SE-Comp, sem-IV (CBSGS) (R-2012), Applied mathematics-IV ①

QP - 37068

Solution

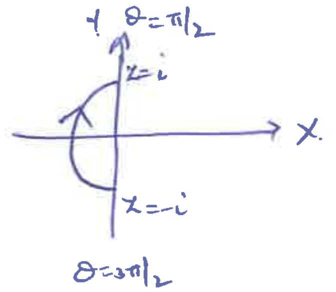
Q.1 (a) ∵ The contour is a semi-circle and $f(z)=|z|=\sqrt{x^2+y^2}$, use polar coordinates.

$x=r\cos\theta$, $y=r\sin\theta$, $\sqrt{x^2+y^2}=r^2$

$z=re^{i\theta}$ ∵ $|z|=1$, $r=1$

$\therefore z=e^{i\theta}$, $dz=ie^{i\theta}d\theta$

Hence, $\int_C |z|dz = \int_{3\pi/2}^{\pi/2} ie^{i\theta}d\theta = 2i$



(b) The characteristic equation is

$\begin{vmatrix} 1-\lambda & 0 \\ 2 & 4-\lambda \end{vmatrix}=0 \Rightarrow \lambda=1, 4$

eigenvalues of A^{-1} are $1, 1/4$

$\therefore 4A^{-1}+3A+2I$ are $\boxed{4(1)+3(1)+2=9}$ & $\boxed{4(\tfrac{1}{4})+3(4)+2=1+12+2=15}$

(c) If the equation of the line of regression of y on x is $y-\bar{y}=b_{yx}(x-\bar{x})$

then b_{yx} is the slope of the line of regression. $\therefore b_{yx}=0.6$

but $b_{yx}=r\dfrac{\sigma_y}{\sigma_x}$ and $\sigma_y=2\sigma_x$

Thus $0.6=r\dfrac{2\sigma_x}{\sigma_x}$

$\therefore 0.6=2r$

$\therefore \boxed{r=0.3}$

(d) The given problem becomes.

Minimise $z=0x_1+x_2+3x_3$

Subject to $-2x_1-x_2+0x_3\geq -3$

$x_1+2x_2+6x_3\geq 5$

$-x_1+x_2+2x_3\geq 2$

$x_1-x_2-2x_3\geq -2$

$x_1, x_2, x_3\geq 0$

∵ The last given equality type constraint is now expressed in the form of two constraints, we write y_3 as $y_3'-y_3''$

Maximise $w=-3y_1+5y_2+2y_3'-2y_3''$

Subject to $-2y_1+y_2-y_3'+y_3''\leq 0$

$-y_1+2y_2+y_3'-y_3''\leq 1$

$0y_1+6y_2+2y_3'-2y_3''\leq 3$

Replacing $y_3' - y_3''$ by y_3 which now is unrestricted, the dual becomes

Maximise $\quad w = -3y_1 + 5y_2 + 2y_3$

Subject to $\quad -2y_1 + y_2 - y_3 \leq 0$

$-y_1 + 2y_2 + y_3 \leq 1$

$6y_2 + 2y_3 \leq 3$

$y_1, y_2 \geq 0$, y_3 is unrestricted.

Q-2 (a) The circle $|z-1| = 3$ has centre at $(1,0)$ and radius 3. And $z=-1$ lies inside the circle. and $f(z) = e^{2z}$ is analytic in C.

$\therefore$ by corollary of Cauchy's Formula.

$$\int_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$$

$$\therefore \int_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i}{3!} f^{3}(z_0) = \frac{2\pi i}{6}\left(\frac{8}{e^2}\right) \quad \left[\because f(z) = e^{2z},\ f^3(z) = 8e^{2z},\ f^3(-1) = 8/e^2\right]$$

$$= \frac{8\pi i}{3e^2}$$

(b) The characteristic equation of A is

$\lambda^3 - 18\lambda^2 + 81\lambda - 108 = 0$

$\therefore (\lambda-3)(\lambda^2 - 15\lambda + 36) = 0$

$\therefore (\lambda-3)(\lambda-12)(\lambda-3) = 0$

$\therefore \lambda = 3, 3, 12$

we know that each characteristic root of A is also a root of the minimal polynomial of A. So if $f(x)$ is the minimal polynomial of A then $x-3$ & $x-12$ are the factors of $f(x)$.

