Revised **(3 Hours)** **[Total Marks: 80**

N.B.: (1) Attempt any **TWO** questions from each Section.
(2) Figures to the right indicate marks for respective subquestions.
(3) Answers to section I and section II should be written in the same answer book

SECTION - I

1. (a) Let A be a rectangle in $\mathbb{R}^n$. If $f : A \to \mathbb{R}$ is a continuous function then prove that f is integrable on A. (6)

   (b) Let A be a rectangle in $\mathbb{R}^n$ and $f, g : A \to \mathbb{R}$ be integrable on A. For any partition P of A and sub-rectangle S, show that $m_S(f)+m_S(g) \leqslant m_s(f+g)$ and $M_S(f)+M_S(g) \geqslant M_s(f+g)$. Deduce that $L(f,P) + L(g,P) \leqslant L(f+g,P)$ and $U(f,P)+U(g,P) \geqslant U(f+g,P)$. Also show that $f+g$ is integrable and $\displaystyle\int_A (f+g) = \int_A f + \int_A g$. (8)

   (c) When a subset A of $\mathbb{R}^n$ is said to have a measure zero? Show that the closed interval $[a,b]$ does not have measure zero. (6)

2. (a) If $\{A_j\}_{j\in J}$ is a countable collection of subsets of $\mathbb{R}^n$ then prove that $m^*\left(\bigcup_{j\in J} A_j\right) \leqslant \displaystyle\sum_{j\in J} m^*(A_j)$. (6)

   (b) If $\{E_k\}_{k=1}^{\infty}$ is a descending collection of measurable subsets of $\mathbb{R}^n$ and $m(E_i) < \infty$ for some i then prove that $m\left(\bigcap_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} m(E_k)$. Give an example to show that the condition $m(E_i) < \infty$ for some i cannot be dropped. (8)

   (c) Show that every closed subset of $\mathbb{R}^n$ is measurable. (6)

3. (a) State and prove Egoroff's theorem. (10)

   (b) Let f be a bounded function defined on a closed and bounded interval $[a,b]$. If f is Reimann integrable over $[a,b]$ then prove that it is Lebesgue integrable over $[a,b]$. Is the converse true? Justify. (10)

4. (a) If f and g are non-negative measurable functions on E then prove that $\displaystyle\int_E (\alpha f + \beta g) = \alpha\int_E f + \beta\int_E g$ for any $\alpha, \beta > 0$. (5)

   (b) State and prove Monotone convergence theorem. (5)

   (c) State Fatau's Lemma. Show by an example that the inequality in Fatau's Lemma may be strict inequality. (4)

   (d) Let f be a measurable function on E. Prove that f^+ and f^- are integrable over E if and only if $|\ f\ |$ is integrable over E. (6)

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SECTION - II

5. (a) Define Dirichlet's Kernel $D_N(x)$. Show that the N-th Dirichlet kernel is given by $D_N(x) = \dfrac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}$. Further show that the Fourier coefficient of D_N is (6)

$$\widehat{D_N}(n) = \begin{cases} 1 & \text{if } |n| \leq N \\ 0 & \text{otherwise} \end{cases}.$$

   (b) State and prove Fejer theorem. (8)

   (c) Find the Fourier coefficient and hence find the Fourier series of the function $f(x) = |x|$, where $-\pi \leq x \leq \pi$. (6)

6. (a) Show that any separable Hilbert space has an orthonormal basis. (8)

   (b) Let S be a closed subspace of a Hilbert space H over $\mathbb{C}$ and $x \in H$. Show that there exists a unique element $a \in S$ such that $\|x - a\| = \inf_{y\in S} \|x - y\|$. (8)

   (c) Let H be a Hilbert space over $\mathbb{C}$ and $x, y \in H$. If x is orthogonal to y, then show that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. Hence show that if $\{x_1, x_2, \ldots, x_n\}$ is an orthonormal set in H, then $\left\|\sum_{i=1}^n x_i\right\|^2 = \sum_{i=1}^n \|x_i\|^2$. (4)

7. (a) Show that $L^2([-\pi, \pi])$ is unitarily isomorphic to $\ell^2(\mathbb{Z})$. (6)

   (b) Let $f \in L^2([-\pi, \pi])$. Then for any collection of complex numbers $\{c_k\}_{k=-N}^N$, show that $\left\| f - \sum\limits_{k=-N}^{N} \widehat{f}(k)e^{ikx} \right\|_2 \leq \left\| f - \sum\limits_{k=-N}^{N} c_k e^{ikx} \right\|_2$. Equality holds if and only if $c_k = \widehat{f}(k)$ for $-N \leq k \leq N$. (8)

   (c) If $f \in L^2([-\pi, \pi])$, then show that $\sum\limits_{-\infty}^{\infty} |\widehat{f}(n)|^2 = \|f\|^2$. (6)

8. (a) Let D be the unit disc and let $f(\theta)$ be a continuous function on the boundary ∂D of D. Show that

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) f(t) dt$$

   is harmonic extension of f to the unit disc D. (8)

   (b) Show that the expression of the Laplacian $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is given in polar coordinates by the formula $\Delta = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$. (8)

   (c) Find the solution of the Dirichlet's problem $\Delta u = 0$ in the unit disc with boundary condition $u(1, \theta) = \cos^3\theta + 3\sin 3\theta$. (4)

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