M.SC.(MATHEMATICS) PART –II
Algebra & Field Theory (R-2017)
(PAPER – II) *(DEC - 2017)*

QP Code : 28089

TOTAL TIME: 3 HRS.
TOTAL MARKS: 80

INSTRUCTIONS: 1) All questions carry equal marks.
2) This paper has TWO sections. Each section carries FOUR questions.
3) Attempt ANY TWO subquestions out of FOUR from each section.

SECTION A

1) a) State and prove Jordan-Holder theorem.
b) Define solvable group. Prove that every group of order eight is solvable.

2) a) With correct justification, write down the charater table for the alternating group on three symbols. Prove that for an abelian group all the characters are one dimensional.
b) State and prove Maschke's theorem.

3) a) State and prove any one of the Noether isomorphism theorems. Explain clearly all the notations used.
b) Let M be an R-module. Prove that a subset N of a M is a submodule of M if and only if N is non-empty and closed under addition as well as scalar multiplication from R.

4) a) Prove that every submodule of a finitely generated module over a principal ideal domain is free.
b) Define rational canonical form. Let V be a finite dimensional vector space over a field F and let T be a linear transformation of V. Prove that there is a basis of V with respect to which the matrix of T is in rational canonical form.

SECTION B

1 a) Define the term: algebraic closure. Prove the existence of an algebraic closure of a field.
b) Prove that if K/F is a finite extension, then it is algebraic. Is the converse always true?

2 a) Prove that separable extensions form a distinguished class.
b) Construct a field of order 8 with correct justification.

3 a) Using Sylow theory prove that the field of complex numbers is algebraically closed.
b) Define the terms: Galois group, fixed field. Determine with correct justification the Galois group of Q(i) over Q. (Here Q denotes the set of rational numbers.)

4 a) Define the term constructible number. Prove that a sum of constructible numbers is constructible.
b) With correct justification, give an example of a polynomial with real coefficients over Q, which is not solvable by radicals.

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M.SC.(MATHEMATICS) PART –II
Algebra - II (Old)
(PAPER – II) (DEC - 2017)

Q.P.Code: 011473

Time: 3 Hours [Total Marks: 100]

N.B. 1) Attempt any **five** questions out of **eight**.
2) All questions carry equal marks.

Q. 1. (a) State and prove the orbit-stabilizer formula. **(10)**
(b) State without proof any one the Sylow theorems. Explain clearly all the notation used. Determine all the Sylow 2-subgroups of S_3. Show that they are all conjugate. **(10)**

Q. 2. (a) State and prove Jordan Hölder theorem for finite groups. **(10)**
(b) (i) Let G be a group and let H be a normal subgroup of G. Prove that if both H and G/H are solvable, then so is G. **(5)**
(ii) Prove that every group of order 9 is abelian. **(5)**

Q. 3. (a) State and prove Hilbert basis theorem. **(10)**
(b) (i) Define the term: Noetherian ring with one example. If k is a field, is the polynomial ring $k[X_1, X_2, \ldots, X_n, \cdots]$ in countably many variables over k Noetherian? Justify. **(5)**
(ii) State and prove (any one) isomorphism theorems for modules over a commutative ring R with unity. **(5)**

Q. 4. (a) Prove that for every prime p and every natural number n, there exists a finite field of order p^n. **(10)**
(b) (i) Let R be a commutative ring with unity and let M be an R-module. Prove that $\text{Ann } R = \{r \in R \mid rn = 0, \text{ for all } n \in N\}$ is an ideal of R. Further show that if $N \subset L$ are submodules of M, then $\text{Ann}(L) \subset \text{Ann}(N)$. **(5)**
(ii) Let F be a field and X be an indeterminate and let $R = F[X]$. Let V be a vector space over F and T be a linear transformation from V to V. Write a formula for making V into an $F[X]$-module. Verify that V is an $F[X]$-module via this formula. **(5)**

Q. 5. (a) (i) Define the terms: algebraic element and transcendental element. Give one example of each with correct justification. **(5)**
(ii) Determine the degree of the field extension $\mathbb{Q}(\sqrt{5}+\sqrt{7})$ over $\mathbb{Q}$ with correct justification. **(5)**
(b) State and prove primitive element theorem. **(10)**

Q. 6. (a) (i) State (without proof) the fundamental theorem for Galois theory. Explain clearly all the notation used. **(5)**
(ii) Define the term: cyclotomic field. Determine the degree of the cyclotomic field over $\mathbb{Q}$ with correct justification. **(5)**
(b) Determine whether the regular 5-gon is constructible by straightedge and compass. Justify. **(10)**

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Q. 7. (a) (i) Prove that the map $a + b\sqrt{2} \mapsto a - b\sqrt{2}$ is an automorphism of $\mathbb{Q}(\sqrt{2})$. Find the fixed field of this automorphism. **(5)**

(ii) Determine the degree of the splitting field of $X^4 - 2$ over $\mathbb{Q}$ with correct justification. **(5)**

(b) Let R be a commutative ring with unity and let M be an R-module. Prove that the following statments are equivalent:

(i) Every non-empty set of submodules of M contains a maximal element under inclusion.

(ii) Every submodule of M is finitely generated.