Let $f(x) = (x-3)(x-12) = x^2 - 15x + 26$

Now, $A^2 - 15A + 36I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore f(x) = x^2 - 15x + 36$ annihilates A

Thus, $f(x)$ is the monic polynomial of lowest degree that annihilates A.

Hence, $f(x)$ is the minimal polynomial of A.

Since the degree is less than the order of A, A is derogatory.

Q2 (c) we have S.N.V

$$Z = \frac{X-m}{\sigma} = \frac{X-2.5}{3.5}$$

(i) when $X=2$, $Z = \frac{2-2.5}{3.5} = -0.14$

when $X=4.5$, $Z = \frac{4.5-2.5}{3.5} = 0.57$

$\therefore P(2 \le X \le 4.5) = P(-0.14 \le Z \le 0.57)$

= Area between ($Z=-0.14$ and $Z=0.57$)

= Area between ($Z=0$ & $Z=0.14$)

+ Area between ($Z=0$ & $Z=0.57$)

$= 0.0557 + 0.2157$

$= 0.2714$

(ii) when $X=-1.5$, $Z = \frac{-1.5-2.5}{3.5} = -1.14$

when $X=5.3$, $Z = \frac{5.3-2.5}{3.5} = 0.8$

$\therefore P(-1.5 \le X \le 5.3) = P(-1.14 \le Z \le 0.8)$

= Area between ($Z=-1.14$ & $Z=0.8$)

= Area between ($Z=0$ & $Z=1.14$)

+ Area between ($Z=0$ & $Z=0.8$)

$= 0.3729 + 0.2881 = 0.6610$

Q3 (a) we have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

ie $\int_0^{\infty} K x e^{-x/3}\,dx = 1$

$\therefore \boxed{K = \frac{1}{9}}$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \frac{1}{9}\int_0^{\infty} x^2 e^{-x/3}\,dx$$

$= 6$

$$\therefore P(X > 6) = \frac{1}{9}\int_6^{\infty} x e^{-x/3}\,dx$$

$= 3e^{-2}$

$= 0.406.$

Q3(b) we first express the given problem in standard form.

Maximise $Z = 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$

i.e. $Z - 4x_1 - 10x_2 + 0s_1 + 0s_2 + 0s_3 = 0$

Subject to $2x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 10$

$2x_1 + 5x_2 + 0s_2 + s_2 + 0s_3 = 20$

$2x_1 + 3x_2 + 0s_1 + 0s_2 + s_3 = 18$

In tabular form.

Iteration Number	Basic Variables	x_1	x_2	s_1	s_2	s_3	R.H.S Solution	Ratio
0	Z	-4	-10	0	0	0	0	
s_2 leaves	s_1	2	1	1	0	0	10	10
x_2 enters.	s_2	2	5^*	0	1	0	20	4 ←
	s_3	2	3	0	0	1	18	6
1	Z	0	0	0	2	0	40	
s_2 leaves	s_1	$8/5$	0	1	$-1/5$	0	6	$15/4$ ←
x_1 enters	x_2	$2/5$	1	0	$1/5$	0	4	10
	s_3	$4/5$	0	0	$-3/5$	1	6	$15/2$

$\therefore x_1 = 0,\ x_2 = 4,\ Z_{max} = 40$

But further considerations show that s_1 may leave & x_1 may enter.

Iteration Number	Basic Variables	x_1	x_2	s_1	s_2	s_3	R.H.S Solution
2	Z	0	0	0	$-1/5$	0	40
	x_1	1	0	$5/8$	$-1/8$	0	$15/4$
	x_2	0	1	$-1/4$	$1/5$	0	$5/2$
	s_3	0	0	$-1/2$	$-1/2$	1	3

$\therefore x_1 = 15/4,\ x_2 = 5/2\quad Z_{max} = 40$

(c) $f(z) = \dfrac{2}{(z-1)(z-2)} = \dfrac{-2}{z-1} + \dfrac{2}{z-2}$

(i) $|z|<1$ clearly $|z|<2$

$$f(z) = \frac{2}{1-z} - \frac{2}{2(1-\frac{z}{2})} = 2(1-z)^{-1} - \left[1-\frac{z}{2}\right]^{-1}$$

$f(z) = 2[1+z+z^2+z^3 \cdots] - [1+\frac{z}{2}+(\frac{z}{2})^2+(\frac{z}{2})^3+\cdots]$ (5)

Case (ii) when $1<|z|<2$, we write.

