(3 Hours) **[Total Marks: 80]**

N.B.: 1) Question No. 1 is Compulsory.

2) Answer any **THREE** questions from **Q.2 to Q.6**.

3) Figures to the right indicate full marks.

Q 1. a) Evaluate the Laplace transform of $\sqrt{1+\sin t}$ [5]

b) Find directional derivative of $\phi = 4xz^2 + x^2yz$, at $(1,-2,-1)$ in direction of $2i - j - 2k$ [5]

c) Find orthogonal trajectories of the family of curves $e^x \cos y - xy = c$.[5]

d) Obtain half range sine series for $f(x)=x, \quad 0<x<2$. [5]

Q 2. a) If $u + v = e^{2x}(x\cos 2y - y\sin 2y)$ then find analytic function $f(z)$ by Milne Thomson Method [6]

b) Find the Fourier series for $f(x) = 9 - x^2$, $-3 \le x \le 3$ [6]

c) Find the Laplace transform of the following

i) $L[t\sqrt{1+\sin t}]$ ii) $L\left[\frac{\sinh 2t}{t}\right]$ [8]

Q 3. a) Using Convolution theorem, find Inverse Laplace of $\frac{s}{(s^2+4)^2}$. [6]

b) Prove that $J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\sin x + \frac{(3-x^2)}{x^2}\cos x\right]$. [6]

c) Find Fourier series for $f(x) = (\pi - x)^2$ in $0 \le x \le 2\pi$. Hence deduce that

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$ [8]

Q 4 a) Find the Fourier transform of $f(t) = e^{-|t|}$ [6]

b) Show that the function $f_1(x) = 1$, $f_2(x) = x$ are orthogonal on $(-1,1)$ and determine the constant A & B so that functions $f_3(x) = 1 + Ax + Bx^2$ is orthogonal to both $f_1(x)$ and $f_2(x)$ on that interval. [6]

c) Find bilinear transformation which maps the points z=1, i,-1 onto the points w=i, 0,-i hence find the image of $|z| < 1$ on to w plane find invariant points of this transformation [8]

Q 5 a) Solve using Laplace Transform $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = te^{-t}$ given $y(0) = 4$ and $y'(0) = 2$. [6]

b)Find Complex form of the Fourier series for $f(x) = e^{ax}$ in $-\pi < x < \pi$ where 'a' is a real constant. Hence deduce that $\frac{\pi}{a\sinh a\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2}$ [6]

c) Verify Green's Theorem in the plane for $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where C is the boundary of the region defined by $y = x^2$ and $y = \sqrt{x}$. [8]

Q 6. a) Prove that $4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$ [6]

b) Find the map of the line x-y=1 by transformation $w = \frac{1}{z}$ [6]

c) Evaluate $\iint_S \bar{F}.d\bar{s}$ where $\bar{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ where S is the region bounded by $x^2 + y^2 = 4$, $z = 0$, $z = 3$ using Gauss divergence theorem. [8]

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Solution of A.M. III (IXTC)

Q.P. Code :- 24393

Q1 :-

a) $L[\sqrt{1+\sin t}]$

$$\Rightarrow L\left[\sqrt{\cos^2 t/2 + \sin^2 t/2 + 2\sin t/2 \cos t/2}\right]$$

$$\Rightarrow L\left[\sqrt{(\cos t/2 + \sin t/2)^2}\right]$$

$$\Rightarrow L[\cos t/2 + \sin t/2]$$

$$\Rightarrow L[\cos t/2] + L[\sin t/2]$$

$$= \frac{s}{s^2+(1/2)^2} + \frac{1/2}{s^2+(1/2)^2} = \frac{4s}{4s^2+1} + \frac{2}{4s^2+1}$$

$$= \frac{4s+2}{4s^2+1}$$

b) $\phi = 4xz^2 + x^2yz$

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + J\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(4xz^2 + x^2yz)$$

$$= i(4z^2 + 2xyz) + J(x^2z) + k(8xz + x^2y)$$

$$\nabla\phi|_{(1,-2,-1)} = i(4+4) + J(-1) + k(-8+(-1))$$

$$= 8\hat{i} - \hat{j} - 9\hat{k}$$

The directional derivative in the direction of the vector $2\hat{i} - \hat{j} - 2\hat{k}$

$$= \frac{(8\hat{i} - \hat{j} - 9\hat{k})\cdot(2\hat{i} - \hat{j} - 2\hat{k})}{\sqrt{4+1+4}} = \frac{16+1+18}{\sqrt{9}}$$

$$= \frac{35}{3}$$

c) Let $u = e^x \cos y - xy$

$$\frac{\partial u}{\partial x} = e^x \cos y - y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y - x$$