(10)

Q. 8. (a) (i) Define the terms: Artinian ring, Artinian module. Give one example of each. **(5)**

(ii) Let G be a group acting on a set X. Prove that the stabilizer in G of an element $x \in X$ is a subgroup of G. **(5)**

(b) Let $\sigma = (1\ 2\ 3\ 4\ 5) \in S_5$. Find $\tau \in S_5$ such that $\tau\sigma\tau^{-1} = \sigma^{-1}$. **(10)**

Q. P. Code: 11901

Revised **(3 Hours)** **[Total Marks: 80**

N.B.: (1) Attempt any **TWO** questions from each Section.
(2) Figures to the right indicate marks for respective subquestions.
(3) Answers to section I and section II should be written in the same answer book

SECTION - I

1. (a) Let A be a rectangle in $\mathbb{R}^n$. If $f : A \to \mathbb{R}$ is a continuous function then prove that f is integrable on A. (6)

(b) Let A be a rectangle in $\mathbb{R}^n$ and $f, g : A \to \mathbb{R}$ be integrable on A. For any partition P of A and sub-rectangle S, show that $m_S(f) + m_S(g) \leqslant m_s(f+g)$ and $M_S(f) + M_S(g) \geqslant M_s(f+g)$. Deduce that $L(f,P) + L(g,P) \leqslant L(f+g,P)$ and $U(f,P) + U(g,P) \geqslant U(f+g,P)$. Also show that $f+g$ is integrable and $\displaystyle\int_A (f+g) = \int_A f + \int_A g$. (8)

(c) When a subset A of $\mathbb{R}^n$ is said to have a measure zero? Show that the closed interval $[a, b]$ does not have measure zero. (6)

2. (a) If $\{A_j\}_{j\in J}$ is a countable collection of subsets of $\mathbb{R}^n$ then prove that $m^*\left(\bigcup_{j\in J} A_j\right) \leqslant \displaystyle\sum_{j\in J} m^*(A_j)$. (6)

(b) If $\{E_k\}_{k=1}^{\infty}$ is a descending collection of measurable subsets of $\mathbb{R}^n$ and $m(E_i) < \infty$ for some i then prove that $m\left(\bigcap_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} m(E_k)$. Give an example to show that the condition $m(E_i) < \infty$ for some i cannot be dropped. (8)

(c) Show that every closed subset of $\mathbb{R}^n$ is measurable. (6)

3. (a) State and prove Egoroff's theorem. (10)

(b) Let f be a bounded function defined on a closed and bounded interval $[a, b]$. If f is Reimann integrable over $[a, b]$ then prove that it is Lebesgue integrable over $[a, b]$. Is the converse true? Justify. (10)

4. (a) If f and g are non-negative measurable functions on E then prove that $\displaystyle\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$ for any $\alpha, \beta > 0$. (5)

(b) State and prove Monotone convergence theorem. (5)

(c) State Fatau's Lemma. Show by an example that the inequality in Fatau's Lemma may be strict inequality. (4)

(d) Let f be a measurable function on E. Prove that f^+ and f^- are integrable over E if and only if $|\ f\ |$ is integrable over E. (6)

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SECTION - II

5. (a) Define Dirichlet's Kernel $D_N(x)$. Show that the N-th Dirichlet kernel is given by $D_N(x) = \dfrac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}$. Further show that the Fourier coefficient of D_N is (6)

$$\widehat{D_N}(n) = \begin{cases} 1 & \text{if } |n| \leq N \\ 0 & \text{otherwise} \end{cases}.$$

(b) State and prove Fejer theorem. (8)

(c) Find the Fourier coefficient and hence find the Fourier series of the function $f(x) = |x|$, where $-\pi \leq x \leq \pi$. (6)

6. (a) Show that any separable Hilbert space has an orthonormal basis. (8)

(b) Let S be a closed subspace of a Hilbert space H over $\mathbb{C}$ and $x \in H$. Show that there exists a unique element $a \in S$ such that $\|x - a\| = \inf_{y \in S} \|x - y\|$. (8)

(c) Let H be a Hilbert space over $\mathbb{C}$ and $x, y \in H$. If x is orthogonal to y, then show that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. Hence show that if $\{x_1, x_2, \ldots, x_n\}$ is an orthonormal set in H, then $\left\|\sum_{i=1}^n x_i\right\|^2 = \sum_{i=1}^n \|x_i\|^2$. (4)

7. (a) Show that $L^2([-\pi, \pi])$ is unitarily isomorphic to $\ell^2(\mathbb{Z})$. (6)

(b) Let $f \in L^2([-\pi, \pi])$. Then for any collection of complex numbers $\{c_k\}_{k=-N}^{N}$, show that $\left\| f - \sum_{k=-N}^{N} \widehat{f}(k) e^{ikx} \right\|_2 \leq \left\| f - \sum_{k=-N}^{N} c_k e^{ikx} \right\|_2$. Equality holds if and only if $c_k = \widehat{f}(k)$ for $-N \leq k \leq N$. (8)

(c) If $f \in L^2([-\pi, \pi])$, then show that $\sum_{-\infty}^{\infty} |\widehat{f}(n)|^2 = \|f\|^2$. (6)

8. (a) Let D be the unit disc and let $f(\theta)$ be a continuous function on the boundary ∂D of D. Show that

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) f(t) dt$$

is harmonic extension of f to the unit disc D. (8)

(b) Show that the expression of the Laplacian $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is given in polar coordinates by the formula $\Delta = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$. (8)

(c) Find the solution of the Dirichlet's problem $\Delta u = 0$ in the unit disc with boundary condition $u(1, \theta) = \cos^3\theta + 3\sin 3\theta$. (4)

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Q. P. Code: 11898

(3 Hours) [Total Marks: 100

N.B.: (1) Attempt any **FIVE** questions.
(2) Figures to the right indicate marks for respective sub-questions.

