Solution of the Paper [CBCGS (R2016)]

SE - Sem - IV Extc . . QP → 36987

Applied Mathematics - IV

1. (a) $F = xy + y^2 - 2y^2 y'$

using Euler's Equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

we get. $x + 2y - 4yy' - \frac{d}{dx}(-2y^2) = 0$

$x + 2y - 4yy' + 4yy' = 0$

$\boxed{y = x/2}$ is required Extremal.

(b) Cauchy - Schwartz inequality in $\mathbb{R}^n$

If $u = (u_1, u_2 - - - u_n)$ and $v = (v_1, v_2, - - - v_n)$ are any two vectors in $\mathbb{R}^n$, then

$$|u \cdot v| \leq ||u|| \cdot ||v||$$

Given, vectors $u = (-4, 2, 1)$ and $v = (8, -4, -2)$

$$|u \cdot v| = 42$$

$$||u|| = \sqrt{21}$$

$$||v|| = \sqrt{84}$$

$$\therefore ||u|| \cdot ||v|| = \sqrt{21}\sqrt{84} = 42$$

Thus $||u|| \cdot ||v|| = |u \cdot v|$

Proved.

(c) If λ is an Eigenvalue of A and X is the Corresponding Eigenvector then,

$$AX = \lambda X$$

$$X = A^{-1}(\lambda X) = \lambda(A^{-1}X)$$

$\frac{1}{\lambda}X = A^{-1}X$, $\therefore \frac{1}{\lambda}$ is an Eigen value of A^{-1}

Hence Proved.

(d) Since $\Sigma P_i = 1$, we have $P(0)+P(1)+P(2)=1$

$$\therefore 3c^3+4c-10c^2+5c-1=0$$

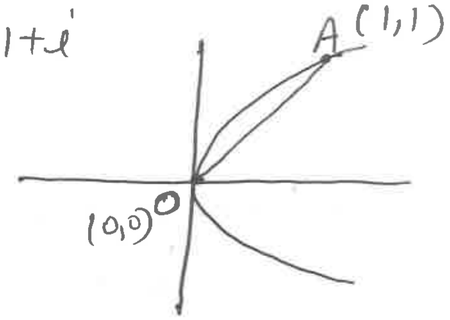
$$\boxed{C = 1/3}$$

Probability distribution is

X:	0	1	2
$P(X=x)$:	$1/9$	$2/9$	$2/3$

$$\therefore \boxed{P(X=1) = P(X=0) = 1/9}$$

2. (a) Let OA be the line from $Z=0$ to $Z=1+i$



(i) Along OA, i.e. $y=x$, $dy=dx$

$$\therefore dz = dx + i\,dy$$
$$dz = (1+i)\,dx$$

$$\therefore I = \int_0^{1+i} (x+iy)^2\,dz$$

$$= \int_0^1 (x^2-y^2+2ixy)(1+i)\,dx$$

$$\boxed{I = \frac{2}{3}(i-1)}$$

(ii) On the arc OA of the Parabola $x=y^2$, $dx = 2y\,dy$

$$dz = dx + i\,dy = (2y+i)\,dy.$$

$$I = \int_0^1 (x^2-y^2+2ixy)(2y+i)\,dy.$$

$$\boxed{I = \frac{2}{3}(i-1)}$$

The two integrals are Equal.
i.e. the given integral is dependent of path.
because $f(z)=z^2$ is an analytic function.

(b) We have,

$$M_0(t) = E(e^{tx}) = \int_0^\infty e^{tx} \cdot 2e^{-2x}\,dx$$

$$M_0(t) = \frac{2}{2-t}, \quad t \neq 2$$

Now, $M_0(t) = \frac{2}{2(1-t/2)} = \left(1-\frac{t}{2}\right)^{-1} = 1 + \frac{t}{2} + \frac{t^2}{2^2} + \frac{t^3}{2^3} + \cdots$

$\mu_1' =$ Coefficient of $t = \frac{1}{2}$, $\mu_2' =$ Coefficient of $\frac{t^2}{2!} = \frac{2}{2^2}$

Mean $= \mu_1' = \frac{1}{2}$,

$$\text{var}(X) = \mu_2' - (\mu_1')^2 = \frac{2}{2^2} - \frac{1}{2^2} = \frac{1}{2^2} = \frac{1}{4}$$

(c) $\Sigma X = 110$, $\Sigma Y = 585$, $N = 5$, $\Sigma D^2 = 8$

$\bar{X} = 22$, $\bar{Y} = 117$; $\Sigma(X-\bar{X})^2 = 250$, $\Sigma(Y-\bar{Y})^2 = 40$

$\Sigma(X-\bar{X})(Y-\bar{Y}) = 60$

$$r = \frac{\Sigma(X-\bar{X})(Y-\bar{Y})}{\sqrt{\Sigma(X-\bar{X})^2\,\Sigma(Y-\bar{Y})^2}} = \frac{60}{\sqrt{250 \times 40}} = 0.6$$

$$R = 1 - \frac{6\Sigma D^2}{N^3 - N} = 0.6$$

$r = R$.

