Q.P. Code 32504 Page No. 1

1 Random Signal analysis

Q.1. (a) A bag contains 7 red & 3 black balls & another bag contains 4 red & 5 black balls. one ball is transfered from the first bag to the second bag & then a ball is drawn from the second bag. If this ball happens to be red, find the prob. that a black ball was transfered.

→ We have P_1 = Prob. of transfering black ball $= \frac{3}{10}$

P_1' = prob. of now drawing a red ball $= \frac{4}{10}$

P_2 = Prob. of transfering red ball $= \frac{7}{10}$

P_2' = Prob. of now drawing red ball $= \frac{5}{10}$

req. Prob. $= \dfrac{P_1 P_1'}{P_1 P_1' + P_2 P_2'}$

$$= \frac{(3/10)\cdot(4/10)}{(3/10)\cdot(4/10) + (7/10)\cdot(5/10)}$$

$$= \frac{12}{47}$$

Q.1. (b) Check whether the R.P. given by ~~x~~ $X(t) = A\sin t + B\cos t$ is ergodic, where A, B are Random variables normally distributed with zero means & unit variances.

→ i) Since A, B are normally distributed with zero means & unit variances,

$E(A) = 0$, $E(B) = 0$, $V(A) = 1$, $V(B) = 1$

now, the ensemble average is,

$$E[X(t)] = E[A\sin t + B\cos t]$$
$$= \sin t \cdot E(A) + \cos t \cdot E(B)$$
$$= 0$$

ii) The time-average of $\{x(t)\}$ is

$$\overline{X}_T = \frac{1}{2T}\int_{-T}^{T}(A\cos t + B\sin t)\, dt$$

$$= \frac{1}{2T}\left[A\sin t - B\cos t\right]_{-T}^{T}$$

$$= \frac{1}{2T}\left[(A\sin T - B\cos T) - (A\sin(-T) - B\cos(-T))\right]$$

$$= \frac{1}{2T}\left[2A\sin T\right]$$

$$= \frac{1}{T}\left[A\sin T\right]$$

Since the fu^n $\sin T$ is bounded ($|\sin t| \le 1$) as $T\to\infty$, the limit of $\overline{X}(t)$ will be 0.

$$\therefore \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}X(t)\,dt = 0$$

iii) Since the time-average $\overline{X}_T$ is equal to the ensemble average $E(X(t))$, $x(t)$ is mean-ergodic.

Q. 1. (c) Write note on "Markov chain"

→ A random process in which the set of state space S & the set of paramete T are discrete taking only a finite or Countable no. of finite values & in which the Prob. P_{ij}, that the proce will go from i^{th} state to j^{th} state in the next step depends only on the present state & not on how it arrived at that state is called Markov chain of the 1^{st} order.

This is written as,

$$P[X_{n+1} = j \mid X_0 = i_0,\ X_1 = i_1,\ X_2 = i_2 \ldots X_{n-1} =$$

$$= P[X_{n+1} = j \mid X_n = i]$$

In the above expression $X_0, X_1 \dots X_{n-1}$ is the 'past', X_n is 'present' & 'X_{n+1}' is future. The statement states that the prob. of the future event (X_{n+1}) depends only on 'present event (X_n)' & not on 'past' $(X_0, X_1 \dots X_{n-1})$. Such a chain called 'memoryless'.

- The conditional probability that the markov chain which is in state i at step n will go to state j at the next step i.e $n+1$ i.e $P[X_{n+1}=j \mid X_n=i]$ is called one-step transition probability.

The condn prob. that the markov chain which is in state i at step k will go to state j at step $k+n$ i.e $P[X_{k+n}=j \mid X_k=i]$ is called n-step transition prob.

- The matrix in which the prob. that from n-th step to $(n+1)$th are given is called one-step transition prob matrix & is denoted by P.

The following matrix is one-step transition prob. matrix.

$$
\text{states of } X_n = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array}
\begin{array}{c} \text{states of } X_{n+1} \\ \begin{array}{cccc} 0 & 1 & 2 & \cdots \end{array} \\ \begin{bmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ P_{20} & P_{21} & P_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \end{array}
$$

The value $P_{ij} = P(X_{n+1}=j \mid X_n=i)$ is the prob. that the process which is in ith state at n-th step will make a transition into the jth state in the next step.

i) $P_{ij} \geq 0$, $i, j \geq 0$,

ii) $\sum_{j=0} P_{ij} = 1$, $i = 0, 1 \dots$

The Cond^n can be Stated in word as
i) all entries are positive &
ii) sum of the entries in each row is equal to 1.