1. (a) Show that there is a non-measurable subset in $\mathbb{R}$. (10)

   (b) (i) Show that exterior measure of any countable subset of $\mathbb{R}$ is zero. Show by an example that the converse is not true. (5)

   (ii) If E_1 and E_2 are Lebesgue measurable subsets of $\mathbb{R}$, then show that $E_1 \cup E_2$ is Lebesgue measurable. (5)

2. (a) If f is a measurable function, then show that

   (i) f^2 is measurable (5)

   (ii) $\lambda + f$ is measurable, where $\lambda \in \mathbb{R}$. (5)

   (b) (i) Let f be a bounded function defined on the closed and bounded interval $[a, b]$. If f is Riemann integrable over $[a, b]$, then show that it is Lebesgue integrable over $[a, b]$ and the two integrals are equal. (5)

   (ii) Show by an example that a Lebesgue integrable function may not be Riemann integrable. (5)

3. (a) State and prove Fatou's lemma. Show by an example that the inequality in Fatou's Lemma may be a strict inequality. (10)

   (b) (i) Let $\{f_n\}$ be an increasing sequence of non-negative measurable functions on E. If $f_n \to f$ pointwise a.e. on E, then show that $\lim_{n\to\infty} \int_E f_n = \int_E f$. (5)

   (ii) Let $E_1 \subseteq E_2 \subseteq \cdots$ be measurable subsets of $\mathbb{R}$ with $E = \cup_{n=1}^{\infty} E_n$. Show that $m(E) = \lim_{n\to\infty} m(E_n)$. (5)

4. (a) State Fubini's theorem. Use Fubini's theorem to evaluate $\int_A (x \sin y - ye^x)dxdy$, where $A = [-1, 1] \times [0, \pi/2]$. (10)

   (b) (i) Let A be a subset of $\mathbb{R}$. Show that the characteristic function χ_A is measurable if and only if the set A is measurable. (5)

   (ii) Let f and g be non-negative integrable functions on a measurable subset E of $\mathbb{R}$. Show that if $f = g$ a.e. then $\int_E f = \int_E g$. (5)

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5. (a) Let (f_n) be a sequence of measurable functions. Show that $\sup_n\{f_n\}$ and $\lim_{n\to\infty} f_n$ are measurable functions. (10)

(b) Let f and g be two non-negative measurable functions on a measaurable set E and c be a non-negative real number. Show that (10)

(i) $\int_E (cf) = c\int_E f$

(ii) $\int_E (f+g) = \int_E f + \int_E g$

6. (a) Show that any separable Hilbert space has a complete orthonormal basis. (10)

(b) State and prove Holder's inequality. (5)

(c) State and prove Parseval's identity (5)

7. (a) Show that $\ell^2(\mathbb{N})$ is a complete metric space. (10)

(b) Let $\{e_n\}_{n\in\mathbb{N}}$ be an arbitrary orthonormal set in $L^2[-\pi,\pi]$ and let $c_1, c_2, \ldots$ be complex numbers such that the series $\sum_{k=1}^{\infty} c_k$ converges. Show that there exist a function $f \in L^2[-\pi,\pi]$ such that $c_k = \langle f, e_k\rangle$ and $\sum_{k=1}^{\infty} c_k^2 = \|f\|^2$ (10)

8. (a) Let f be an integrable function on the circle which is differentiable at a point x_0. Show that $S_N(f)(x_0) \to f(x_0)$ as $N \to \infty$, where $S_N f(x)$ is the N-th partial sum of the Fourier series of f. (10)

(b) (i) Find the solution of the Dirichlet's problem $\Delta u = 0$ on the unit disc, with the boundary condition $u(1,\theta) = \cos^2\theta$. (5)

(ii) Find the Fourier series of the function $f(x) = |x|$ in $-\pi \le x \le \pi$. (5)

Q. P. Code : 29070

[Maximum Marks: 80

N.B. 1) Attempt any **two** questions from each section.
2) All questions carry equal marks.
3) Answer to Section-I and Section-II should be written in same answer book.
4) K denotes either the set of real numbers $I\!R$ or set of complex numbers $\mathbb{C}$.

SECTION-I

1. (a) (i) Prove that a metric space X is complete if every Cauchy sequence in X has convergent subsequence. (5)
(ii) Show that a space $I\!R^n$ is complete with its usual metric (5)
(b) (i) Define a Baire space. Give the example of Baire space with proper justification. (5)
(ii) Let X equal the countable union $\cup B_n$. Show that if X is a Baire space, at least one of the sets $\overline{B_n}$ has nonempty interior. (5)

2. (a) (i) Let T be a set and $B(T)$ denote the set of all bounded scalar valued functions on T. For $\boldsymbol{x}$ in $B(T)$,let $\|\boldsymbol{x}\|_\infty = \sup\{|\boldsymbol{x}(t)| : t \in T\}$. Then show that $\|.\|_\infty$ is a norm on $B(T)$. (5)
(ii) Let X be a normed space, Y be a closed subspace of X and $X \neq Y$. Let r be real number such that $0 < r < 1$. Then show that there exists some $\boldsymbol{x_r} \in X$ such that (5)
$$\|\boldsymbol{x_r}\| = 1 \quad \text{and} \quad r \leq \text{dist}(\boldsymbol{x_r}, Y) \leq 1.$$
(b) Prove that every finite dimensional subspace Y of a normed space X is complete. (10)

3. (a) Let X be a normed space and f be a non zero linear functional on X. Then show that f is discontinuous if and only if $Z(f)$ (zero space of f) is dense in X. (10)
(b) Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For a fixed $y \in L^q$, define $f_y : L^p \to K$ by $f_y(x) = \int_a^b xydm$, $x \in L^p$. Then show that $f_y \in (L^p)'$ and $\|f_y\| = \|y\|_q$. (10)