By C.R. equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$\therefore \frac{\partial v}{\partial y} = e^x \cos y - y = \frac{\partial u}{\partial x}.$$

$$v = e^x \sin y - y^2/2 + f(x)$$

$$\frac{\partial v}{\partial x} = e^x \sin y + f'(x)$$

$$\because \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$-e^x \sin y - x = -e^x \sin y - f'(x)$$

$$\therefore f'(x) = x$$

$$f(x) = \frac{x^2}{2} + c$$

$$\therefore v = e^x \sin y - y^2/2 + \frac{x^2}{2} + c.$$

d) Given $f(x) = x$

To obtain half range sine series of this function, consider this function as odd function in $(-2, 2)$

Total length of interval

$$2l = 4 \Rightarrow l = 2$$

$\because f(x)$ is odd $\Rightarrow a_0 = a_n = 0$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x) \sin \frac{n\pi x}{2}\, dx$$

$$= \frac{2}{2}\int_0^2 x \sin\frac{n\pi x}{2}\,dx$$

$$= \left[x\left(\frac{-\cos\frac{n\pi x}{2}}{n\pi/2}\right) - 1\left(\frac{-\cancel{\sin\frac{n\pi x}{2}}}{(n\pi/2)^2}\right)\right]_0^2$$

$$= \frac{2\,(-\cos n\pi)}{n\pi/2}$$

$$= -\frac{4}{n\pi}(-1)^n = \frac{4(-1)^{n+1}}{n\pi}.$$

$$\therefore\ f(x) = \sum_{n=0}^{\infty} \frac{4(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{2}$$

Q 2:-

a) $u + v = e^{2x}(x\cos 2y - y\sin 2y)$

Let $f(z) = u + iv$

$if(z) = iu - v$

$(1+i)f(z) = (u-v) + i(u+v)$

Let $(1+i)f(z) = F(z)$

$F(z) = U + iV$ where $U = u-v$ & $V = u+v$

$\because$ $F(z)$ is analytic as $f(z)$ is analytic.

$\therefore\ \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$ & $\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$ (C.R. equation)

$U = u - v$

$V = u + v = e^{2x}(x\cos 2y - y\sin 2y)$

$$\frac{\partial V}{\partial x} = (e^{2x} + 2e^{2x}\cdot x)\cos 2y - 2e^{2x}y\sin 2y$$

$$\frac{\partial V}{\partial y} = x e^{2x}(-2\sin 2y) - e^{2x}(\sin 2y + 2y\cos 2y)$$

$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$$

$$= \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x}$$

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$= \{x e^{2x}(-2\sin 2y) - e^{2x}(\sin 2y + 2y\cos 2y)\}$

$+ i\{(e^{2x} + 2e^{2x}.x)\cos 2y - 2e^{2x}\, y\sin 2y\}$

Put $x = z$ & $y = 0$

$F'(z) = i(e^{2z} + 2z\, e^{2z})$

$F(z) = i\, z e^{2z} + c$

b) $f(x) = 9 - x^2$, $-3 \le x \le 3$

Total length of interval $2l = 6$

$l = 3$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{3}$

$\because f(x) = 9 - x^2$

$f(-x) = 9 - (-x)^2 = 9 - x^2 = f(x)$

$\therefore f(x)$ is even function

$\Rightarrow b_n = 0$

$a_0 = \frac{1}{3}\int_{-3}^{3} f(x)\,dx = \frac{2}{3}\int_{0}^{3} f(x)\,dx$

$= \frac{2}{3}\int_0^3 (9 - x^2)\,dx = \frac{2}{3}\left[9(x)_0^3 - \left(\frac{x^3}{3}\right)_0^3\right]$

$= \frac{2}{3}\left[27 - \frac{27}{3}\right] = \frac{2}{3} \times 27\,(1 - \frac{1}{3})$