Thus, the values of R and r are equal. The values of X increases by 5 and the values of Y, when arranged in ascending order also increases by the same amount 2 every time.

3. (a) $0 = (0,0,0)$ belongs to W, since $0+0+0=0$.

Suppose $u = (a,b,c)$ and $v = (a',b',c')$ belong to W:

Then $a+b+c=0$ and $a'+b'+c'=0$. Then, for any scalers k and k', we have

$$ku + kv = k(a,b,c) + k'(a',b',c')$$
$$= (ka+k'a',\ kb+k'b',\ kc+k'c')$$

Furthermore,

$$(ka+k'a') + (kb+k'b') + (kc+k'c') = k(a+b+c) + k'(a'+b'+c')$$
$$= 0$$

Thus $ku+k'v$ belongs to W

$\therefore$ W is subspace of V

(b) Residue at $(z=-1) = \lim_{z\to -1} \frac{1}{3!} \frac{d^3}{dz^3}\left[(z+1)^4 \frac{e^{2z}}{(z+1)^4}\right]$

$$= \frac{8}{6} e^{-2}$$

Residue at $(z=-1) = \frac{8\pi i}{3} e^{-2}$

(c) $$A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$$

$|A-\lambda I| = \lambda^3 - 11\lambda^2 + 39\lambda - 45$

Eigen values $\lambda = 3, 3, 5$

Eigen vectors Corresponding to $\lambda = 3$

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},\ X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Corresponding to $\lambda = 5$

$$X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

A is diagonalizable since it has three linearly independent eigenvectors.

Transforming Matrix $P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

Diagonal Matrix $D = P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

4. (a) $f = 2xy + y'''^2$

$\frac{\partial f}{\partial y} = 2x$, $\frac{\partial f}{\partial y'} = 0$; $\frac{\partial f}{\partial y''} = 0$; $\frac{\partial f}{\partial y'''} = 2y'''$

Hence, the equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) - \frac{d^3}{dx^3}\left(\frac{\partial f}{\partial y'''}\right) = 0$$

becomes $2x - \frac{d^3}{dx^3}(2y''') = 0$

$$2x - \frac{d^3}{dx^3}\left(2\frac{d^3y}{dx^3}\right) = 0$$

$$\frac{d^6y}{dx^6} = x$$

$\therefore$ C.F. is $y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4 + c_6 x^5$

Now P.I.: $y = \frac{1}{D^6} x$

$$P.I. = \frac{x^7}{7!}$$

Hence the solution is $y = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4 + c_6x^5 + c_6x^5 + \frac{x^7}{7!}$

(b) (I) Binomial Distribution:

$p = 0.01,\ q = 1 - p = 0.99,\ n = 10,$

$$P(X \geq 1) = 1 - P(X \leq 1)$$
$$= 1 - {}^{10}C_0 (0.01)^0 (0.99)^{10} - {}^{10}C_1 (0.01)^1 (0.99)^9$$
$$= 0.00425$$

(II) Poisson Distribution

$$m = np = 10(0.01) = 0.1$$
$$P(X = x) = e^{-m} \cdot \frac{m^x}{x!} = e^{-0.1} \frac{(0.1)^x}{x!}$$

$$P(X \geq 1) = 1 - P(X \leq 1)$$
$$= 1 - P(X=0) - P(X=1)$$
$$= 0.0047$$

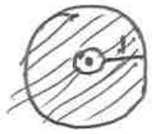
(c) Let $f(z) = \dfrac{z-1}{(z+1)(z-3)} = \dfrac{1/2}{z+1} + \dfrac{1/2}{z-3}$