Q.1. (d) With usual notation find 'p' of Binomial distribution if $n=6$ & $9P(X=4)=P(X=2)$

→ In the usual notation for binomial distribution we have,

$$P(X=x) = {}^{n}C_{x} \cdot p^{x} \cdot q^{n-x} = {}^{6}C_{x}\, p^{x} q^{6-x}$$

but $9\,P(X=4) = P(X=2)$

$$9 \times {}^{6}C_{4} \cdot p^{4} \cdot q^{2} = {}^{6}C_{2} \cdot p^{2} \cdot q^{4}$$

$$9p^{2} = q^{2} \qquad [\because {}^{n}C_{r} = {}^{n}C_{n-r}$$

$$3p = q$$

$$3p = 1-p \quad \ldots [p+q=1]$$

$$4p = 1$$

$$p = \frac{1}{4} = 0.25$$

8.2 (a) The power spectral density of a WSS process is given by

$$S_X(\omega) = \begin{cases} \frac{b}{a}(a-|\omega|), & |\omega| \le a \\ 0, & |\omega| > a \end{cases}$$

find the autocorrelation function.

→ The autocorrelation funⁿ is given by inverse fourier transform of $S(\omega)$.

$$R(\tau) = F^{-1}[S(\omega)]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, e^{i\tau\omega}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-a}^{a} \frac{b}{a}(a-|\omega|)e^{i\tau\omega}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-a}^{a} \frac{b}{a}(a-|\omega|)(\cos\tau\omega + i\sin\tau\omega)\, d\omega$$

Since the 1st integral is even & the second is odd.

$$R(\tau) = \frac{1}{\pi}\int_{0}^{a} \frac{b}{a}(a-\omega)\cos\tau\omega\, d\omega$$

.... [by integratⁿ by parts]

$$= \frac{1}{\pi}\cdot\frac{b}{a}\left[(a-\omega)\frac{\sin\tau\omega}{\tau} - \int \frac{\sin\tau\omega}{\tau}(-1)\, d\omega\right]_0^a$$

$$= \frac{b}{\pi a}\left[(a-\omega)\frac{\sin\tau\omega}{\tau} - \frac{\cos\tau\omega}{\tau^2}\right]_0^a$$

$$= \frac{b}{\pi a}\left[-\frac{\cos a\tau}{\tau^2} + \frac{1}{\tau^2}\right]$$

$$= \frac{b}{\pi a\tau^2}(1-\cos a\tau)$$

Q.2 (b) Let $X_1, X_2, X_3 \ldots$ be a sequence of random variable.
define: i) Convergence almost everywhe
2) Convergence in probability
3) convergence in mean $\mathcal{Z}$
4) Convergence in distribution

→ i) Convergence almost Everywher (A.E
Consider a sequence of random Variable $\{x_n(z)\}$. If the sequence of fu^n & $X_n(z)$ converges to the fu^n $X(z)$ for all z as $n \to \infty$ & if $P[x_n(z) \to X(z)] = 1$ as $n \to \infty$ then We say that the sequence $\{x_n(z)\}$ Converges almost everywhere to $X(z)$

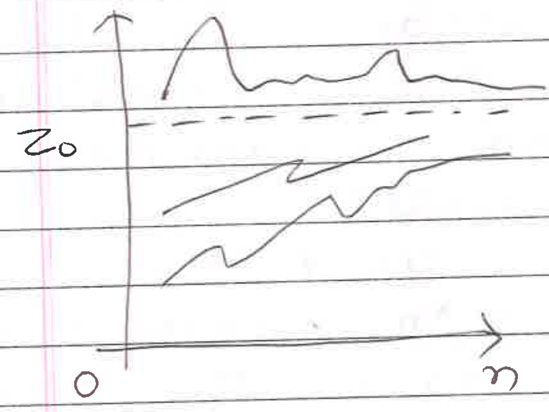
Example:
Consider the following experiment. An Urn contain 2 red balls & 2 yellow balls At time n a ball is drawn at random from the urn & its colour is noted. I the number of balls of this colour is greater than the no. of balls of the other colour then the ball is put back in the urn; otherwise the ball is kept out. Let $X_n(z)$ denote the no. of red balls in the urn after n^{th} draw. discuss the Convergence of Seq. of R.

