(3 Hours) [Total Marks : 100]

N.B. **1**. All questions are compulsory.

2. Figures to the right indicate marks for respective parts

Q.1 Choose correct alternative in each of the following: (20)

i. The set $S = \{(x,y) \in \mathbb{R}^2 \ / \ 1 < x^2 + y^2 < 2\}$ is

(a) A closed set (b) Neither Open nor closed set

(c) An open set (d) None of these

ii. Let $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & if\ (x,y) \neq (0,0) \\ 0 & otherwise \end{cases}$ and let

$l_1 = \lim_{x \to 0} \lim_{y \to 0} f(x,y)$ and $l_2 = \lim_{y \to 0} \lim_{x \to 0} f(x,y)$. Then

(a) $l_1 = l_2$ (b) $l_1 \neq l_2$

(c) f is continuous at $(0,0)$ (d) None of these

iii. Let $x_n: \mathbb{N} \to \mathbb{R}^3$ be defined by $x_n = \left(\frac{1}{n}, 2n, \frac{1}{2n^2}\right)$ then x_n is_______

(a) Convergent (b) Divergent

(c) Bounded (d) None of these

iv. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are bounded .Then

(a) f may or may not be continuous at all points (b) f is continuous at all points

(c) f is differentiable at all points. (d) None of these

v. If $f: \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function such that $\frac{\partial f}{\partial y} = 0$.Then

(a) f is independent of x and z (b) f depends on x and z only

(c) f is constant (d) None of these.

vi. Which of the following is the level set of $f(x,y,z) = x^2 + y^2 + z^2$ for $k = 9$?

(a) Sphere of radius 3 centered at origin (b) Circle of radius 3 centered at origin

(c) Sphere of radius 4 centered at origin (d) Sphere of radius 2 centered at (2,0,0)

vii. Let A: Every partial derivative is the directional derivative in the direction of unit coordinate vector.

B: Every continuous scalar field is differentiable.

Then which of the following is true?

(a) A is true, B is false. (b) A is false, B is true.

(c) Both A & B are true. (d) Both A & B are false.

viii. The linear approximation of $f(x,y) = x\sqrt{y}$ at $(1,1)$ is

(a) $x + \frac{y}{2} - \frac{1}{2}$ (b) $x + \frac{y}{2} - 1$

(c) $x + \frac{y}{2} - \frac{3}{2}$ (d) $x + y - \frac{1}{2}$

ix. Let $A = f_{xx}(a,b), B = f_{xy}(a,b), C = f_{yy}(a,b)$. Then (a,b) is a saddle point of $f(x,y)$ if

(a) $AC < B^2$ (b) $AC = B^2$

(c) $AC > B^2$ (d) None of these

x. The minimum value of $f(x,y) = x^2 + y^2$ where $x + y = 1$ is

(a) 0 (b) 1

(c) $\frac{1}{2}$ (d) None of these

Q.2 a) Attempt any ONE question from the following: (08)

i. Let $f, g: \mathbb{R}^n \to \mathbb{R}$ be two real valued functions. Let $a \in \mathbb{R}^n$ such that $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$. Then prove that

(I) $\lim_{x \to a} (f-g)(x) = l - m$

(II) $\lim_{x \to a} (\lambda f)(x) = \lambda l$, $\lambda \in \mathbb{R}$.

ii. State and prove Mean value theorem for a real valued function of n variables.

b) Attempt any TWO questions from the following: (12)

i. Using $\epsilon - \delta$ definition to show that

$$\lim_{(x,y)\to(1,1)} \frac{xy - y - 2x + 2}{x - 1} = -1$$

ii. Define continuity of $f: \mathbb{R}^n \to \mathbb{R}$ at $a \in \mathbb{R}^n$. If f is continuous at a, then show that

(a) $\exists\, \delta > 0$ such that f is bounded on $B(a, \delta)$.

