QP Code : 79055

(3 hours) **Total Marks: 100**

N.B: (1) All questions are compulsory
(2) Attempt any two sub-questions from part (a), part (b) and part (c)
(3) Figures to the right indicate marks for respective sub-questions.

8249BC3DB426BA1554D464856AE74C8F

1. (a) (i) Let $f:[0,1] \to \mathbb{R}$ be defined by $f(x) = x$. Let $\{P_n\}$ be a sequence of partitions of $[0,1]$ given by $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots\cdots, \frac{n-1}{n}, 1\right\}$. Find $U(P_n, f)$ and $L(P_n, f)$ and show that $\lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} L(P_n, f)$. Hence find $\int_0^1 f(x)dx$. **5**

(ii) State the change of variable formula for double integral clearly stating the conditions under which it is valid. Explain further how will you use it to express the double integral in polar co-ordinates. **5**

(b) (i) Show that the sequence $f_n(x) = \frac{1}{nx+1}$ does not converge uniformly on [0,1] but converge uniformly on $[a, 1]$ where $a > 0$. **5**

(ii) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = |xy|$. Show that f is continuous at (0,0). **5**

(c) (i) Use chain rule and find $\frac{\partial w}{\partial s}, \frac{\partial w}{\partial t}$ at $s = 0, t = 1$ where $w = xy + yz + zx,\ x(s,t) = s\,t, y(s,t) = e^{st}, z(s,t) = t^2$. **5**

(ii) Find the surface integral $\iint_S xy\, dS$ where S is the surface of a tetrahedron with sides $x = 0, y = 0, z = 0, x + y + z = 1$. **5**

2. (a) Let $f:[a,b] \to \mathbb{R}$ be a bounded function. Show that f is Riemann integrable on $[a,b]$ iff for each $\epsilon > 0$ there is a partition P_ϵ of $[a,b]$ such that $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$. **10**

(b) (i) Show that if $f:[a,b] \to \mathbb{R}$ is a monotone function then f is Riemann integrable on $[a,b]$. **6**

(ii) Show that $f:[0,3] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 0 \text{ if } x \in [0,3]\setminus\mathbb{Z} \\ -1 \text{ if } x \in [0,3] \cap \mathbb{Z} \end{cases}$ is Riemann integrable on [0,3]. **4**

(c) (i) State and prove the second fundamental theorem of integral calculus. **6**

(ii) Evaluate $\lim_{x\to 0} \frac{1}{x^3} \int_0^x \frac{t^2}{1+t^4} dt$. Justify each step. **4**

3. (a) Let $\{f_n\}$ be a sequence of continuous real valued functions defined on a non-empty subset S of $\mathbb{R}$. If $\{f_n\}$ converges uniformly to a function f on S then show that f is continuous on S. Further show that $\lim_{n\to\infty} \lim_{x\to p} f_n(x) = \lim_{x\to p} \lim_{n\to\infty} f_n(x)$ for each $p \in S$ **10**

(b) (i) Let $\{f_n\}$ be a sequence of continuous real valued functions defined on a non-empty subset S of $\mathbb{R}$. Show that $\{f_n\}$ converges uniformly to a function f on S if and only if for given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$, for $m, n \geq n_0$ and each $x \in S$. **6**

(ii) Let $f_n(x) = x^n$ for $x \in [0,1]$. Find $f(x) = \lim_{n\to\infty} f_n(x)$. Show that $\int_0^1 f(x)dx = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx$ but $\{f_n\}$ does not converge uniformly to f on $[0,1]$. **4**

VZ-Con. 1085-17.

3 (c) (i) State and prove Weierstrass M-test. 6

(ii) Show that the series $\sum_{n=1}^{\infty} e^{-nx}x^n$ converge uniformly on $[0,\infty)$. 4

4 (a) Let S be an open subset of $\mathbb{R}^n$ and $f: S \to \mathbb{R}^m$ with $f(x) = (f_1(x), f_2(x), \cdots\cdots f_m(x))$. Let $a \in S$. Then show that f is continuous at a iff each f_k is continuous at $a, 1 \le k \le m$. 10

(b) (i) Let S be an open subset of $\mathbb{R}^n$ and $a \in S$. If $f: S \to \mathbb{R}$ is differentiable at a then show that the total derivative of f is unique. 6

(ii) If f: $\mathbb{R}^3 \to \mathbb{R}^2$ is differentiable at (0,0) and $f(0,0,0) = (1,2)$, $Df(0,0,0) = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$. If g: $\mathbb{R}^2 \to \mathbb{R}^2$ is defined by $(x, y) = (x + 2y + 1, \ 3xy)$, then find $D(g.f)(0,0,0)$. 4

(c) (i) Let f: $\mathbb{R}^2 \to \mathbb{R}$ be a non-constant differentiable function. Suppose $f(x, y) = k$ (constant) describes a curve C then shoe that 6

(1) ∇f is normal to C.

(2) The direction derivative of f is 0 along C.

(3) The direction derivative has the largest value in the direction normal to C.

(ii) Find the real value $\theta \varepsilon (0,1)$ if it exists satisfying 4

$f(b) - f(a) = \nabla f(a + \theta(b - a)) \cdot (b - a)$ where

$f(x, y) = x^2 - y^2, a = (0,0), b = (2,1)$.

5. (a) State Divergence theorem for a solid in 3-space bounded by an orientable closed surface with positive orientation and prove the Divergence theorem for cubical region. 10

(b) (i) Verify divergence theorem for the vector field $\bar{F}$ and the region V where $\bar{F}(x, y, z) = 3x\hat{\imath} + xy\hat{\jmath} + 2xz\hat{k}$, V is the cube bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$. 6

(ii) Use stokes' Theorem to evaluate $\int_C \bar{F}.d\bar{r}$, where $\bar{F}$=y $\bar{\imath}$ +xz $\bar{\jmath}$ +$z^2\bar{k}$ and C is the boundary of the surface of the plane z=x+4 cut by the cylinder $x^2 + y^2 = 4$. 4

(c) (i) If S and C satisfy hypothesis of Stokes' Theorem and f, g have continuous second order partial derivatives ,prove with usual notation 6

1)$\int_C (f\nabla g) \cdot d\bar{r} = \iint_S (\nabla f \times \nabla g) \cdot \hat{n}ds$

2)$\int_C (f\nabla f) \cdot d\bar{r} = 0$

3)$\int_C (f\nabla g + g\nabla f) \cdot d\bar{r} = 0$

(ii) Evaluate the surface integral $\iint_S \bar{F} \cdot \hat{n}\, dS$, where $\bar{F}(x, y, z) = x\hat{\imath} + xy\hat{\jmath} + xz\hat{k}$, and S is the part of the plane $3x + y + z = 6$ that lies in the first octant. 4