$= \frac{2\times 27}{3} \times \frac{2}{3} = 4 \times 3 = 12$

$a_n = \frac{1}{3}\int_{-3}^{3} f(x)\cos\frac{n\pi x}{3}\,dx$

$= \frac{2}{3}\int_0^3 (9 - x^2)\cos\frac{n\pi x}{3}\,dx$

$= \frac{2}{3}\left[(9-x^2)\left(\frac{\cancel{\sin n\pi x/3}}{n\pi/3}\right) - (-2x)\left(\frac{-\cos n\pi x/3}{(n\pi/3)^2}\right) + (-2)\left(\frac{-\cancel{\sin n\pi x/3}}{(n\pi/3)^3}\right)\right]_0^3$

$= \frac{2}{3}\left[\frac{2(3)(-\cos n\pi)}{(n\pi/3)^2}\right] = \frac{4\times 9}{n^2\pi^2}(-1)(-1)^n = \frac{36}{n^2\pi^2}(-1)^{n+1}$

$$\therefore f(x) = 6 + \sum_{n=1}^{\infty} \frac{36}{n^2\pi^2} (-1)^{n+1} \cos \frac{n\pi x}{3}$$

c) i) $L\left[t\sqrt{1+\sin t}\right]$

$$= (-1)^1 \frac{d}{ds} L\left[\sqrt{1+\sin t}\right]$$

$$= (-1) \frac{d}{ds} L\left[\cos t/_2 + \sin t/_2\right]$$

$$= (-1) \frac{d}{ds}\left[\frac{s}{s^2+1/4} + \frac{1/2}{s^2+1/4}\right]$$

$$= (-1) \frac{d}{ds}\left[\frac{4s}{4s^2+1} + \frac{2}{4s^2+1}\right]$$

$$= (-1) \frac{d}{ds}\left[\frac{4s+2}{4s^2+1}\right]$$

$$= (-1)\left[\frac{(4s^2+1)4 - (4s+2)(8s)}{(4s^2+1)^2}\right]$$

$$= 4(-1)\left[\frac{4s^2+1-8s^2-4s}{(4s^2+1)^2}\right] = (-4)\left[\frac{-4s^2-4s+1}{(4s^2+1)^2}\right]$$

$$= \frac{4(2s+1)^2}{(4s^2+1)^2}$$

ii) $L\left[\frac{\sinh 2t}{t}\right] = \int_s^{\infty} L[\sinh 2t]\,ds$

$$= \int_s^{\infty} L\left[\frac{e^{2t}-e^{-2t}}{2}\right] ds$$

$$= \frac{1}{2}\int_s^{\infty}\left[L(e^{2t}) - L(e^{-2t})\right] ds$$

$$= \frac{1}{2}\int_s^{\infty}\left[\frac{1}{s-2} - \frac{1}{s+2}\right] ds$$

$$= \frac{1}{2}\left[\log(s-2) - \log(s+2)\right]_s^{\infty} = \frac{1}{2}\left[\log\frac{s-2}{s+2}\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\log\frac{(1-2/s)}{(1+2/s)}\right]_s^{\infty} = \frac{1}{2}\left[\log 1 - \log\frac{(s-2)}{(s+2)}\right] = \frac{1}{2}\log\left(\frac{s+2}{s-2}\right)$$

Q 3 :-

a) $L^{-1}\left[\frac{s}{(s^2+4)^2}\right]$

$= L^{-1}\left[\frac{s}{s^2+4}\cdot\frac{1}{s^2+4}\right]$

$= \int_0^t \frac{\sin 2u}{2}\cdot \cos 2(t-u)\, du$

$= \frac{1}{4}\int_0^t 2\sin 2u \cos(2t-2u)\, du$

$= \frac{1}{4}\int_0^t [\sin(\cancel{2u}+2t-\cancel{2u}) + \sin(2u-2t+2u)]\, du$

$= \frac{1}{4}\int_0^t [\sin 2t + \sin(4u-2t)]\, du$

$= \frac{1}{4}\left[\sin 2t\,(u)_0^t + \left(\frac{-\cos(4u-2t)}{4}\right)_0^t\right]$