Sol^n. — suppose in the 1st draw a red ball is selected & by rule it is to be kept. Now at any other draw if yellow ball is drawn since the no. of yellow balls (=2) is greater than the no. of red balls (=1) the yellow ball will be put in the urn. This will continue for every draw if a yellow ball is drawn.

ii) Convergence in probability –

<u>defn</u> – A seq. of R.V. $\{x_n(z)\}$ is said to converge in prob. to the random variable $X(z)$, if for any $\varepsilon > 0$, we have

$$P[\,|x_n(z) - X(z)| > \varepsilon\,] \to 0 \text{ as } n \to \infty$$



The fig. illustrates the convergence in Prob. where the limiting random variable $X(z)$ is constant z_0.

<u>explanation</u> – we have a seq. of R.V. $X_1(z), X_2(z), X_3(z) \dots$ if as $n \to \infty$.

$$P[\,|X_n(z) - X(z)| > \varepsilon\,] \to 0$$

then we say that the seq. of R.V. $\{x_n(z)\}$ converges in Prob. to $X(z)$.

iii) Mean square convergence –

Consider a seq. of Random variable $\{x_n(z)\}$ if $E\{x_n(z) - X(z)\}^2] \to 0$ as $n \to \infty$, we say that the seq. of R.V. $\{x_n(z)\}$ converges to $X(z)$ in mean sq. sense.

We denote mean sq. convergence by limit in the mean.

$$\text{l.i.m } X_n(z) \to X(z) \text{ as } n \to \infty$$

example: Let z be a no. selected at Random from interval $S = [0, 1]$. Let the Probability that z lies in a sub-interval be equal to the length of the sub-interval. Let us define a new seq. of Random variable by

$$V_n(z) = z\left(1 - \frac{1}{n}\right), \qquad n = 1, 2, 3 \dots$$

then we can show that seq. $\{V_n(z)\}$ converges in mean sq. sense.

iv) Convergence in distribution -

def^n - A seq. of Random variables $\{x_n(z)\}$ is said to converge in distribution if the seq. of their distribut fu^n $\{f_n(z)\}$ converges to the distribution fu^n $F(z)$ for all z at which $F(z)$ is continuous i.e if $F_n(z) \to F(z)$ as $n \to \infty$

Central limit thm. is example of convergence in distribut^n.

example: Central limit thm.

If $\bar{X} = X_1 + X_2 + \cdots + X_n$ where $E[x_i] = \mu$ & $var(X_i) = \sigma^2$, $i = 1, 2 \ldots n, \ldots$ then by central limit thm. the distribution of $\bar{X}$ tends to normal distribution with mean μ & Std. deviation $\sigma/\sqrt{n}$.

Q. 3 (a) P.T. if input LTI is WSS, the output is also WSS.

→ Let the time-invariant linear system

$$Y(t) = \int_{-\infty}^{\infty} h(u)\, X(t-u)\, du$$

$$E[Y(t)] = \int_{-\infty}^{\infty} h(u)\, E[X(t-u)]\, du$$

since $X(t)$ is WSS, $E[X(t-u)]$ is constant i.e independent of t say μ_x.

$$E[Y(t)] = \int_{-\infty}^{\infty} h(u) \cdot \mu_x\, du = \mu_x \int_{-\infty}^{\infty} h(u)\, du$$

$$= \text{independent of } t.$$

further, the autocorrelation fu^n of $Y(t)$ is given by

$$R_Y(t_1, t_2) = E[Y(t_1) \cdot Y(t_2)]$$

$$= E\left[\int_{-\infty}^{\infty} h(u_1)\, X(t_1 - u_1)\, du_1 \cdot \int_{-\infty}^{\infty} h(u_2)\, X(t_2 - u_2)\, du_2\right]$$

$$= \int_{-\infty}^{\infty} h(u_1)\, du_1 \int_{-\infty}^{\infty} h(u_2)\, du_2 \cdot E[X(t_1 - u_1) \cdot X(t_2 - u_2)]$$