$\therefore f(z)$ is not analytic at $z = -1$ & $z = 3$

$f(z)$ is analytic (i) $|z| < 1$ (II) $1 < |z| < 3$ (III) $|z| > 3$

case-I when $|z| < 1$ & $|z| < 3$

$$f(z) = \frac{1}{2} \cdot \frac{1}{1+z} + \frac{1}{2} \frac{1}{(-3)} \cdot \frac{1}{1 - (z/3)}$$
$$= \frac{1}{2}(1+z)^{-1} - \frac{1}{6}\left(1 - \frac{z}{3}\right)^{-1}$$
$$= \frac{1}{3} - \frac{5}{9} z + \frac{13}{27} z^2 + \cdots$$



$0 < |z| < 1$

Case-II When $1<|z|<3$ we get $\left|\frac{1}{z}\right|<1$ & $\left|\frac{z}{3}\right|<1$

$$f(z)=\frac{1}{2}\cdot\frac{1}{1+z}+\frac{1}{2}\cdot\frac{1}{z-3}$$

$$=\frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}-\frac{1}{6}\left(1-\frac{z}{3}\right)^{-1}$$

$$=\frac{1}{2}\left[\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}\cdots\right]-\frac{1}{6}\left[1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+\cdots\right]$$

This is required Laurent's series

Case-III when $|z|>3$, clearly $|z|>1$

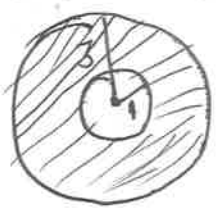
$\left|\frac{z}{3}\right|>1$ & $\frac{|z|}{1}>1$ $\therefore \frac{3}{|z|}<1$ & $\frac{1}{|z|}<1$

$$f(z)=\frac{1}{2}\cdot\frac{1}{z+1}+\frac{1}{2}\frac{1}{z-3}$$

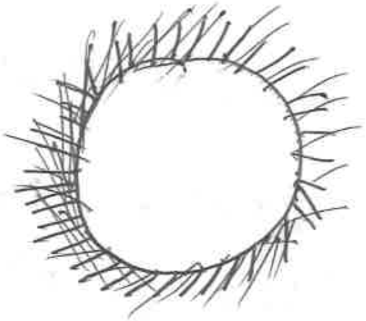
$$=\frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}+\frac{1}{2z}\left(1-\frac{3}{z}\right)^{-1}$$

$$=\frac{1}{2z}\left[2+\frac{2}{z}+\frac{10}{z^2}+\frac{26}{z^3}+\cdots\right]+\cancel{\tfrac{1}{6}}\{$$

This is required Laurent's series



$1<|z|<3$



$|z|>3$

5. (a)

$$A=\begin{bmatrix}0 & c & -b\\ -c & 0 & a\\ b & -a & 0\end{bmatrix}$$

$$|A-\lambda I|=\lambda^3+(a^2+b^2+c^2)\lambda=0$$

$$A^2=\begin{bmatrix}-c^2-b^2 & ab & ac\\ ab & -c^2-a^2 & bc\\ ac & bc & -b^2-a^2\end{bmatrix};\quad A^3=-\begin{bmatrix}0 & c & -b\\ -c & 0 & a\\ b & -a & 0\end{bmatrix}(a^2+b^2+c^2)$$

$$=-(a^2+b^2+c^2)A$$

$A^3 + (a^2+b^2+c^2)A = 0$

Hence A Satifies Cayley - Hamiton theorem

NOW,
$$|A| = \begin{vmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{vmatrix} = abc - abc = 0$$

Since A is Singular, A^{-1} does not Exist

(b) Step1: $v_1 = u_1 = (1,1,1)$

Step 2: $v_2 = u_2 - \text{proj}\, u_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} \cdot v_1$

$\langle u_2, v_1 \rangle = 0$ and $\|v_1\|^2 = 3$;

$V_2 = (-1, 1, 0)$

Step 3: $v_3 = u_3 - \text{proj}\, u_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} \cdot v_1 - \frac{\langle u_3 \cdot v_2 \rangle}{\|v_2\|^2} \cdot v_2$

$\langle u_3, v_1 \rangle = 4$; $\langle u_3, v_2 \rangle 1$; $\|v_1\|^2 = 3$; $\|v_2\|^2 = 2$

$v_3 = \left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)$

Orthonormal basis $v_1 = (1,1,1)$; $v_2 = (-1,1,0)$; $v_3\left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)$

Norms are $\|v_1\| = \sqrt{3}$, $\|v_2\| = \sqrt{2}$; $\|v_3\| = \frac{1}{\sqrt{6}}$

