality for the functions $f(x) = x$ and $g(x) = x^3$ in $C[-1,1]$ using $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$

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