Since $X(t)$ is WSS, its autocorrelation is fu^n of time-difference bet^n $t_1 - u_1$ & $t_2 - u_2$.

$$\therefore E[X(t_1 - u_1) \cdot X(t_2 - u_2)] = R_X[(t_1 - u_1) \cdot (t_2 - u_2)]$$

$$= R_X(t_1', t_2')$$

where $t_1' = t_1 - u_1$, $t_2' = t_2 - u_2$

$$E[X(t_1 - u_1) \cdot X(t_2 - u_2)] = R_X(t_1' - t_2')$$

$$= R_X[(t_1 - t_2) - (u_1 - u_2)]$$

$$= X(t) \text{ is WSS}$$

$$R_Y(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1) \cdot h(u_2) \cdot R_X[(t_1 - t_2) - (u_1 -$$
$$du_1 \cdot du_2$$

$$= R_Y(\tau)$$

thus mean of $Y(t)$ is independent of -
& the autocorrelation fu^n of $Y(t)$ is
a fu^n of the time - difference.
Hence, $Y(t)$ also is WSS.

Q.3.b- A binary communication transmitter sends data as one of two types of s/g denoted by 0 or 1. due to noise, sometimes a transmitted 1 is received as 0 & vice versa. If the prob. that a transmitted 0 is correctly received as 0 is 0.9 & the Prob. that the 1 is received as 1 is 0.8 & if the Prob. of transmitting 0 is 0.45 find the prob. that (i) a 1 is received (2) a 0 is received (iii) a 1 was transmitted given that 1 was received (iv) 0 was transmitted given that a 0 was received. (v) the error has occurred.

→ $P(T_0)$ = 0 is transmitted = 0.45

$P(T_1)$ = a 1 is transmitted = $1 - P(T_0)$

= 0.55

$P(R_0/T_0)$ = 0 is received when 0 is transmitted

= 0.9

$P(R_1|T_0)$ = a 1 is received when 0 was transmitted = $1 - 0.9$

= 0.1

$P(R_1/T_1)$ = a 1 is received when 1 was transmitted

= 0.8

$P(R_0/T_1)$ = a 0 is received when a 1 was transmitted

= $1 - 0.8$ = 0.2

1) P(1 is received) = $P(R_1)$

$$P(R_1) = P(R_1/T_1) \cdot P(T_1) + P(R_1/T_0) \cdot P(T_0)$$

$$= 0.8 \times 0.55 + 0.1 \times 0.45$$

$$\boxed{P(R_1) = 0.485}$$

ii) P(0 is received) = P(0 is received when 0 is transmitted) + P(0 is received when 1 is tx)

$P(R_0) = P(R_0/T_0) \cdot P(T_0) + P(R_0/T_1) \cdot P(T_1)$

$= 0.9 \times 0.45 + 0.2 \times 0.55$

$\boxed{P(R_0) = 0.515}$

iii) P (1 was transmitted given that 1 was received i.e.

$P(T_1/R_1) = \frac{P(R_1/T_1) \cdot P(T_1)}{P(R_1)}$

$P(T_1/R_1) = \frac{0.8 \times 0.55}{0.485} = 0.907$

4) P (0 was transmitted given that 0 was received) $= P(T_0/R_0)$

$= \frac{P(R_0/T_0) \cdot P(T_0)}{P(R_0)} = \frac{0.9 \times 0.45}{0.515}$

$= 0.786$

v) $P(\text{error}) = P(R_0/T_1) \cdot P(T_1) + P(R_1/T_0) \cdot P(T$

$= 0.2 \times 0.55 + 0.1 \times 0.45$

$P(\text{error}) = 0.155.$

Q. 4 (a) If $f(x) = K \cdot e^{-|x|}$, $-\infty < x < \infty$ represents a pdf then find
(1) the value of K (2) mean & variance

