QP - 37579

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(1) a) $F = y'^2 - y^2 + 2xy$

$\frac{\partial F}{\partial y} = -2y + 2x$

$\frac{\partial F}{\partial y'} = 2y'$

The Euler's eqⁿ is

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$-2y + 2x - \frac{d}{dx}(2y') = 0$

$-2y + 2x - 2y'' = 0$

$x = y'' + y$

$= (D^2+1)y = x$

$D^2 + 1 = 0$

$D = \pm i$

$\therefore CF = C_1 \cos x + C_2 \sin x$

$P.I = \frac{1}{D^2+1} x = (1+D^2)^{-1} x = (1 - D^2 + D^4 \cdots) x$

$= x - 0 + 0 \cdots$

$= x$

$\therefore$ The complete solⁿ is $y = CF + P.I$

$y = C_1 \cos x + C_2 \sin x + x$

given $y(0) = 0$ & $y(\pi/2) = 0$

$C_1 = 0 \quad C_2 = -\frac{\pi}{2}$

$\therefore y = x - \frac{\pi}{2} \sin x$

2b) $\int_0^\infty k x e^{-x/3} dx = 1$

$k\left[\frac{x e^{-x/3}}{-1/3} - \frac{e^{-x/3}}{1/9}\right]_0^\infty = 1$

$k[9] = 1$

$k = 1/9$

mean $= E[X] = \int_0^\infty x \cdot k x e^{-x/3} dx = \int_0^\infty x^2 \frac{1}{9} e^{-x/3} dx = \underline{\underline{6}}$

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b)

3) The characteristic eqn is $(\lambda-2)(\lambda-2)(\lambda-4)$

Let us assume $(\lambda-2)(\lambda-4)$ is the minimal polynomial

and $A^2-6A+8I=0$

$\therefore$ the minimal polynomial is $(\lambda-2)(\lambda-4)$

deg of minimal poly <3.

$\therefore$ A is derogatory.

d) $\displaystyle\int_c \frac{z^2-2z+4}{z^2-1}\,dz \qquad c: |z-1|=1$

$z^2-1=0$

$z=\pm 1$

$z=1$ is inside & $z=-1$ is outside.

$$\therefore \int_c \frac{z^2-2z+4}{z^2-1}\,dz = \int_c \frac{\frac{z^2-2z+4}{z+1}}{z-1}\,dz = 2\pi i\left[\frac{z^2-2z+4}{z+1}\right]_{z=1}$$

$$= 2\pi i\left[\frac{1-2+4}{1+1}\right]$$

$$= 2\pi i\left[\frac{3}{2}\right] = \underline{3\pi i}$$

Q2) a) characteristic eqn

$\lambda^2-9=0$

$\lambda=\pm 3$ are the eigen values.

for $\lambda=3$ eigen vector is $\begin{bmatrix}1\\1\end{bmatrix}$

for $\lambda=-3$ — " — $\begin{bmatrix}2\\-1\end{bmatrix}$

$M=\begin{bmatrix}1 & 2\\1 & -1\end{bmatrix} \quad M^{-1}=-\frac{1}{3}\begin{bmatrix}-1 & -2\\-1 & 1\end{bmatrix} \quad D=\begin{bmatrix}3 & 0\\0 & -3\end{bmatrix}$

$\tan A = M f(D) M^{-1} = \begin{bmatrix}1 & 2\\1 & -1\end{bmatrix}\begin{bmatrix}\tan 3 & 0\\0 & \tan(-3)\end{bmatrix}\left(-\frac{1}{3}\right)\begin{bmatrix}-1 & -2\\-1 & 1\end{bmatrix}$

$3\tan A = \begin{bmatrix}\tan 3 & 4\tan 3\\2\tan 3 & \tan 3\end{bmatrix} = \tan 3\begin{bmatrix}1 & 4\\2 & 1\end{bmatrix} = \tan 3\text{-}A$

b) singularities are $z^2+z=0$ (3)

$z(z+1)=0$

$z=0,\ z=-1$ both are inside.

$\therefore \frac{1}{z^2+z} = \frac{A}{z} + \frac{B}{z+1}$

$= \frac{1}{z} - \frac{1}{z+1}$

$\int_c \frac{\sin\pi z + \cos\pi z}{z^2+z}\,dz = \int_c \frac{(\sin\pi z+\cos\pi z)}{z}\,dz - \int_c \frac{\sin\pi z + \cos\pi z}{z+1}\,dz$

$= 2\pi i\{\sin\pi(0) + \cos\pi(0) - \sin\pi(-1) - \cos(-\pi)\}$

$= 2\pi i\{1+1\}$

$= 4\pi i$

c) C_1: $(a,b)+(c,d) = (a+c,\ b\cdot d) \in V$

$\therefore C_1$ is satisfied

C_2: $\alpha(a,b) = (\alpha a, b^{\alpha})$

$\alpha a, b^{\alpha} \in R$ & $b^{\alpha} > 0$

$\therefore$ closed under scalar multiplication.