$= \frac{1}{4}\left[t\sin 2t - \frac{1}{4}\{\cos 2t - \cos 2t\}\right]$

$= \frac{t}{4}\sin 2t.$

b) $\because\ 2n J_n(x) = x J_{n-1}(x) + x J_{n+1}(x)$

$\therefore\ x J_{n-1}(x) = 2n J_n(x) - x J_{n+1}(x)$

$J_{n-1}(x) = \frac{2n}{x} J_n(x) - J_{n+1}(x)$

Put $n = -3/2$

$J_{-5/2}(x) = -\frac{3}{x} J_{-3/2}(x) - J_{-1/2}(x)$ — ①

$\because\ J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$

& $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

$\therefore$ from (i)

$$J_{-5/2}(x) = -\frac{3}{x}\left[-\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)\right] - \sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3\cos x}{x^2} + \frac{3}{x}\sin x - \cos x\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\sin x + \left(\frac{3-x^2}{x^2}\right)\cos x\right]$$

c) $f(x) = (\pi - x)^2$ in $0 \le x \le 2\pi$

$2l = 2\pi$

$l = \pi$

Let fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}(\pi - x)^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{(\pi - x)^3}{(-3)}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi^3}{-3} + \frac{\pi^3}{3}\right] = \frac{1}{\pi}\left[\frac{2\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}(\pi - x)^2\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(\pi - x)^2\left(\cancel{\frac{\sin nx}{n}}\right) - (-2)(\pi - x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(\cancel{-\frac{\sin nx}{n^3}}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[2\frac{(-\pi)(-\cos 2n\pi)}{n^2} - \frac{2\pi(-1)}{n^2}\right]$$

$$= \frac{2\cancel{\pi}}{\cancel{\pi} n^2}\left[1 + 1\right] = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \sin nx \, dx$$

$$= \frac{1}{\pi}\left[(\pi-x)^2\left(\frac{-\cos nx}{n}\right) - 2(\pi-x)(-1)\cancel{\left(\frac{-\sin nx}{n^2}\right)} + 2(-1)(-1)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left\{\frac{(\pi^2)\ (-\cos 2n\pi)}{n} - \frac{\pi^2(-1)}{n}\right\} + 2\left\{\frac{\cos 2n\pi - 1}{n^3}\right\}\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{\pi^2}{n} + \frac{\pi^2}{n}\right) + 2\left(\frac{1-1}{n^3}\right)\right] = 0$$

$$\therefore f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nx$$

$$(\pi - x)^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx \quad \text{— (I)}$$

Put $n = 0$

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}$$

$$\frac{2\pi^2}{3} = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - - - -\right)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + - - - - - - \text{— (II)}$$

Put $n = \pi$ in (I)

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n$$

$$\frac{-\pi^2}{3} = 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - - -\right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} - - - - - \text{(III)}$$

(II) + (III)

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \frac{2}{7^2} - - -$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} - - - - -$$

Q4:-

a) $f(t) = e^{-|t|}$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|t|} (\cos st + i \sin st)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|t|} \cos st\, dt + \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|t|} \sin st\, dt$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-|t|} \cos st\, dt + 0$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t} \cos st\, dt$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{e^{-t}}{s^2+1^2} (-\cos st + s \sin st) \right]_0^{\infty}$$

$$= \frac{2}{\sqrt{2\pi}} \left[0 - \frac{1}{s^2+1} (-1) \right]$$

$$= \sqrt{2/\pi}\ \frac{1}{s^2+1}$$

b) $$\int_{-1}^{1} f_1(x) f_1(x)\, dx = \int_{-1}^{1} 1.1\, dx = (x)_{-1}^{1} = 1+1 = 2$$

$$\int_{-1}^{1} f_2(x) f_2(x)\, dx = \int_{-1}^{1} x.x\, dx = 2\int_{0}^{1} x^2 dx = 2\left(\frac{x^3}{3}\right)_0^1 = 2/3$$

$$\int_{-1}^{1} f_1(x) f_2(x)\, dx = \int_{-1}^{1} 1.x\, dx = 0$$

$\Rightarrow f_1(x)$ & $f_2(x)$ are orthogonal in $[-1, 1]$

$\int_{-1}^{1} f_1(x) f_3(x)\, dx = 0$ (if f_1 & f_3 are orthogonal)

$$\Rightarrow \int_{-1}^{1} 1.(1 + Ax + Bx^2)\, dx = 0$$

$$\int_{-1}^{1} (1 + Ax + Bx^2)\, dx = 0$$

$$(x)_{-1}^{1} + A\left(\frac{x^2}{2}\right)_{-1}^{1} + B\left(\frac{x^3}{3}\right)_{-1}^{1} = 0$$

