(3 Hours) [Total Marks: 100]

Note: (i) All questions are compulsory.

(ii)Figures to the right indicate marks for respective parts.

Q.1 Choose correct alternative in each of the following (20)

$i.$ The number of elements in S_5 is

(a) 5 (b) 24

(c) 120 (d) 60

$ii.$ A permutation in which every element goes to itself is called ___________ Permutation.

(a) cyclic (b) Transposition

(c) identity (d) odd.

$iii.$ What is the number of even permutations in S_3?

(a) 3 (b) 0

(c) 4 (d) 6

$iv.$ The number of functions from a set with m elements to a set with n elements are

(a) m^n (b) n^m

(c) $m \times n$ (d) None of these

$v.$ The number of ways to pick a sequence of two different letters of the alphabet that appear in the word BOAT is

(a) 21 (b) 12

(c) 8 (d) None of these

vi. Let $S(n, k)$ denote the Stirling number of second kind on n-set into k-disjoint nonempty unordered subsets, then $S(n, 1)$ is

(a) n (b) n^2

(c) 1 (d) None of these

vii. How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

(a) 101 (b) 102

(c) 100 (d) None of these

viii. The letters of the word CHORD can be arranged in a row in how many ways?

(a) 120 (b) 48

(c) 24 (d) None of the above

ix. The number of terms in the expansion of $(a + 3b + 7c)^8$ is

(a) $\binom{10}{8}$ (b) $\binom{11}{8}$

(c) $\binom{10}{3}$ (d) None of the above

x. At a party, seven gentlemen check their hats. In how many ways can their hats be returned so that no gentleman receives his own hat?

(a) 7! (b) D_7

(c) $\frac{D_7}{7}$ (d) None of the above

Q2. Attempt any **ONE** question from the following: (08)

a) i. Prove that any permutation can be expressed as a product of transpositions

ii. Define Linear Homogeneous recurrence relation of degree n.

Show that if the characteristic equation $x^2 - a_1 x - a_2 = 0$ of the recurrence relation $h_n = a_1 h_{n-1} + a_2 h_{n-2}$ has two distinct non-zero roots q_1 and q_2 then $h_n = c_1 q_1^n + c_2 q_2^n$ is the general solution of the recurrence relation $h_n = a_1 h_{n-1} + a_2 h_{n-2}$.

Q.2 Attempt any **TWO** questions from the following: (12)

b) i. For the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 5 & 1 & 8 & 3 & 2 & 9 & 6 & 4 \end{pmatrix}$

(I) Express σ in one row notation. (II) Find the inverse of σ.

(III) Express σ as a product of transposition and find the sign of σ.

ii. Let α = (1325)(143)(25) $\in S_5$. Find α^{-1} and express it as a product of disjoint cycles. State whether α^{-1} is even or odd.

iii. A young pair of rabbits (one of each gender) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month. Find a recurrence relation and solve it for the number of pairs of rabbits on the island after *n* months, assuming that no rabbits ever die.

iv. Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$, $a_0 = 1, a_1 = 6$.

Q3. Attempt any **ONE** question from the following: (08)

a) i. Show that $\mathbb{N} \times \mathbb{N}$ is countable

ii. State the recurrence relation for $S(n, k)$ and find the value of $S(6, 3)$. State the value of $S(n, n)$ and $S(n, 0)$.

Q3. Attempt any **TWO** questions from the following: (12)

b) i. Show that [0, 1] ~ (0, 1)

ii. State addition and Multiplication Principles. 3 persons enter into a drama theater. There are 5 chairs are available. How many ways are there to occupy the seats?

iii. Prove by mathematical induction, $S(n, 2) = 2^{n-1} - 1, n \geq 2$

iv. State Pigeonhole principle. In Algebra class, 32 of the students are boys. Each boy knows five of the girls in the class and each girl knows eight of the boys. How many girls are in the class?

Q4. Attempt any **ONE** question from the following: (08)

a) i. Prove by giving a Combinatorial argument:

$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}.$$

ii. State and prove The Inclusion Exclusion principle.

Q4. Attempt any **TWO** questions from the following: (12)

b) i. State and prove The Binomial Theorem

ii. Consider the multiset $S = \{3.a, 2.b, 4.c\}$ of 9 objects of 3 types. Find the number of $8-$permutations of S.

iii. How many numbers from 1 to 500 (both inclusive) are divisible by 4 or 5 or 7?

iv. How many integer solutions are there to equation

$x_1 + x_2 + x_3 + x_4 = 15$ satisfying $x_1 \geq -2,\ x_2 \geq 3,\ x_3, x_4 \geq 1$

Q5. Attempt any **FOUR** questions from the following: (20)

a) Show that two cycles given by the same permutation are either identical or disjoint.

b) Find a recurrence relation for the number of ways to arrange *n* distinct objects in a row. Find the number of arrangements of eight objects.

c) Given eight different English books, seven different French books, and five different German books:

(a) How many ways are there to select one book?

(b) How many ways are there to select three books, one of each language?

d) If 5 points are chosen at random in the interior of a square of side length 2 units, show that at least 1 pair of points has a separation of less than $\sqrt{2}$ units

e) How many 11-letter words can be made using the letters of the word ABRACADABRA?

f) Define a derangement and hence calculate the values of D_2 , D_3 and D_4.

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