4. (a) (i) Let X and Y be Banach spaces and $F : X \to Y$ be a linear map which is closed and surjective. Then show that F is continuous and open. (5)
(ii) Let X be a normed space over K and Y be a subspace of X and $g \in Y'$. Then show that there is some $f \in X'$ such that $f_{|Y} = g$ and $\|f\| = \|g\|$. (5)
(b) State and prove Uniform boundedness principle. (10)

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SECTION-II

5. (a) (i) Show that a matrix A represents a rotation of $\mathbb{R}^2$ if and only if $A \in SO_2$. (5)

(ii) Show that every element $A \in SO_3$ has the eigenvalue 1. (5)

(b) (i) Define Isometry and show that every isometry is the composition of an orthogonal linear operator and a translation. (5)

(ii) Prove that every line is intersection of two planes. (5)

6. (a) State and prove fundamental theorem for plane curve. (10)

(b) (i) State and prove Frenet-Serret equations. (5)

(ii) Verify Frenet-Serret equations for $\gamma(t) = (\frac{4}{5}\cos t, 1 - \sin t, \frac{-3}{5}\cos t)$. (5)

7. (a) (i) Let $S \subset \mathbb{R}^3$ be the set obtained by rotating a regular plane curve C about an axis in the plane which do not meet the curve. Prove or disprove the surface of revolution S is regular surface. (5)

(ii) Show that the transition maps of a smooth surface are smooth. (5)

(b) (i) Prove that the unit sphere $S^2 = \{(x, y, z)/x^2 + y^2 + z^2 = 1\}$ is orientable. (5)

(ii) Find the values of c for which the set $f(x, y, z) = c$ is a regular surface, where
$f(x, y, z) = (x + y + z - 1)^2$. (5)

8. (a) (i) Show that $k_n = k_1 \cos^2\theta + k_2 \sin^2\theta$ where k_n is normal curvature and k_1, k_2 are principal curvature. (5)

(ii) Show that any normal section of a surface is a geodesic. (5)

(b) Consider parametrized equation of elliptic paraboloid $\sigma(u, v) = (u, v, u^2 + v^2)$. Calculate

(i) The coefficients of the first fundamental form (2)

(ii) The coefficients of the Second fundamental form (2)

(iii) The Gaussian curvature (2)

(iv) The Principal curvatures (2)

(v) The Mean curvature (2)

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M.SC.(MATHEMATICS) PART –II
Differential Geometry (OLD)
(PAPER – III) (DEC - 2017)

QP CODE : 28878

Duration:[3 Hours] [Marks: 100]

N.B. 1) All questions carry equal marks.
2) Attempt any **five** questions.

1. (a) Let V be an inner product space. State and prove following properties for norm function $\|\cdot\| : V \longrightarrow \mathbb{R}$
   (i) Triangle inequality (3)
   (ii) Cauchy-Schwarz inequality (4)
   (iii) Pythagoras theorem (3)

   (b) (i) Define Special orthogonal group SO_n and rotation of $\mathbb{R}^3$. (2)
   (ii) Show that a matrix A represents a rotation of $\mathbb{R}^3$ if and only if $A \in SO_3$. (8)

2. (a) (i) Show that an isometry which fixes the origin is a linear operator. (5)
   (ii) Give an example of isometry which do not fix the origin. Justify. (5)

   (b) Let A be a $n \times n$ real matrix. Show that the following conditions are equivalent (10)
   (i) A is orthogonal.
   (ii) Multiplication by A preserves dot product.
   (iii) The columns of A are mutually orthogonal unit vectors.

3. (a) Consider initial value problem $\dfrac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.
   (i) State Picard's theorem for the existence and uniqueness of solution of initial value problem. (2)
   (ii) Explain Picard's scheme of approximation for the solution of initial value problem. (6)
   (iii) Explain why Picard's theorem known for local existence and uniqueness of solution of initial value problem. (2)

   (b) Find approximate solution upto t^4 of the initial value problem (10)
   $\dfrac{dx}{dt} = 2x + ty$, $\dfrac{dx}{dt} = xy$ with $x(0) = 1$ and $y(0) = 1$.

4. (a) Let S be a regular surface and $p \in S$. Prove that there exists a neighborhood V of p in S such that V is the graph of a differentiable function which has the form $z = f(x, y)$ or $y = g(x, z)$ or $x = h(y, z)$. (10)

   (b) (i) Let $\sigma : U \to \mathbb{R}^3$ be a patch of a surface S containing a point p of S and (u, v) be co-ordinates in U. Then show that the tangent space to S at p is the vector subspace of $\mathbb{R}^3$ spanned by the vectors σ_u and σ_v. (5)
   (ii) Prove or disprove: Every plane in $\mathbb{R}^3$ is a regular surface. (5)

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5. (a) State and prove the fundamental theorem for plane curve. (10)

   (b) The parametric equation of a circular helix with the $z-$ axis is given by

$$\gamma(\theta) = (a\cos\theta, a\sin\theta, b\theta), -\infty < \theta < \infty$$

   (i) Find the curvature of a circular helix. (5)

   (ii) Find the torsion of a circular helix. (5)

6. (a) State and prove the generalized Stoke's theorem for the integration of exterior forms. (10)

   (b) (i) Prove that the local maxima and local minima of function f are critical points of f. (5)

   (ii) Let $f(x, y, z) = (x + y + z - 1)^2$. Locate the critical points and critical values of f. (5)

7. (a) Prove or disprove

   (i) The differential $dN_p : T_p(S) \longrightarrow T_p(S)$ of the Gauss map is a self adjoint linear map. (5)

   (ii) The value of the second fundamental form for a unit vector $v \in T_p(S)$ is equal to the normal curvature of a regular curve passing through p and tangent to v. (5)

   (b) Consider surface covered by the parametrization $\sigma(u, v) = (u + v, u - v, uv)$. Calculate following at the point $(2, 0, 1)$.