$$2 + \frac{A}{2}(1-1) + \frac{B}{3}(1+1) = 0$$

$$2 + \frac{2B}{3} = 0$$

$$\frac{2B}{3} = -2 \Rightarrow \boxed{B = -3}$$

$$\int_{-1}^{1} f_2(x) f_3(x)\, dx = 0 \qquad (\text{as } f_1 \text{ \& } f_3 \text{ are orthogonal})$$

$$\int_{-1}^{1} x(1 + Ax + Bx^2)\, dx = 0$$

$$\int_{-1}^{1} (x + Ax^2 + Bx^3)\, dx = 0 \Rightarrow 0 + A\left(\frac{x^3}{3}\right)_{-1}^{1} + 0 = 0$$

$$A\left(\tfrac{1}{3} + \tfrac{1}{3}\right) + 0 = 0$$

$$A = 0$$

$\therefore A = 0$ & $B = -3$

c) $z = 1, i, -1$ & $w = i, 0, -i$

let the bilinear transformation is

$$w = \frac{az + b}{cz + d}.$$

If $z = 1$ then $w = i$

$$i = \frac{a+b}{c+d} \Rightarrow a + b = ci + di$$

$$\Rightarrow a + b - ci - di = 0 \quad \text{—(i)}$$

If $z = i$ & $w = 0$

$$0 = \frac{ai + b}{ci + d} \Rightarrow ai + b = 0 \quad \text{—(ii)}$$

If $z = -1$ then $w = -i$

$$-i = \frac{-a+b}{-c+d} \Rightarrow -a + b = ic - id$$

$$\Rightarrow -a + b - ic + id = 0 \quad \text{—(iii)}$$

from (III) $b = -ai$

(I) + (III)

$2b - 2ic = 0$

$b = ic$

$\therefore \; ic = -ai \Rightarrow c = -a$

(I) − (III)

$2a - 2di = 0$

$a = di$

$d = \frac{a}{i} = -ai$

$$\therefore \quad w = \frac{az+b}{cz+d}$$

$$w = \frac{az-ai}{-az-ai}$$

$$w = \frac{z-i}{-z-i}$$ is the bilinear transformation.

$\Rightarrow -wz - wi = z - i$

$\Rightarrow \quad i - wi = z + wz$

$\Rightarrow \quad i(1-w) = z(1+w)$

$$\Rightarrow \quad z = \frac{i(1-w)}{(1+w)}$$

$\because \; |z| < 1$

$\Rightarrow \left| \frac{i(1-w)}{(1+w)} \right| < 1 \Rightarrow |i|\,|1-w| < |1+w|$

$\Rightarrow |1-u-iv| < |1+u+iv|$

$\Rightarrow \sqrt{(1-u)^2 + v^2} < \sqrt{(1+u)^2 + v^2}$

$\Rightarrow (1-u)^2 + v^2 < (1+u)^2 + v^2$

$\Rightarrow \cancel{1} - 2u + \cancel{u^2} + \cancel{v^2} < \cancel{1} + \cancel{u^2} + 2u + \cancel{v^2}$

$\Rightarrow \quad 4u > 0$

$\Rightarrow u > 0$ i.e. right half plane.

The invariant point of this transformation is

$$z = \frac{z-i}{-z-i}$$

$$-z^2 - iz = z - i$$

$$z^2 + (1+i)z - i = 0$$

$$z = \frac{-(1+i) \pm \sqrt{(1+i)^2 + 4i}}{2}$$

$$= \frac{-(1+i) \pm \sqrt{1-1+2i+4i}}{2}$$

$$= \frac{(-1-i) \pm \sqrt{6i}}{2}$$

$= \frac{-1-i \pm \sqrt{6i}}{2}$ are the invariant points of this transformation

Q5:-

a) $$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = t\bar{e}^{t}$$

taking laplace of it

$$L\left[\frac{d^2y}{dt^2}\right] + 2L\left[\frac{dy}{dt}\right] + L[y] = L[t\bar{e}^{t}]$$

$$[s^2 L(y) - s y(0) - y'(0)] + 2[sL(y) - y(0)] + L[y] = \frac{1}{(s+1)^2}$$

$$(s^2 + 2s + 1) L(y) - s \times 4 - 2 + 2(-4) = \frac{1}{(s+1)^2}$$

$$(s+1)^2 L(y) = \frac{1}{(s+1)^2} + 4s - 10$$

$$L(y) = \frac{1}{(s+1)^4} + \frac{(4s+4)}{(s+1)^2} - \frac{14}{(s+1)^2}$$

$$L(y) = \frac{1}{(s+1)^4} + \frac{4}{(s+1)} - \frac{14}{(s+1)^2}$$

$$y = L^{-1}\left(\frac{1}{(s+1)^4}\right) + 4L^{-1}\left(\frac{1}{s+1}\right) - 14L\left(\frac{1}{(s+1)^2}\right)$$

