(3 Hours) **[Total Marks: 100]**

Note: (i) All questions are compulsory.

(ii) Figures to the right indicate marks for respective parts.

Q.1 Choose correct alternative in each of the following (20)

$i.$ Rank Nullity Theorem states that if $T:V \to W$ is a linear transformation, then

(a) Dim V = dim(Im T) + dim$W$
(b) Dim V - dim(Im T) = dim (ker T)
(c) Dim V = dim W
(d) Dim V/W = dim$V$ - dim W

$ii.$ Which of the following is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$?

(a) $T(x,y)=(x - y , y)$
(b) $T(x,y)=(|x| , y + 1)$
(c) $T(x,y)=(x^2 + y, x - y)$
(d) All the above

$iii.$ Let A be a $m\times n$ matrix and let row rank = p and column rank = q. Then the relation p and q is

(a) $p = q$
(b) $p > q$
(c) $p < q$
(d) Cannot be determined

$iv.$ Let $E = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ then E^{-1} is

(a) $\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
(b) $\begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
(c) $\begin{pmatrix} 1 & 0 & 1/5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
(d) $\begin{pmatrix} 1 & 0 & -1/5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$v.$ A $m \times n$ non-homogeneous system of linear equations $AX = b$ has a solution if and only if

(a) $Rank\ A = Rank[A|b]$
(b) $Rank\ A > Rank[A|b]$
(c) $Rank\ A < Rank[A|b]$
(d) none of these

$vi.$ The rank of the identity matrix of order n is

(a) $n - 1$
(b) n
(c) $n + 1$
(d) None of these

$vii.$ Which of the following is **not true**

(a) $\det(AB) = \det(A^t B^t)$
(b) $\det(AB) = \det(BA)$
(c) $\det(AB) = \det(AB^t)$
(d) $\det(AB) = \det(A) + \det(B)$

$viii.$ In a group G, $(a^{-1}b)^{-1} =$

(a) ab^{-1}
(b) $b^{-1}a$
(c) $a^{-1}b$
(d) ba^{-1}

$ix.$ Consider the pairs (i) $(\mathbb{N}, +)$, (ii) $(\mathbb{R}^+, \times)$ and (iii) $(\mathbb{Q}^*, \times)$ then

(a) (i), (ii) & (iii) are groups.
(b) Only (iii) is a group.
(c) (ii) & (iii) are groups.
(d) None of the above

x. In an abelian group G which of the following is true?

(a) $a = a^{-1}, \forall a \in G$ (b) $a = a^2, \forall a \in G$

(c) $(ab)^2 = a^2b^2, \forall a, b \in G$ (d) None of the above

Q2. Attempt any **ONE** question from the following: (08)

$a)$ i. Define p) A linear transformation q) kernel of a linear transformation. Further, prove that if $T : V \to V'$ is a linear transformation then T is injective if and only if $\ker T = \{0\}$.

ii. Let V be a finite dimensional vector spaces over $\mathbb{R}$, S, T: $V \to V$, be linear maps and $B = \{e_1, e_2, ..., e_n\}$ be the basis of V, then prove that $m(S \circ T) = m(S) \cdot m(T)$

Q.2 Attempt any **TWO** questions from the following: (12)

$b)$ i. If $F : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $F(x,y,z) = (x+y+z, x+2y-z, 3x+5y-z)$, show that F is a linear transformation. Find whether F is non-singular.

ii. If $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(x,y,z) = (x, 2y, 0)$, find $\ker T$, basis of $\ker T$ and nullity T.

iii. Find the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, such that $T(1 , 2) = (2 , 3)$ and $T(0, 1) = (1 , 4)$

iv. $P_3[\mathbb{R}]$ denote the vector space of all polynomials over $\mathbb{R}$ of degree atmost 3 and D denote the differentiation mapping. Consider the basis $B = \{1, 1+x, 1+x^2, 1+x^3\}$ of $P_3[\mathbb{R}]$. Find $[m(D)]_B^B$.

Q3. Attempt any **ONE** question from the following: (08)

$a)$ i. Define an elementary matrix and show that it is invertible.

ii. Define a bilinear function. Further, let $\phi : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a bilinear function with $\phi(A^1, A^1) = 0, \forall A^1 \in \mathbb{R}^2$ and $\phi(E^1, E^2) = 1$ where E^1, E^2 are the standard unit vectors of $\mathbb{R}^2$. Then, show that $\phi(A^1, A^2) = \det(A^1, A^2)$ for any column vectors $A^1, A^2 \in \mathbb{R}^2$.

Q3. Attempt any **TWO** questions from the following: (12)

$b)$ i. Use Cramer's rule to solve the following system

$$2x + y + z = 3, x - y - z = 0, x + 2y + z = 0$$

ii. Let $A \in M_{m\times n}(\mathbb{R})$ and $T: \mathbb{R}^n \to \mathbb{R}^m, T(X) = AX, \forall X \in \mathbb{R}^n$ be a linear transformation. Show that Rank T=Rank A

iii. Define adjoint of a matrix. Find $\boldsymbol{A}^{-1}$ for $\boldsymbol{A} = \begin{pmatrix} 0 & -3 & -2 \\ 1 & -4 & -2 \\ -3 & 4 & 1 \end{pmatrix}$ using adjoint method.

iv. Express the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ as product of elementary matrices.

Q4. Attempt any **ONE** question from the following: (08)

$a)$ i. Find the cube roots of unity and show that they constitute a group under multiplication.

ii. Let G be a group. For any $a, x \in G$, prove that $(xax^{-1})^n = xa^n x^{-1}, \forall n \in \mathbb{Z}$

Q4. Attempt any **TWO** questions from the following: (12)

$b)$ i. Let G be a group. Prove that $o(a) = o(bab^{-1}), \forall a, b \in G$.

ii. Prove that $\mathbb{Z}_6$ is a group under addition modulo 6. Is it a group under multiplication modulo 6? Justify your answer.

iii. Let H and K be subgroups of a group$G$. Prove that $HK = \{hk | h \in H, k \in K\}$ is a subgroup of G if and only if $HK = KH$.

iv. Let H and K be subgroups of a group G.

(p) Prove that $H \cap K$ is a subgroup of G.

(q) Show that $H \cup K$ need not be a subgroup of G.

Q5. Attempt any **FOUR** questions from the following: (20)

$a)$ Prove that inverse (if it exist) of linear transformation is also a linear transformation.

$b)$ Let $T : V \rightarrow W$ be a linear transformation. Show that ker T is a subspace of V.

$c)$ For $A, B \in M_n(\mathbb{R})$, if A is invertible show that

I) $\det(A^{-1}) = (\det A)^{-1}$

II) $\det(ABA^{-1}) = \det B$

III) $\det(A^t B^t) = \det A . \det B$

$d)$ Find Rank of a matrix $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & -1 & -1 \end{pmatrix}$. What can you say about the rank of the matrix, which is obtained from A by interchanging 2nd and 3rd columns of A?

$e)$ Let G be a group and $a \in G$ such that $o(a) = n$. Then show that $a^m = e$ if and only if $o(a)$ divides m.

$f)$ Define the centre of a group $G$and show that it is a subgroup of G.

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