   (i) The coefficients of the first fundamental form (2)

   (ii) The coefficients of the Second fundamental form (2)

   (iii) The Gaussian curvature (2)

   (iv) The Principal curvatures (2)

   (v) The Mean curvature (2)

8. (a) Let $\sigma(u, v) = (ucosv, usinv, u)$ where $u = e^{\lambda t}$, $v = t$ and λ is constant. Find the length of part of the curve with $0 \leq t \leq \pi$. (5)

   (b) Compute the tangent, normal and binormal to the curve $\gamma(t) = (\frac{1}{3}(1+t)^{\frac{3}{2}}, \frac{1}{3}(1-t)^{\frac{3}{2}}, \frac{t}{\sqrt{2}})$. (5)

   (c) The parametric equation of Möbius band is $\sigma(t, \theta) = ((1-tsin\frac{\theta}{2})cos\theta, (1-tsin\frac{\theta}{2})sin\theta, tcos\frac{\theta}{2})$ Is the Möbius band orientable? Justify. (5)

   (d) Show that the angle of intersection of two curves γ and $\bar{\gamma}$ on a surface S is given by $cos\theta = \frac{F}{EG}$ where E,F and G are notations as in first fundamental form. (5)

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M.SC.(MATHEMATICS) PART –II
Graph Theory (REV-2017)
(PAPER – IV) (DEC - 2017)

Q.P. Code: 13190

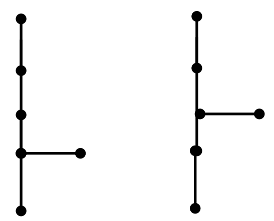
Duration: 3 hrs **Marks: 80**

N.B. 1) Both the sections are compulsory.

2) Attempt **ANY TWO** questions from each section.

Section I

1 (a) Define isomorphic graphs and decide which of the following pairs of graphs are isomorphic. Justify your answer. 4

i.


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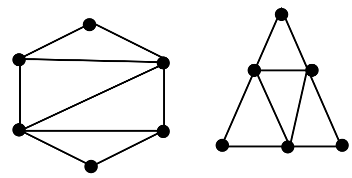
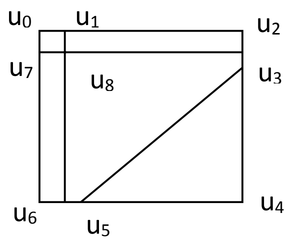


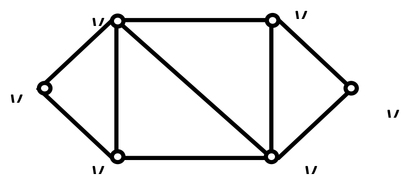
(b) State and prove Menger's theorem – Vertex form. 8

(c) i. Prove that a finite graph is bipartite if and only if it has no odd cycle. 4

ii. Draw line graphs of the following graphs. 4



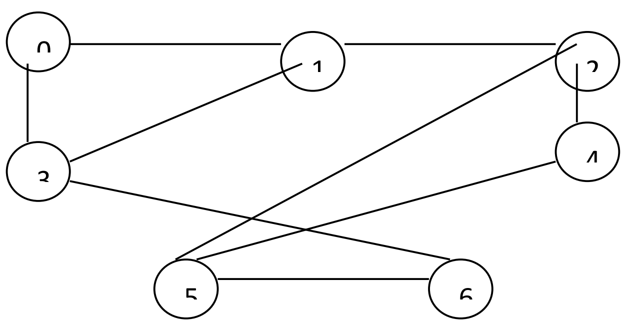
G_1


G_2

2 (a) Prove the following results. 6
i. Every tree is bipartite.
ii. A graph is connected if and only if it has a spanning tree.

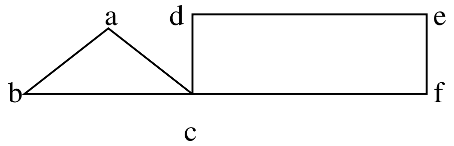
(b) Use Huffman coding to encode these symbols with given frequencies: 6
a: 0.20, *b*: 0.10, *c*: 0.15, *d*: 0.25, *e*: 0.30, *f*: 0.12. What is the average number of bits required to encode a character?

(c) i. Draw Breadth First Search (BFS) tree for the following graph. 4



ii. Decode the Prufer sequence (2, 3, 3, 4, 4, 5).and draw corresponding tree. 4

3 (a). Discuss Fleury's algorithm for the following graph. 6



(b) Prove that "If a connected graph G has $n \geq 3$ vertices and for every pair u, v of non-adjacent vertices, $deg(u) + deg(v) \geq n$, then G is Hamiltonian". 6

(c) i. If the degree sequence of the simple graph G is (2, 2, 3, 3, 4), find degree sequence of G^C. Hence if the degree sequence of the simple graph G is $(d_1, d_2, ..., d_p)$, find degree sequence of G^C. 4

ii. Prove that a graph is Hamiltonian if and only if its closure is Hamiltonian. 4

4 (a) Prove that if G is a bipartite graph with bipartition X and Y , then G has a matching of X into Y if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$. 10

(b) Prove that R(3, 4) = 9. Where R(s, t) is Ramsey number. 10

Section II

5. (a) For any graph G prove that, $\chi(G) \leq \Delta(G)+1$. 10

(b) Prove that every k - chromatic graph with n-vertices has a least ${}^kC_2$ edges. 10

.6. (a) Prove that a graph G is planar if and only if it contains no contraction of K_5 or $K_{3,3}$ 10

(b) If G is a (connected) simple, finite planar graph with n vertices, ($n \geq 3$), then prove that G has at most $3n - 6$ edges. Also if G contains no triangles, then prove that G has at most $2n - 4$ edges. 10

7 (a) Define the following. 10

i) Diagraph
ii) Weakly connected
iii) Strongly connected
iv)Tournament
v) Hamiltonian Path.