$$y = \bar{e}^{t}\frac{t^3}{3!} + 4\bar{e}^{t} - 14\bar{e}^{t}.t$$

b) $f(x) = e^{ax} \quad -\pi < x < \pi$

$2l = 2\pi \Rightarrow l = \pi$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ax} e^{-inx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(a-in)x} dx$$

$$= \frac{1}{2\pi} \left[\frac{e^{(a-in)x}}{(a-in)} \right]_{-\pi}^{\pi} = \frac{1}{2\pi} \left[\frac{e^{(a-in)\pi} - e^{-(a-in)\pi}}{(a-in)} \right]$$

$$= \frac{1}{2\pi} \left[\frac{e^{a\pi} e^{-in\pi} - e^{-a\pi} e^{in\pi}}{(a-in)} \right]$$

$$= \frac{1}{2\pi} \left[\frac{e^{a\pi}(-1)^n - e^{-a\pi}(-1)^n}{(a-in)} \right] = \frac{(-1)^n}{2\pi(a-in)} \left(\frac{e^{a\pi} - e^{-a\pi}}{2} \right)$$

$$= \frac{(-1)^n \sinh a\pi}{\pi(a-in)}$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh a\pi}{\pi(a-in)} e^{inx}$$

$$e^{ax} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (\sinh a\pi)(a+in)}{\pi(a^2+n^2)} e^{inx}$$

$$\frac{\pi e^{ax}}{\sinh a\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a+in)}{\pi(a^2+n^2)} (\cos nx + i \sin nx)$$

$$\frac{\pi e^{ax}}{\sinh a\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a\cos nx - n\sin nx)}{\pi(a^2+n^2)}$$

Put $a = 0$

$$\frac{\pi}{\sinh a\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n a}{(a^2+n^2)}$$

$$\frac{\pi}{a \sinh a\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(a^2+n^2)}$$

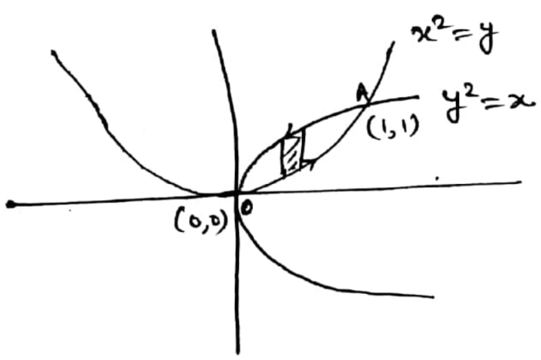
c) $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$

According to Greens theorem

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = \iint_R (-6y + 16y)\,dx\,dy \quad \text{—} \circledast$$

the region R is bounded by $y = x^2$ & $y = \sqrt{x}$
$y^2 = x$



L.H.S. of $\circledast$

1) along OA : $y = x^2 \Rightarrow dy = 2x\,dx$

$$\int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)\,2x\,dx$$

$$= \int_0^1 (3x^2 - 8x^4)\,dx + (8x^3 - 12x^4)\,dx$$

$$= 3\left(\frac{x^3}{3}\right)_0^1 - 8\left(\frac{x^5}{5}\right)_0^1 + 8\left(\frac{x^4}{4}\right)_0^1 - 12\left(\frac{x^5}{5}\right)_0^1$$

$$= \frac{3}{3} - \frac{8}{5} + \frac{8}{4} - \frac{12}{5} = 1 - \frac{8}{5} + 2 - \frac{12}{5} = 3 - \left(\frac{20}{5}\right) = -\frac{5}{5} = -1$$