→ We know $|x| = x$ if $x \geq 0$
$|x| = -x$ if $x \leq 0$

i) x is cont. R.V. whose pdf is defined in $[-\infty, \infty]$ we have

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_{-\infty}^{\infty} K \cdot e^{-|x|} = 1$$

$$2\int_{0}^{\infty} K \cdot e^{-x}\, dx = 1 \qquad \left[\because \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx\right]$$
if $f(x)$ is even

$$2K\left[-e^{-x}\right]_0^{\infty} = 1$$

$$-2K[0-1] = 1$$

$$2K = 1 \qquad \boxed{K = 1/2}$$

ii) $$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \cdot \left[0.5e^{-|x|}\right] dx$$

$$= 0 \qquad \left[\because \int_{-a}^{a} f(x)\,dx = 0\right.$$ if $f(x)$ is odd]

$$E[x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^2 \times 0.5\, e^{-|x|}\, dx$$

$$= 0.5 \times 2 \int_{0}^{\infty} x^2 \cdot e^{-x}\, dx$$

$$= \int_{0}^{\infty} x^2 \cdot e^{-x}\, dx \quad = \Gamma 3 = 2!$$

$$= 2$$

mean $E(x) = 0$

Var. $E(x^2) - E[x)]^2 = 2 - 0 = 2$

Q.4. (b) The transition prob. matrix of Markov chain is

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.5 & 0.4 & 0.1 \\ 2 & 0.3 & 0.4 & 0.3 \\ 3 & 0.2 & 0.3 & 0.5 \end{array}$$

find the limiting prob.

→ Limiting Prob. be $\pi = [\pi_1, \pi_2, \pi_3]$ then we have $\pi P = \pi$ such that $\Sigma \pi_i =$

$$[\pi_1 \ \pi_2 \ \pi_3] \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} = [\pi_1 \ \pi_2$$

$$0.5\pi_1 + 0.3\pi_2 + 0.2\pi_3 = \pi_1$$
$$0.4\pi_1 + 0.4\pi_2 + 0.3\pi_3 = \pi_2$$
$$0.1\pi_1 + 0.3\pi_2 + 0.5\pi_3 = \pi_3$$
& $\pi_1 + \pi_2 + \pi_3 = 1$

from above eq^n,

$$-0.5\pi_1 + 0.3\pi_2 + 0.2\pi_3 = 0 \quad — (1)$$
$$0.4\pi_1 - 0.6\pi_2 + 0.3\pi_3 = 0 \quad — (2)$$
$$0.1\pi_1 + 0.3\pi_2 - 0.5\pi_3 = 0 \quad — (3)$$

Putting $\pi_3 = 1 - \pi_1 - \pi_2$ in (1) & (2) we g

$$-0.5\pi_1 + 0.3\pi_2 + 0.2(1 - \pi_1 - \pi_2) = 0$$
$$-0.7\pi_1 + 0.1\pi_2 = -0.2 \quad — (4)$$

$$0.4\pi_1 - 0.6\pi_2 + 0.3(-\pi_1 - \pi_2 + 1) = 0$$
$$0.1\pi_1 - 0.9\pi_2 = -0.3 \quad — (5)$$

multiply (4) by 0.9 & (5) by 0.1 & add

$$-0.63\pi_1 + 0.09\pi_2 = -0.18$$
$$0.01\pi_1 - 0.09\pi_2 = -0.03$$
$$-0.62\pi_1 = -0.21$$

$$\pi_1 = \frac{0.21}{0.62} = 0.34$$

from ④, $0.1\pi_2 = -0.2 + 0.7\pi_1$

$$= -0.2 + 0.238$$

$$-0.1\pi_2 = 0.038$$

$$\pi_2 = 0.38$$

& $\pi_3 = 1 - \pi_1 - \pi_2$

$$= 1 - 0.34 - 0.38 = 0.28$$

$\pi_1 = 0.34$, $\pi_2 = 0.38$ & $\pi_3 = 0.28$

Q. 5. (a) The joint Prob. density funⁿ of two cont. random variable X & Y is given by

$$f_{XY}(x,y) = \begin{cases} c\, e^{-x}\cdot e^{-y} & 0 < x < \infty \\ & 0 < y < \infty \\ 0 & \text{elsewhere} \end{cases}$$

① find the value of C. (1)

~~② $f_X(x)$, f_Y~~ ② $f_X(x)$, $f_Y(y)$ (1)