(b) Every tournament D contains a vertex from which every other vertex is reachable by a path of length at most two. 10

8. (a) Define Eigen value of a graph G and prove that if G be a connected graph with k distinct eigen values and let d be the diameter of G then k > d. 10

(b) If G is bipartite and λ is an eigenvalue of G with multiplicity m, then –λ is also an eigenvalue with multiplicity m. 10

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M.SC.(MATHEMATICS) PART –II

Graph Theory (Old)

(PAPER – IV) (DEC - 2017)

Q.P.Code:13374

(3 hours) **Total Marks : 100**

N.B. : 1) Answer any FIVE questions.

2)All questions carry EQUAL marks.

1. (a) State and prove Havel Hakimi theorem about graphic degree sequence.

   (b) Let G be a graph with vertex set $\{v_1, v_2, \ldots\ldots, v_p\}$ and adjacency matrix $A = (a_{ij})$. Show that

   (i) $(i,j)^{th}$entry of A^kdenotes the number of (v_i, v_j) walks of length k

   (ii) $\frac{1}{6}(trace\ (A^3))$ is number of triangles in G.

2. (a) Let e be any edge of a graph G, then prove that $\tau(G) = \tau(G.e) + \tau(G - e)$ where $\tau(G)$ denotes number of spanning trees of a graph G.

   (b) Show that a vertex v of a tree T is a cut vertex if and only if its degree is more than one. Show that any non trivial loopless connected graph on at least two vertices contains at least two non cut vertices.

3. (a) Define closure of a graph G. Show that closure of a graph G, is well defined. Show further that a simple graph G is Hamiltonian if and only if its closure is Hamiltonian.

   (b) Let κ, κ'denote vertex and edge connectivity of a graph G, then show that $\kappa \leq \kappa' \leq \delta$. Give an example to show that inequyality can be strict.

4. (a) Let G be a bipartite graph with a bipartition(X, Y). Show that G contains a matching that saturates every vertex of X if and only if for any subset S of V, $|N(S)| \geq |S|$.

   (b) Determine perfect matching in the graphs K_{2n} and $K_{n,n}$.

5. (a) Let $\pi_k(G)$ is number proper k colorings of G. Show that $\pi_k(G) = \pi_k(G - e) - \pi_k(G.e)$ for any edge e of G.

   (b) Define critical graph. Show that only 1critical graph is K_1, only 2critical graph is K_2 and only 3critical graphs are C_{2t+1}.

6. (a) State Kruskasl's algorithm. Prove that any spanning tree constructed by Kruskal's algorithm is optimal.

   (b) Define line graph of a graph. Show that line graph of connected graph is isomorphic to the graph if and only if it is cycle.

7. (a) State and prove Euler's theorem about planar graph. Hence deduce that K_5 and $K_{3,3}$ are not planar.

   (b) Define dual of a planar graph. Let G^*denote dual of a planar graph G. Show that if $G^* \cong G$ then $|E(G)| = 2|V(G)| - 2$. Construct such a graph on $n \geq 4$ vertices.

8. (a) Define Ramsey number $r(p,q)$; $p,q \geq 2$. Show that $r(p,q) \leq \binom{p+q-2}{p-1}$.

   (b) If T is a tree on m vertices then show that $r(T, K_n) = (m-1)(n-1) + 1$.

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M.SC.(MATHEMATICS) PART –II
Numerical Analysis. (R-2017)
(PAPER – V) (DEC - 2017)

Q. P. Code : 11370

Revised **(3 Hours)** **Total Marks : 80**

Instructions :

- **Attempt any two questions from each section**
- **All questions carry equal marks. Scientific calculator can be used.**
- **Answers to section I and section II should be written in the same answer book**

Section I(Attempt any two questions)

Q1 a) Define : Absolute error, Relative error and Percentage error. If 0.3333 is the approximate value of $\frac{1}{3}$ find the absolute and relative errors and percentage error

b) Convert decimal number $(43)_{10}$ to corresponding binary, octal and hexadecimal number.

Q2 a) Let $g : [a,b] \to [a,b]$ be a continuous function such that $|g'(x)| \leq k < 1$ For all $x \in [a,b]$. If $p_0 \in [a,b]$ be any number, then show that the sequence defined by $p_n = g(p_{n-1}), n \geq 1$ converges to the unique fixed point p in $[a,b]$. Also find the root of the equation $2xe^x = 1$ using fixed point iteration method with the accuracy 10^{-4} and belongs to $[0,1]$.

b) Derive the Newton-Raphson Method iterations formula to find a root of the algebraic or transcendental equation $f(x) = 0$. Hence Use the formula to solve $f(x) = cosx - xe^x$ with initial approximation $x_0 = 1$ upto three decimal places.

Q3 a) Solve the following system by using the Gauss-Jordan elimination method.

$$x + y + z = 5$$
$$2x + 3y + 5z = 8$$
$$4x + 5z = 2$$

b) Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Find the smallest Eigen value in magnitude of matrix A using 4 iterations of Inverse power method.

Q4 a) Prove that if f(x) and g(x) are the function of x then

i. $\Delta[f(x)g(x)] = f(x)\Delta g(x) + g(x+h)\Delta f(x) = f(x+h)\Delta g(x) + g(x)\Delta f(x)$

ii. $\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)g(x+h)}$

b) Derive Cubic Spline interpolation formula.