L.H.S. of ⊛ along AO: $x = y^2 \Rightarrow dx = 2y\,dy$

$$\int_1^0 (3y^4 - 8y^2)\,2y\,dy + (4y - 6y^3)\,dy$$

$$= \int_1^0 (6y^5 - 16y^3)\,dy + (4y - 6y^3)\,dy$$

$$= 6\left(\frac{y^6}{6}\right)_1^0 - 16\left(\frac{y^4}{4}\right)_1^0 + 4\left(\frac{y^2}{2}\right)_1^0 - 6\left(\frac{y^4}{4}\right)_1^0$$

$$= \frac{6}{6}(-1) - \frac{16}{4}(-1) + \frac{4}{2}(-1) - \frac{6}{4}(-1)$$

$$= -1 + 4 - 2 + \frac{3}{2}$$

$$= -1 + 2 + \frac{3}{2} = 1 + \frac{3}{2} = \frac{5}{2}$$

L.H.S. of ⊛ is $= -1 + \frac{5}{2} = \frac{3}{2}$

R.H.S. of ⊛

$$\Rightarrow \iint_R 10y\,dx\,dy = \int_0^1 \int_{y=x^2}^{y=\sqrt{x}} 10y\,dx\,dy$$

$$= 10\int_0^1 \left(\frac{y^2}{2}\right)_{x^2}^{\sqrt{x}} dx = 5\int_0^1 (x - x^4)\,dx$$

$$= 5\left(\frac{x^2}{2} - \frac{x^5}{5}\right)_0^1 = 5\left(\frac{1}{2} - \frac{1}{5}\right) = 5\left(\frac{5-2}{10}\right)$$

$$= 5 \times \frac{3}{10} = \frac{3}{2}$$

$\therefore$ L.H.S. = R.H.S.

Hence Greens theorem is verified.

Q6 :-

a) $\because\ 2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$ — (i)

differentiating both sides w.r.t. x

$2J_n''(x) = J_{n-1}'(x) - J_{n+1}'(x)$ — (ii)

~~put n~~ replace n by $n-1$ in (1)

$$2J'_{n-1}(x) = J_{n-2}(x) - J_n(x) \quad \text{— (iii)}$$

replace n by $n+1$ in (1)

$$2J'_{n+1}(x) = J_n(x) - J_{n+2}(x) \quad \text{— (iv)}$$

from (iii) & (iv) in (ii)

$$4J''_n(x) = 2J'_{n-1}(x) - J'_{n+1}(x)$$

$$= J_{n-2}(x) - J_n(x) - J_n(x) + J_{n+2}(x)$$

$$\therefore 4J''_n(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$$

b) Given transformation is

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$$\Rightarrow x+iy = \frac{1}{u+iv}$$

$$\Rightarrow x+iy = \frac{u-iv}{(u+iv)(u-iv)}$$

$$\Rightarrow x+iy = \frac{u-iv}{u^2+v^2} \Rightarrow x = \frac{u}{u^2+v^2},\ y = \frac{-v}{u^2+v^2}$$

Image of $x-y=1$

$$\frac{u}{u^2+v^2} + \frac{v}{u^2+v^2} = 1$$

$$u+v = u^2+v^2$$

$$\Rightarrow u^2+v^2-u-v=0$$

Hence the image of $x-y=1$ is the circle

$u^2+v^2-u-v=0$

c) $F = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$

$$\nabla\cdot\bar{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}\right)$$

$= (4 - 4y + 2z)$

By Gauss divergence theorem

$$\iint_S \bar{F} \cdot d\bar{s} = \iiint_V (\nabla \cdot \bar{F})\, dv$$

$$= \iiint_V (4 - 4y + 2z)\, dx\, dy\, dz$$

$$= \iint \int_{z=0}^{3} (4 - 4y + 2z)\, dz\, dx\, dy$$

$$= \iint \left\{ 4(z)_0^3 - 4y(z)_0^3 + 2\left(\frac{z^2}{3}\right)_0^3 \right\} dx\, dy$$

$$= \iint \{ 12 - 12y + 6 \}\, dx\, dy = \iint_{x^2+y^2=4} (18 - 12y)\, dx\, dy$$

$$= \int_0^{2\pi} \int_{r=0}^{2} (18 - 12 r \sin\theta)\, r\, dr\, d\theta \quad \text{(Using Polar co-ordinates)}$$

$$= \int_0^{2\pi} \left[18\left(\frac{r^2}{2}\right)_0^2 - 12\left(\frac{r^3}{3}\right)_0^2 \sin\theta \right] d\theta$$

$$= \int_0^{2\pi} \left[9 \times 4 - 4(8) \sin\theta \right] d\theta$$

$$= 36\, (\theta)_0^{2\pi} - 32\, (-\cos\theta)_0^{2\pi}$$

$$= 36\,(2\pi) - 32\,(0)$$

$$= 72\pi$$