③ $f_{X/Y}(x/y)$, $f_{Y/X}(y/x)$ (4)

④ $E[Y/X = x]$, $E[X/Y = y]$ (4)

→ ①
$$\int_0^\infty \int_0^\infty f(x,y)\, dx\, dy = 1$$

$$c \int_0^\infty \int_0^\infty e^{-x}\cdot e^{-y}\, dx\, dy = c\int_0^\infty e^{-x} dx \int_0^\infty e^{-y} dy$$

$$= c\left[-\frac{e^{-x}}{1}\right]_0^\infty \left[-\frac{e^{y}}{1}\right]_0^\infty$$

$$= c$$

$$= 1$$

② now $f_X(x) = \int_0^\infty f_{XY}(x,y)\, dy$

$$= \int_0^\infty e^{-x} e^{-y}\, dy$$

$$= e^{-x}\left(-e^{y}\right)_0^\infty$$

$$= e^{-x}$$

||ly $f_Y(y) = e^{-y}$

③
$$f_{X/Y}(x/y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{e^{-x}\cdot e^{-y}}{e^{-y}}$$

$$= e^{-x}$$

$$f_{Y/X}(y/x) = \frac{f_{XY}(x,y)}{f_X(x)} = e^{-y}$$

Q. 5.

(4) $E(y/x = x) = \int_0^\infty y \cdot f_{y/x}(y/x)\, dy$

$= \int_0^\infty y \cdot e^{-y}\, dy$

$= \left[y(-e^{-y}) - \int -e^{-y} \cdot 1 \cdot dy \right]_0^\infty$

$= \left[-ye^{-y} - e^{-y} \right]_0^\infty$

$= 1$

|||ly $E(x/y = y) = 1$

Q. 5. (b) Write note on "Little's formula".

→ - Let $N(t)$ denote the no. of customers in the s/m at time t & Let $P_n(t)$ denote the prob. distributⁿ of the no. of customers in the s/m. Both $N(t)$ & $P_n(t)$ are dependent on time 't'. This state of the s/m is called transient state.

- but as $t \to \infty$, $P_n(t)$ may tend to P_n, a limit, if it exists. When this happens the s/m is said to have reached equilibrium or steady state.

There are certain general relations in queueing theory which hold good under general conditions.

1. Little's Results

This is a very simple but very elegant result in queueing theory. If L denotes the average (expected) no. of customers (units) in the s/m, i.e. the no. in the queue as well as those being served; W denotes the avg. (expected) time a customer has to be in the s/m (waiting time + serving time) in steady state, λ is the long-term arrival rate i.e the no. of customers arriving at the service centre per unit time, then

$$\boxed{L = \lambda W} \quad \text{— (a)}$$

2. Utilisation factor

The ratio λ/μ where λ is the arrival rate & μ is the service rate is called the utilisation factor & is denoted by ρ.

$$\boxed{\rho = \frac{\lambda}{\mu}} \quad \text{— (b)}$$

It gives an estimate of fraction of the total time for which the service centre is busy.

The fraction of time $\boxed{\text{The s/m is busy} = \rho = \frac{\lambda}{\mu}}$

the fraction of time $\boxed{\text{The s/m is Idle} = 1-\rho}$ — c

3. Derivation of Little formula
Let λ_a be the mean arrival rate of entering customers. Suppose a customer is req. to pay on an avg. an entrance fee 'E' according to certain rule of the s/m; T denotes time, then the no. of customers entering the s/m is $\lambda_a T$ & total amount paid by the customer to the s/m is $\lambda_a TE$.
E' is the avg. amount the s/m earned in unit time. then the total amount earned by s/m in time T is E'T.
but two amounts must be equal.

$$E'T = \lambda_a ET$$

$$E' = \lambda_a E \quad \text{— d}$$

from d we get $L = \lambda_a W$ — i.e A

if we suppose that each customers pay Re.1 per being in the s/m then E' will be the mean arrival rates of customers per unit time which we denote by L. E will be the avg. time a customer spends in the s/m which we denote by W.
- if we suppose the customer pay Re. 1 for being in the queue then d gives

$$\boxed{L_q = \lambda_a W_q} \quad \text{— B}$$

Where L_q & W_q has the meaning given above.