(b) $|f|$ is continuous at a. Explain with an example that converse of this is not true.

iii. Define directional derivative of a scalar field f at a point a in the domain in the direction of u. Calculate the directional derivative of the function $f(x,y,z) = 3x^2 - 3y^2 + 3z^2$ at $(1,2,3)$ in the direction of $(0,1,0)$ using the definition and also using the relationship between directional derivative and partial derivative.

iv. Let $f: \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$. Define $D_i f(a)$, the i-th partial derivative of f at $a, 1 \le i \le n$. Determine whether the partial derivatives of f exist at $(0,0)$. For the following function. In case they exist, find them.

$$f(x,y) = \begin{cases} \dfrac{x^3 y - x y^3}{x^2 + y^2} & if\ (x,y) \neq (0,0) \\ 0 & if\ (x,y) = (0,0) \end{cases}$$

Q.3 a) Attempt any ONE question from the following: (08)

i. Suppose $f: U \to \mathbb{R}$, where U is an open set in $\mathbb{R}^n$.Show that if f is differentiable at $a \in U$ then for any direction $u \in \mathbb{R}^n$, $D_u f(a) = Df(a)(u)$.

ii. Prove that a differentiable scalar field is continuous.

b) Attempt any TWO questions from the following: (12)

i Find total derivative as linear transformation T for the function $f(x,y,z) = e^{x+y+z}$ at point $a = (0,0,0)$

ii. Find directional derivative of $f(x,y,z) = x^2 + y^2 - z^2$ at $(3,4,5)$ along the curve of intersection of two surfaces $S_1: 2x^2 + 2y^2 - z^2 = 25$ and $S_2: x^2 + y^2 = z^2$

iii. Find the equation of the tangent plane and normal line to the surface $x^3 + 7x^2 z + z^3 = 4$ at $(2,1,-2)$.

iv. Check whether the second order mixed partial derivatives are equal, for each of the following functions.

1. $f(x,y) = x^3 + xy^2 - 5xy$
2. $f(x,y) = \sqrt{xy}$

Q.4 a) Attempt any ONE question from the following: (08)

i. Let U be an open set in $\mathbb{R}^n$ and $f: U \to \mathbb{R}^n$ be given by $f(x) = (f_1(x), f_2(x), \dots f_m(x)), \forall x \in U$. Prove that f is differentiable at $a \in U$ if and only if each f_i is differentiable at a and for any $u \in \mathbb{R}^n$.

$$Df(a)(u) = (Df_1(a)(u), Df_2(a)(u), \dots, Df_m(a)(u))$$

ii. Let $f: S \subseteq \mathbb{R}^n$ be a scalar field where S is a non-empty open subset of $\mathbb{R}^n$. Let $a \in S$ and f is differentiable at a. Prove that if f has a local maximum or local minimum at a then $\nabla f(a) = 0$.

b) Attempt any TWO questions from the following: (12)

i. Define $Df(a)$, the total derivative at $a \in \mathbb{R}^n$ for a function $f: \mathbb{R}^n \to \mathbb{R}^m$ in terms of a linear transformation. Show that if f is differentiable at a then f is continuous at a.

ii. Given $u = f(x, y)$ has continuous second order partial derivatives w.r.t. x and y, if $x = r\cos\theta$, $y = r\sin\theta$, Show that $u_x^2 + u_y^2 = u_r^2 + \frac{1}{r^2} u_\theta^2$

iii. Find the critical points, saddle points and local extrema ,if any, for the function $f(x, y) = x^3 + y^3 - 3axy$.

iv. Divide 120 into three parts so that the sum of their product taken two at a time shall be maximum.

Q.5 Attempt any FOUR questions from the following: (20)

a) Show that for the following functions the limit does not exists.

(i) $\lim_{(x,y)\to(0,0)} \frac{x^3 y}{2x^6 + y^2}$ (ii) $\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2 + y^4 + z^4}$

b) For the following function find the real $\theta \in (0,1)$ if it exists satisfying

$f(b) - f(a) = \nabla f\big(a + \theta(b - a)\big) \cdot (b - a)$

$f(x, y, z) = xy + yz + zx$; $a = (0,0,0)$; $b = (1,1,1)$

c) Find the maximum rate of change of the function $f(x, y, z) = \log(x + y + z)$ at $(1,2,3)$. Also find the direction in which maximum rate of change occurs.

d) Find the total derivative of $f(x, y) = 3x^3 y + 7y\sin x + e^{xy}$ at $(1,1)$ using gradient.

e) Determine the second order Taylor's formula for the function
$f(x,y) = e^x \, cosy$ at $(0, \frac{\pi}{2})$.

f) Find the Hessian matrix of $f: \mathbb{R}^3 \to \mathbb{R}$ given by
$f(x,y,z) = 2x^3 + 4xyz + 3y^3 + z^3$ at $(1,1,1)$.

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