$$\frac{2}{(z-1)(z-2)} = -\frac{2}{(z-1)} + \frac{2}{(z-2)}$$

$$= \frac{-2}{z[1-\frac{1}{z}]} - \frac{2}{2[1-\frac{z}{2}]}$$

$$= -\frac{2}{z}[1-\frac{1}{z}]^{-1} - [1-\frac{z}{2}]^{-1}$$

$$= -\frac{2}{z}[1+\frac{1}{z}+\frac{1}{z^2}+\cdots] - [1+(\frac{z}{2})+(\frac{z}{2})^2+\cdots]$$

Case (iii) when $|z|>2$, $\frac{|z|}{2}>1$

Also when $|z|>2$, $|z|\geq 1$ $\therefore \frac{1}{|z|}<1$ we write

$$f(z) = -\frac{2}{z}\frac{1}{(1-\frac{1}{z})} + \frac{2}{z}\frac{1}{(1-\frac{2}{z})}$$

$$= -\frac{2}{z}(1-\frac{1}{z})^{-1} + \frac{2}{z}(1-\frac{2}{z})^{-1}$$

$$= -\frac{2}{z}(1+\frac{1}{z}+\frac{1}{z^2}+\cdots) + \frac{2}{z}(\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\cdots)$$

$$= -2(\frac{1}{z}+\frac{1}{z^2}+\cdots) + 4(\frac{1}{z^2}+\frac{2}{z^3}+\frac{4}{z^4}+\cdots)$$

Q4 (a) we have $p = 20\% = \frac{20}{100} = 0.2$, $q=1-p=0.8$, $n=6$

$$\therefore P(X=x) = {}^nC_x\, p^x q^{n-x} = 6C_x \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{6-x}$$

$$\therefore P(X\geq 4) = P(X=4) + P(X=5) + P(X=6)$$

$$= 6C_4\left(\frac{1}{5}\right)^4\left(\frac{4}{5}\right)^2 + 6C_5\left(\frac{1}{5}\right)^5\left(\frac{4}{5}\right)^1 + 6C_6\left(\frac{1}{5}\right)^6\left(\frac{4}{5}\right)^0$$

$$= \frac{1}{5^6}[15.4^2+6.4+1] = \frac{205}{5^6} = \frac{41}{3125}$$

(b)

x_i	x_i^2	y_i	y_i^2	x_iy_i
3	9	3	9	9
5	25	4	16	20
4	16	5	25	20
6	36	2	4	12
2	4	6	36	12
$\Sigma x_i=20$	$\Sigma x_i^2=90$	$\Sigma y_i=20$	$\Sigma y_i^2=90$	$\Sigma x_iy_i=73$

$\because \bar{x} = \frac{20}{5} = 4$, $\bar{y} = \frac{20}{5} = 4$

$$r = \frac{\Sigma x_iy_i - n\bar{x}\bar{y}}{\sqrt{(\Sigma x_i^2 - n\bar{x})^2(\Sigma y_i^2 - n\bar{y})^2}}$$

$$= \frac{73-5(4)(4)}{\sqrt{(90-5(4))^2(90-5(4))^2}} = \frac{73-80}{10}$$

$$= -0.7$$

Q4 (c) The characteristic equation is

$$|A-\lambda I| = \begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$

$\lambda = -1, -1, 3$

For $\lambda = -1$, $(A-\lambda I)X = 0$

$(A+I)X = 0.$

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore -2x_1 - x_2 - x_3 = 0$

$x_2 = 2t, \quad x_3 = 2s$

$$X_1 = \begin{bmatrix} s+t \\ 2t \\ 2s \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$X_1 = [1 \ 0 \ 2]' \qquad X_2 = [1 \ 2 \ 0]'$

For $\lambda = 3$ $\quad (A-\lambda I)X = 0$

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving.

$\therefore 3x_1 - x_2 - x_3 = 0$ & $x_1 - x_2 = 0$

$x_1 = x_2 = t$

$x_3 = -2t$

$$\therefore X_3 = \begin{bmatrix} t \\ t \\ 2t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$X_3 = [1 \ 1 \ 2]'$

$\therefore$ A is diagonalisable. $\because$ geometric multiplicity of each eigenvalue coincide with Algebraic multiplicity.