if we suppose that a customer pay Re. 1 for receiving the service, then (D) gives

$$\boxed{L_s = \lambda_a W_s} \qquad (C)$$

(A), (B) & (C) are known as Little's formula

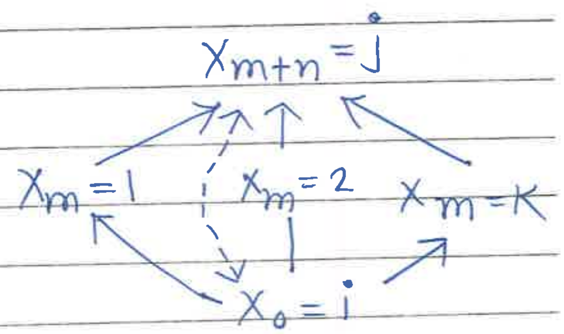
Q. 6. (a) State & prove Chapman - Kolmogorov equation.

→ It helps us to find prob. that the s/m will go from i^{th} state to j^{th} state in $(m+n)$ steps.

It states that we have to go from i^{th} state to some intermediate K^{th} state in m steps & then from K^{th} state we go to j^{th} state in next n steps.

The prob. that the s/m which is in i^{th} state at $n=0$ i.e $X_0 = i$ goes to j^{th} state in $m+n$ steps i.e to $X_{m+n} = j$ is denoted by $P(X_{m+n}=j \mid X_0 = i)$. but this prob. is also denoted by $P^{m+n}(i,j)$

now, the prob. that the s/m goes from $X_0 = i$ to $X_m = K$ & $X_{m+n} = j$ in n steps can be denoted by



$$\sum P(X_{m+n}=j, X_m = K \mid X_0 = i)$$

Thus, Chapman - Kolmogorov eq^n states

$$P^{m+n}(i,j) = \sum_K P^m(i,K) P^n(K,j)$$

proof - The above eq^n states that to go from i^{th} state to the j^{th} state in $(m+n)$ steps, we have to go from i^{th} state to some intermediate k^{th} state in m steps & then from k-th state to j^{th} state in n steps. by the Markov property the two prob. are independent.

$$P(X_{m+n}=j \mid X_0 = i) = \sum_K P(X_{m+n} = j, X_m = K \mid X_0 = i)$$

but $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

$$P(X_{m+n}=j, X_m=k \mid X_0=i) = \frac{P(X_{m+n}=j, X_m=k, X_0=i)}{P(X_0=i)}$$

$$= \frac{P(X_{m+n}=j, X_m=k, X_0=i)}{P(X_m=k, X_0=i)} \cdot \frac{P(X_m=k, X_0=i)}{P(X_0=i)}$$

$$= P(X_{m+n}=j \mid X_m=k, X_0=i) \cdot P(X_m=k \mid X_0=i)$$

by markov property 1^{st} prob depends only on the present state X_m & not on the past state X_0.

i.e $P[X_{m+n}=j \mid X_m=k, X_0=i] = P[X_{m+n}=j \mid X_m=k$

$$\therefore P(X_{m+n}=j, X_m=k \mid X_0=i) = P(X_{m+n}=j \mid X_m=k) \cdot P(X_m=k \mid X_0=i) \quad \text{— (A)}$$

$$= p^n(k,j) \cdot p^m(i,k)$$

$$= p^m(i,k) \cdot p^n(k,j)$$

$$P(X_{m+n}=j \mid X_0=i) = \sum_k p^m(i,k) \cdot p^n(k,j)$$

Q. 6. (b) Write a short note on the following special distribution
1) poisson distribution
2) Gaussian distribution.

→ i) poisson distribution
A distributⁿ of the form

$$P(X=x) = e^{-m}\frac{m^x}{x!} \qquad x = 0, 1, 2 \cdots$$

is called poisson distribution with parameter m.

$$\text{mean} = E(x) = m$$

$$Var(x) = E(x^2) - [E(x)]^2$$
$$= m.$$

ii) Gaussian distribution or Normal distribution.
→ a distribution of the form

$$F(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \qquad -\infty < x < \infty$$

is called Gaussian distributⁿ with parameter m & σ^2.

$$\text{mean} = E(x) = m$$

$$Var(x) = \sigma^2$$