Section II(Attempt any two questions)

Q5 a) Derive Newton-Cotes Quadrature formula and use it to derive Simpson's 1/3rd rule for numerical integration.

b) Evaluate $\int_0^1 \int_0^1 \frac{Sin\, xy}{1+xy} dxdy$ using Trapezoidal Rule with h = k=0.5

Q6 a) Determine the constant a and b by the method of least squares such that $y = ae^{bx}$ fit the following data

x	2	4	6	8	10
y	4.077	11.084	30.128	81.897	222.62

TURN OVER

b) Obtain the first four orthogonal polynomials $f_n(x)$ on $[-1,1]$ with respect to weight function $w(x) = 1$

Q7 a) Derive the Adam-Bashforth predictor and corrector formula to solve the differential equation $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.

b) Using Milne's Method, solve the differential equation $(1 + x)\frac{dy}{dx} + y = 0$ with $y(0) = 2$, for $x = 1.5$ and $x = 2.5$

Q8 a) Derive a Crank Nicolson method to obtain the numerical solution of one dimensional heat equation

b) Solve the heat conduction equation subject to the boundary conditions $u(0, t) = u(1, t) = 0$, and $u(x, 0) = x - x^2$. Take $h = \frac{1}{4}, k = 0.025$

Q.P.Code: 011485

External (Scheme A) (3 Hours) [Total Marks: 100]
Internal (Scheme B) (2 Hours) [Total Marks: 40]

Note:
(1) External (Scheme A) students answer any five questions.
(2) Internal (Scheme B) students answer any three questions.
(3) All questions carry equal marks. Scientific calculator can be used.
(4) Write on top of your answer book the scheme under which your appearing

Q1 a) Define: absolute error, relative error and percentage error. Find absolute error, relative error and percentage error in calculation of $Z = 3x^2 + 2x$ by taking approximate value of x as 3.45, and true value of x as 3.4568.

b) Convert decimal number $(43)_{10}$ to corresponding binary, octal and hexadecimal number.

Q2 a) Explain Ramanujan Method. Using Ramanujan's method, obtain the first four convergence of $x + x^2 = 1$

b) Derive the Newton-Raphson Method iterations formula to find a root of the algebraic or transcendental equation $f(x) = 0$. Hence Use the formula to solve $f(x) = cosx - xe^x$ with initial approximation $x_0 = 1$ upto three decimal places.

Q3 a) Solve the following system by using the Crout's triangularization method.

$$x_1 + x_2 + x_3 = 9$$
$$2x_1 - 3x_2 + x_3 = 13$$
$$3x_1 + 4x_2 + 5x_3 = 40$$

b) Let A = $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Find the smallest Eigen value in magnitude of matrix A using 4 iterations of Inverse power method.

Q4 a) Derive Newton's Divided difference formula of interpolation.

b) Derive Cubic Spline interpolation formula.

Q5 a) Derive Newton-Cotes Quadrature formula and use it to derive Trapezoidal rule for numerical integration.

b) Evaluate $\int_0^1 \int_0^1 \frac{Sin\, xy}{1+xy} dxdy$ using Trapezoidal Rule with $h = k = 0.5$

Q6 a) Using the least –squares method , obtain the normal equation to find the values of a, b and c when the curve $y = c + bx + ax^2$ is to be fitted for the data points (x_i, y_i) $i = 1,2,3, \dots \dots \dots n$.

b) Obtain the first four orthogonal polynomials $f_n(x)$ on $[-1,1]$ with respect to weight function $w(x) = 1$

Q7 a) i) Given $\frac{dy}{dx} - 1 = xy$ with $y(0) = 1$, obtain the Taylor series for $y(x)$.

ii) Using Picard's method, obtain the solution of $\frac{dy}{dx} = x(1 + x^3 y)$, $y(4) = 4$.

TURN OVER

b) Using Milne's Method, solve the differential equation
$(1+x)\frac{dy}{dx}+y=0$ with $y(0)=2$, for $x=1.5$ and $x=2.5$

Q8 a) Derive a Jacobi Iteration formula to obtain the numerical solution of one dimensional heat equation

b) Solve the heat conduction equation subject to the boundary conditions $u(0,t)=u(1,t)=0$, and $u(x,0)=x-x^2$. Take $h=\frac{1}{4}, k=0.025$ and $u_t(x,0)=0, u(x,0)=\sin^3 \pi x \; \forall \; x \in [0,1]$

Q. P. Code : 28442

Duration: 3 **Hours]** **[Maximum Marks: 100**

N.B. 1) Attempt any **five** questions out of **eight**.
2) All questions carry equal marks.
3) K denotes either the set of real numbers $I\!R$ or set of complex numbers $\mathbb{C}$.

1. (a) (i) Let Y be a closed subspace of a normed space X. For $x+Y$ in the quotient space X/Y, let $|\,\|x+Y\|\,| = \inf\{\|x+Y\| : y \in Y\}$. Then show that $|\,\|.\|\,|$ is a norm on X/Y. (5)

(ii) Prove that a sequence $(x_n + Y)$ converges to $(x+Y)$ in X/Y if and only if there is a sequence (y_n) in Y such that $(x_n + y_n)$ converges to x in X. (5)

(b) Let X be a normed space. Then prove that following are equivalent. (10)

(i) Every closed and bounded subset of X is compact.

(ii) The subset $\{x \in X : \|x\| \leq 1\}$ of X is compact.

(iii) X is finite dimensional.