Orthonormal basis for R^3 is

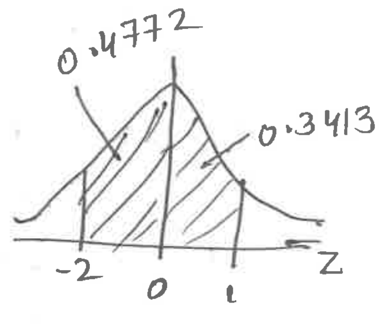
$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$; $q_2 = \frac{v_2}{\|v_2\|} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$q_3 = \frac{v_3}{\|v_3\|} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$

(c) we have S.N.V. $Z = \frac{X - m}{\sigma} = \frac{X - 70}{5}$

(i) when $X = 60$, $Z = -2$

when $X = 75$, $Z = 1$



$P(60 \le X \le 75) = P(-2 < Z < 1)$

$= \text{Area from } (z=0 \text{ to } z=2) + \text{area from } (z=0 \text{ to } z=1)$

$= 0.4772 + 0.3413$

$= 0.8185$

Number of student getting b/w 60 & 75 $= NP = 818$

(ii) $P(X \geq 75) = P(Z \geq 1)$

$= 0.5 - (\text{area b/w } Z=0 \ \& \ Z=1)$

$= 0.5 - 0.3413$

$= 0.1587$

Number of student getting more then 75 $= NP = 159$

6-(a) $I = \int_0^1 F(x, y, y')\, dx$ —— (1)

$F = 2xy + y^2 - y'^2$

Trial solution $\bar{y}(x) = c_0 + c_1 x + c_2 x^2$

By data $\bar{y}(0) = 0, \therefore c_0 = 0$

$\bar{y}(1) = 0, \quad c_2 = -c_1$

$\therefore \bar{y}(x) = c_1 x - c_1 x^2 = c_1 x(1-x)$

$\bar{y}'(x) = c_1 - 2c_1 x = c_1(1-2x)$

Put in (1)

$I = c_1 \int_0^1 2(x^2 - x^3) + c_1(-1 + 4x - 3x^2 - 2x^3 + x^4)\, dx$

$I = \frac{c_1}{6} - \frac{3}{10} c_1^2$

$\frac{dI}{dc_1} = 0, \quad \frac{1}{6} - \frac{3}{5} c_1 = 0 \Rightarrow c_1 = \frac{5}{18}$

$\bar{y}(x) = \frac{5}{18} x(1-x)$

(b) $A=\begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

$$|A-\lambda I| = \lambda^3 - 4\lambda^2 + 5\lambda - 2$$
$$= (\lambda-2)(\lambda-1)^2$$

$$f(A) = (A-2I)(A-I)$$
$$= A^2 - 3A + 2I$$

$f(x) = x^2 - 3x + 2$ anihilates A

$f(x)$ is the monic polynomial of lowes degree that anhililates A

minimal polynomial $= (x^2 - 3x + 2)$.

Hence A is derogatory

(c) (a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$,

$z = e^{i\theta}$; $d\theta = \dfrac{dz}{iz}$, $\sin\theta = \dfrac{z^2-1}{2iz}$

$$\int_c \frac{2\,dz}{3z^2+10iz-3} = \int_c \frac{2\,dz}{(3z+i)(z+3i)}$$ where C is $|z|=1$

$\therefore z = -\frac{i}{3}$ lies inside & $z=-3i$ lies outside of $|z|=1$

$$\text{Residue at } \left(z = -\frac{i}{3}\right) = \lim_{z \to -\frac{i}{3}} \left[z+\frac{i}{3}\right] \frac{2}{(3z+i)(z+3i)}$$

$$= \frac{1}{4i}$$

$$I = 2\pi i \left(\frac{1}{4i}\right) = \frac{\pi}{2}$$

(b) $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx$

(i) $z f(z) = \frac{z^3}{(z^2+a^2)(z^2+b^2)} \to 0$ as $|z| \to \infty$

Now $(z^2+a^2)(z^2+b^2)=0$

$z = \pm ai, \pm bi$

$z = ai$ & $z = bi$ lies upper of the z-plane

Residue at $(z=ai) = \lim_{z\to ai} (z-ai) \cdot \frac{z^2}{(z-ai)(z+ai)(z-bi)(z+bi)}$

$= \frac{a}{2i(a^2-b^2)}$

Similarly Residue at $(z=bi) = \frac{-b^2}{2i(a^2+b^2}$

Sum of Residue $= \frac{\pi}{a+b}$