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix} \quad \& \quad M^{-1}AM = D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Q.5 (c) we write the problem as

$f(x_1, x_2) = x_1^2 + x_2^2$

$h_1(x_1, x_2) = x_1 + x_2 - 4$

$h_2(x_1, x_2) = 2x_1 + x_2 - 5$

Kuhn-Tucker conditions for maxima are

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial h_1}{\partial x_1} - \lambda_2 \frac{\partial h_2}{\partial x_1} = 0$$

$\therefore 2x_1 - \lambda_1 - 2\lambda_2 = 0 \quad — (1)$

$$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial h_1}{\partial x_2} - \lambda_2 \frac{\partial h_2}{\partial x_2} = 0$$

$\therefore 2x_2 - \lambda_1 - \lambda_2 = 0 \quad — (2)$

$\lambda_1 (x_1 + x_2 - 4) = 0 \quad — (3)$

$\lambda_2 (2x_1 + x_2 - 5) = 0 \quad — (4)$

$x_1 + x_2 - 4 \leq 0 \quad — (5)$

$2x_1 + x_2 - 5 \leq 0 \quad — (6)$

Q.5 (a) Null Hypothesis H_0 : $\mu = 70$ years.

Alternative Hypothesis H_a : $\mu \neq 70$ years.

Test statistic : $Z = \dfrac{\bar{x} - \mu}{\sigma / \sqrt{n}}$

$\because$ we are given standard deviation of the sample, we put

$\bar{x} = 71.8, \quad \mu = 70, \quad \sigma = 8.9, \quad n = 100$

$$\therefore |Z| = \left| \frac{71.8 - 70}{8.9 / \sqrt{100}} \right| = 2.02$$

level of Significance : $\alpha = 0.05$

critical value : The value of Z_α at 5% level of significance is 1.96.

Decision : Since the computed value of $|Z| = 2.02$ is greater than the critical value $Z_\alpha = 1.96$, the null hypothesis is rejected.

$\therefore$ The hypothesis is rejected.

(b) Let $z = e^{i\theta}$, $dz = ie^{i\theta} d\theta$, $dz = iz\,d\theta$

$\therefore d\theta = \dfrac{dz}{iz}$ & $\cos\theta = \dfrac{(z^2+1)}{2z}$

$$\therefore \int_0^{2\pi} \frac{d\theta}{3 + 2\cos\theta} = \int_C \frac{1}{3 + 2\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

where, C is the unit circle $|z| = 1$

$$\therefore \int_0^{2\pi} \frac{d\theta}{3 + 2\cos\theta} = \frac{1}{i} \int_C \frac{1}{z^2 + 3z + 1} dz$$

The roots of $z^2 + 3z + 1 = 0$

$$\therefore z = \frac{-3 \pm \sqrt{5}}{2}$$

Let $\alpha = \dfrac{-3 + \sqrt{5}}{2}$, $\beta = \dfrac{-3 - \sqrt{5}}{2}$

clearly α lies within the unit circle & β lies outside it

$\therefore$ Residue of $f(z)$ (at $z = \alpha$) $= \dfrac{1}{\sqrt{5}\, i}$

Residue of $f(z)$ (at $z = \beta$) =

$$\therefore \int_0^{2\pi} \frac{d\theta}{3 + 2\cos\theta} = 2\pi i \,(\text{Residue of } f(z) \text{ at } z = \alpha)$$

$$= 2\pi i \frac{1}{\sqrt{5}\, i} = \frac{2\pi}{\sqrt{5}}$$

Case-3 $\lambda_1 \neq 0, \lambda_2 = 0$

From ① & ②

$2x_1 = \lambda_1, \; 2x_2 = \lambda_1 \quad \therefore x_1 = x_2$

from ③, $x_1 + x_2 = 4$

$\therefore x_1 = 2$

$x_1 = 2$

$\lambda_1 = 2x_1 = 4$

Since these values satisfy ⑤, ⑥, ⑦ & ⑧, we get

$Z_{max} = x_1^2 + x_2^2 = 4+4 = 8.$

⑧

Case-4 $\lambda_1 \neq 0, \lambda_2 \neq 0$

Then we get from ③ & ④

$x_1 + x_2 = 4 \quad \& \; 2x_1 + x_2 = 5$

$\therefore x_1 = 1 \; \& \; x_2 = 3$

Putting these values in ① & ②, we get

$\lambda_1 + \cancel{2\lambda_2} = 2 \quad \& \; \lambda_1 + \lambda_2 = 6$

solving we get

$\lambda_2 = -4 \; \& \; \lambda_1 = 10$

$\therefore \lambda_1$ is +ve & λ_2 is −ve

Hence, we reject this pair.

