(3 Hours) **[Total Marks : 100]**

Q.1 Choose correct alternative in each of the following: (20)

i. Let T: V →V' be a linear transformation, then ker(T) and Im(T) are subspaces of

(a) V and V' respectively (b) V' and V respectively

(c) V (d) V'

ii. Let T be the linear transformation that T(1, 0) = (4, 3) and T(0, 1) = (7, 2). Then T(3,-2) =

(a) (12,-6) (b) (-14,-4)

(c) (11,5) (d) None of the above

iii. If $T:\mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $[m(T)]_B^B = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, where $B = \{e_1, e_2\}$ is the standard basis of $\mathbb{R}^2$ then $[m(T)]_{B'}^{B'} =$ ______ , where $B' = \{e_2, e_1\}$

(a) $\begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ (d) None of the above.

iv. Consider the following statements:

p. Every elementary matrix is invertible.

q. Every matrix **A** can be expressed as a product of elementary matrices.

r. Row rank and column rank of a matrix are equal.

Then

(a) only (p) is true (b) (p) and (r) are true.

(c) All are true. (d) None of the above.

v. For a homogeneous system of m equations in n unknowns **AX = 0**, the dimension of solution space is

(a) $m - rank\ A$ (b) $rank\ A - m$

(c) $n - rank\ A$ (d) $rank\ A$

vi. Adjoint of $\begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}$ is

(a) $\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}$ (d) None of the above

vii. Determinant of $\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 6 & -1 \\ 2 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \\ 3 & 1 & 1 & 7 \end{pmatrix}$ is

(a) 5 (b) 10
(c) 15 (d) None of the above

viii. The order of the group $U(22)$ is

(a) 7 (b) 10
(c) 14 (d) 9

ix. Which of the following is cyclic?

(a) S_3(Permutation Group) (b) V_4 (Klein -4 Group)
(c) μ_5(Group of Fifth roots of unity) (d) None of the above

x. In the group $(\mathbb{Z}_{18}, +)$, order of $\bar{5}$ is

(a) 1 (b) 5
(c) 18 (d) None of these

Q.2 a) Attempt any ONE question from the following: (08)

i. State and prove Rank-Nullity theorem.

ii. Show that the vector space of all polynomials in x of degree less than or equal to n is isomorphic to $\mathbb{R}^{n+1}$

b) Attempt any TWO questions from the following: (12)

i. Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ $T(x,y) = (x+y,x-y)$ be a linear transformation. Check whether T is one-one.

ii. Let $T: \mathbb{R}^3 \to \mathbb{R}^2, T(x,y,z) = (x+z, 2x+y)$ be a linear transformation. Find the matrix of this linear transformation with respect to the standard basis.

iii. Find ker(T) and Im(T) for the linear transformation $T: \mathbb{R}^2 \to: \mathbb{R}^2$ given by $T(x, y) = (x+y, x-y)$.

iv. Let V be a finite dimensional vector spaces over $\mathbb{R}$, $S, T : V \to V$, be linear maps and $B = \{e_1, e_2, ..., e_n\}$ be the basis of V, then prove that $m(S \circ T) = m(S) \cdot m(T)$

Q.3 a) Attempt any ONE question from the following: (08)

i. Prove that an elementary matrix is invertible.

ii. State and prove the Cramer's rule for $n \times n$ linear system $AX = b$.

b) Attempt any TWO questions from the following: (12)

i. Express $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix}$ as a product of elementary matrices.

ii. Define the rank of a matrix, hence find rank of $\boldsymbol{A} = \begin{pmatrix} 5 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

iii. Let A be an $n \times n, n > 1$ invertible matrix. Prove that $\det(adj\ A) = (\det A)^{n-1}$. Further using adjoint, find the inverse of $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$

iv. State the Laplace expansion formula for determinant. Use Laplace expansion to find the determinant of the matrix

$$\begin{pmatrix} 1 & -1 & 3 & 1 \\ -1 & 2 & -1 & 0 \\ 2 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Q.4 a) Attempt any ONE question from the following: (08)

i. Define Subgroup. Let G be a group and H, K be subgroups of G. Prove that $H \cap K$ is a subgroup of G but $H \cup K$ may not be a subgroup of G.

ii. Let G be an abelian group and $a, b \in G$ be such that $o(a) = n, o(b) = m$ and $(m, n) = 1$. Prove that $o(ab) = o(a)o(b)$.

b) Attempt any TWO questions from the following: (12)

i. Define Group. Prove that a group G is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}, \forall\ a, b \in G$.

ii. Show that $H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| a, b \in \mathbb{R}, a > 0 \right\}$ is subgroup of $GL(2, \mathbb{R})$ under multiplication.

iii. Construct composition table of μ_4 (group of 4^{th} roots of unity). Also find order and inverses of all the elements of μ_4.

iv. Explain all six symmetries of an equilateral triangle.

Q.5 Attempt any FOUR questions from the following: (20)

a) Let $T: V \rightarrow V'$ be a bijective linear transformation. Prove that $T^{-1}: V' \rightarrow V$ is also a linear transformation.

b) Check whether $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $T(x,y,z) = (x+y, x-z, y+2z)$ is an isomorphism?

c) Determine whether the following system of linear equations is consistent.

$$2x + 3y + z = 3$$
$$x - 2y + 3z = 5$$
$$3x + y + 4z = 4$$

d) State Cramer's Rule for a $n \times n$ linear system. Use it to solve

$$\begin{pmatrix} 1 & 4 & 2 \\ 3 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 15 \\ 10 \\ 5 \end{pmatrix}$$

e) Prove that $G = \{2^n | n \in \mathbb{Z}\}$ is a group under multiplication.

f) Define the order of an element in a group and find the order of $\sigma = (1\ 2\ 3\ 4)(2\ 3\ 4\ 5)(3\ 4\ 5\ 6)(4\ 5\ 7\ 8) \in S_8$.