2. (a) (i) Define the equivalent norms. Hence, prove that on a finite dimensional vector space X, any norm $\|.\|$ is equivalent to any other norm $\|.\|_0$ (6)

(ii) Show that every finite dimensional subspace Y of a normed space X is closed in X. (4)

(b) (i) Prove or disprove a linear map on a linear space X is continuous with respective any norm define on X. (5)

(ii) Let Y and Z be subspaces of a normed space X, and suppose that Y is closed and is a proper subset of Z. Then prove that for every real number θ in the interval $(0,1)$ there is a $z \in Z$ such that $\|z\| = 1$, $\|z-y\| \geq \theta$ for all $y \in Y$. (5)

3. (a) (i) Define the inner product on a linear space X. Let $H = L^2([a,b])$, the linear space of all equivalence classes of scalar valued square-integrable functions on $[a,b]$, obtained by identifying functions which are equal almost everywhere. For any x and y in H, let (5)

$$\langle x, y\rangle = \int_a^b x\bar{y}\,dm$$

where m is a Lebesgue measure. Then show that H is a Hilbert space.

(ii) Shoe that a linear map F from a normed space X to a normed space Y is a homeomorphism if and only if there are $\alpha, \beta > 0$ such that $\beta\|x\| \leq \|F\| \leq \alpha\|x\|$ for all x in X. (5)

[TURN OVER

(b) (i) Let X be a linear space over K and Y be a subspace of X which is not hyperspace in X. If x_1, x_2 are in X but not in Y, then show that there is some x in Xsuch that for all $t \in [0,1]$, (6)

$$tx_1 + (1-t)x \notin Y \quad tx_2 + (1-t)x \notin Y.$$

(ii) If X is normed space and Y is a subspace of x has the property as desired in above part, then show that the complement Y^c is connected. (4)

4. (a) let X denote a subspace of $B(T)$ (set of scalar valued bounded functions on a set T)with the sup norm, $1 \in X$ and f be a linear functional on X. If f is continuous and $\|f\| = f(1)$, then show that f is positive. Conversely, if Re$x \in X$ whenever $x \in X$ and if f is positive, then show that f is continuous and $\|f\| = f(1)$. (10)

(b) (i) Let E be a measurable subset of $I\!R$ and for $t \in E$, let $x_1(t) = t$. Let $X = \{x \in L^2(E) : x_1 x \in L^2(E)\}$ and $F : X \to L^2(E)$ be defined by $F(x) = x_1 x$. if $E = [a,b]$, then show that F is continuous. (6)

(ii) Let X be a normed space over K, and f be a non zero functional on X. If E is an open subset of X, then show that $f(E)$ is an open subset of K. (4)

5. (a) (i) Let X be a normed space over K and E_1, E_2 be non empty disjoint subsets of X, whenever E_1 is open in X. Then show that there is a real hyperplane in X which separates E_1 and E_2. (6)

(ii) Let X be a normed space over K, $f \in X'$ (dual of normed space X) and $f \neq 0$. Let $a \in X$ with $f(a) = 1$ and $r > 0$. Then show that (4)

$$U(a,r) \cap Z(f) = \emptyset \quad \text{if and only if} \quad \|f\| \leq \frac{1}{r}.$$

Note that $U(a,r)$ open ball of radius r about a.

(b) (i) If a normed space X is finite dimensional, then show that every linear operator on X is bounded. (5)

(ii) Prove that for a normed space Y, $BL(X,Y) = \{0\}$ if and only if $Y = \{0\}$ (5)

6. (a) (i) Let H be a Hilbert space and $A \in BL(H)$. Then show that there is a unique $B \in BL(H)$ such that for all $x, y \in H$, $\langle A(x), y\rangle = \langle x, B(y)\rangle$. (5)

(ii) Let H be a Hilbert space. Consider $A \in BL(H)$ and A^* is the adjoint of A then prove that (5)

$$\|A^*\| = \|A\| \quad \text{and} \quad \|A^*A\| = \|A\|^2 = \|AA^*\|.$$

(b) (i) Let X and Y be normed spaces and $F : X \to Y$ be linear. Then prove that F is compact if and only if for every bounded sequence (x_n) in X, $(F(x_n))$ has a subsequence which converges in Y. (5)

(ii) Let $k \in K$ and $F, G \in CL(X,Y)$ (Space of compact liner maps). Then show that kF and $F + G$ belongs to $CL(X,Y)$. (5)

[TURN OVER

7. (a) Let T be a closed densely defined operator in Hilbert space H. Then prove the following: (10)
   1. D_{T^*} (Domain of operator T^*) is dense in H and $T^{**} = T$.
   2. Given z and w in H, there exist unique $x \in D_T$ and $y \in D_{T^*}$ such that $T(x) + y = z$ and $x - T^*(y) = w$.
   3. Given $w \in H$, there is unique $x \in D_{T^*T}$ such that $x + T^*T(x) = w$.

   (b) If X is a normed space and $B = \{x \in X : \|x\| \leq 1\}$ a closed ball, then prove that ball B^* (Dual of ball B) is weak-star compact. (10)

8. (a) (i) Let $X = C[0,1]$ and $T : \mathcal{D}(T) \to X$ defined as $T(x) = x'$ where the prime denote differentiation and $\mathcal{D}(T)$ is a subspace of functions $x \in X$ which have a continuous derivative. Then prove that T is closed but not bounded. (5)

   (ii) Show that the differentiation operator $D : \mathcal{D}(D \to L^2(-\infty, \infty)$ defined as $D(x) = ix'$ is self-adjoint. (5)

   (b) (i) State the Fredholm Alternative. (5)

   (ii) Let $T : X \to X$ be a compact linear operator on a normed space X, and let $\lambda \neq 0$. Then prove that $T_\lambda = T - \lambda I$ satisfies the Fredholm alternative. (5)

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