$\therefore$ The solution is

$\boxed{x_1 = 2, x_2 = 2, \; Z_{max} = 6.}$

Q-6 (a). Null Hypothesis H_0: The die is unbiased

Alternative Hypothesis H_a: The die is not unbiased.

Calculation of test statistic: On the hypothesis that the die is unbiased we should expect the frequency of each number to be $132/6 = 22$.

Calculation of $(O-E)^2/E$

No	O	E	$(O-E)^2$
1	15	22	49
2	20	22	4
3	25	22	9
4	15	22	49
5	29	22	49
6	28	22	36
		Total	196

$\therefore \chi^2 = \sum \frac{(O-E)^2}{E} = \frac{196}{22} = 8.91$

Level of significance: $\alpha = 0.05$

number of degrees of freedom $= n-1$
$= 6-1 = 5$

Critical value: For 5 d.f. at 5% level of significance the table value of χ^2 is 11.07

Decision: Since the calculated value of $\chi^2 = 8.91$ is less than the table value of $\chi^2 = 11.07$, the hypothesis is accepted.

$\therefore$ The die is unbiased.

Q6 (b) $\bar{x_1} = 17$, $\bar{x_2} = 16$, $S_1 = 2.12$, $S_2 = 1.69$

$$S_p = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1+n_2-2}} \; ; \; S.E = S_p \times \sqrt{\frac{1}{n_1}+\frac{1}{n_2}}$$

$$= 1.073$$

~~∴ t = 0.93~~

$$t = \frac{\bar{x_1}-\bar{x_2}}{S.E} = 0.93$$

~~Accept ...~~ Level of significance : $\alpha = 0.05$

Critical value : The table value of t at $\alpha = 0.05$ for $\nu = 8+7-2 = 13$ degrees of freedom is $t_\alpha = 2.16$

Decision : Since the computed value $|t| = 0.93$ is less than the table value $t_\alpha = 2.16$, the hypothesis is Accepted.

Q6 (c). Introduce the artificial variable A_1 in the first constraint and big penalty in the object function.

Maximise $Z = 3x_1 - x_2 - 0S_1 - 0S_2 - 0S_3 - MA_1$ — (1)

Subject to
$2x_1 + x_2 - S_1 + 0S_2 + 0S_3 + A_1 = 2$ — (2)
$x_1 + 3x_2 + 0S_1 + S_2 + 0S_3 + 0A_1 = 3$ — (3)
$0x_1 + x_2 + 0S_1 + 0S_2 + S_3 + 0A_1 = 4$ — (4)

Now eliminate the term $-MA_1$ from (1) by adding M times the 1^{st} constraint to it

$\therefore Z = (3+2M)x_1 + (-1+M)x_2 - MS_1 - 0S_2 - 0S_3 - 0A_1 - 2M$

$\therefore Z - (3+2M)x_1 - (-1+M)x_2 + MS_1 + 0S_2 + 0S_3 + 0A_1 = -2M$

Setting decision variables $x_1 = 0, x_2 = 0$ & $S_1 = 0$ as the basis feasible solution is $A_1 = 2, S_2 = 3, S_3 = 4$, A_1 is greater than zero, in this case 2. But it must not appear in the final solution ∴ we assign a large Penalty $(-M)$ to A_1 in the object function (1).

→ P.T.O.

Simplex Table

Iteration Number	Basic Var.	x_1	x_2	s_1	s_2	s_3	A_1	RHS Sol.	Ratio.
0	Z	$-3-2M$	$1-M$	M	0	0	0	$-2M$	
A_1 leaves	A_1	2^*	1	-1	0	0	1	2	$2/2=1$ ←
x_1 enters.	s_2	1	3	0	1	0	0	3	$3/1=3$
	s_3	0	1	0	0	1	0	4	$4/0=\ldots$
1	Z	0	$5/2$	$-3/2$	0	0		3	
s_2 leaves	x_1	1	$1/2$	$-1/2$	0	0		1	-2
s_1 enters.	s_2	0	$5/2$	$1/2^*$	1	0		2	4 ←
	s_3	0	1	0	0	1		4	—
2	Z	0	10	0	3	0		9	
	x_1	1	3	0	1	0		3	
	s_1	0	5	1	2	0		4	
	s_3	0	1	0	0	1		4	

$\therefore x_1 = 3,\ x_2 = 0,\ Z_{max} = 9.$