A_1: $u, v \in V$ $u+v =$
$(a,b)+(c,d) = (a+c,\ b\cdot d)$

$= (c+a,\ d\cdot b)$ for real nos.

$= (c,d)+(a,b)$

$= v+u$

A_2: $|||^{ly}$ $(u+v)+w = u+(v+w)$

A_3: additive identity is $(1,0)$

$u+0 = u$

$(a,b)+(0,1) = (a,b)$

A_4: additive inverse

$u+(-u) = 0$

$(a,b)+(-a,\frac{1}{b}) = (1,0)$

M_1: $k(u+v) = k\{(a,b)+(c,d)\}$

$$= k\{(a+c, b.d)\}.$$

$$= \{k(a+c) . (bd)^k\}$$

$$= \{ka+kc, b^k . d^k\}$$

$$= (ka, b^k) + (kc, d^k)$$

$$= k(a,b) + k(c,d)$$

$(ii)^{ly}$ $(k+l)u = ku + lu.$

$(kl)u = k(lu)$

the multiplicative inverse is a vector space.

Q3) a) $r = \dfrac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\Sigma(x-\bar{x})^2 \Sigma(y-\bar{y})^2}}$

$\Sigma(x-\bar{x})(y-\bar{y}) = 97$ $\quad \Sigma(x-\bar{x})^2 = 216 \quad \Sigma(y-\bar{y})^2 = 162$

$$r = 0.5186$$

b) $F = y\sqrt{1+y'^2}$

$G = \sqrt{1+y'^2}$

$H = F + \lambda G = y\sqrt{1+y'^2} + \lambda\sqrt{1+y'^2}$

H is free from x

$\therefore$ the Euler's eq^n is

$$H - y'\frac{\partial H}{\partial y'} = c$$

$$(y+\lambda)\sqrt{1+y'^2} - y'\frac{(y+\lambda)y'}{\sqrt{1+y'^2}} = c$$

$$\frac{(y+\lambda)}{\sqrt{1+y'^2}} = c \quad \therefore y'^2 = \frac{(y+\lambda)^2 - c^2}{c^2}$$

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$$\frac{c}{\sqrt{(y+\lambda)^2-c^2}}\,dy = dx.$$

Integrating we get

$$c\cosh^{-1}\left(\frac{y+\lambda}{c}\right) = x+c'$$

$$y = c\cosh\left(\frac{x+c'}{c}\right)-\lambda$$ is the required curve

c) consider $f(z) = \frac{1}{(z^2+1)(z^2+4)}$

$$z\cdot f(z) = \frac{z}{(z^2+1)(z^2+4)} \to 0.$$

$(z^2+1)(z^2+4) = 0$

$z = \pm i, \pm 2i$

$$\text{Res } f(z)\big|_{z=i} = \lim_{z\to i}\left[(z-i)\frac{1}{(z^2+1)(z^2+4)}\right] = \frac{-1}{6i}$$

$$\text{Res } f(z)\big|_{z=2i} = \lim_{z\to 2i}\left[(z-2i)\frac{1}{(z^2+1)(z^2+4)}\right] = \frac{1}{3i}$$

$$I = 2\pi i\left[\frac{-1}{6i}+\frac{1}{3i}\right] = \frac{\pi}{3}$$

Q4) a) $v_1 = u_1 = (1,1,1)$

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle v_1}{\|v_1\|^2} = (0,1,1) - \frac{2}{3}(1,1,1) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$v_3 = u_3 - \frac{\langle u_3, v_1\rangle v_1}{\|v_1\|^2} - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2}v_2 = (0, -1/2, 1/2)$$

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$q_3 = \frac{v_3}{\|v_3\|} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

b) $\bar{x} = 68$ $\quad \bar{y} = \frac{\Sigma y}{n} = 68.75$

regression line of y on x is

$$(y-\bar{y}) = b_{yx}(x-\bar{x})$$

$$y = 19.79 + 0.72x$$

regression line of x on y is

$$(x-\bar{x}) = b_{xy}(y-\bar{y})$$

$$x = 33.29 + 0.5y.$$

c) $\mu = \frac{\Sigma x_i f_i}{\Sigma f_i} = 0.89$

$p(0) = \frac{e^{-\mu}\mu^0}{0!} = e^{-0.89} = 0.41066$ $\qquad f(0) = 123.198 \simeq 123$

$p(1) = \frac{e^{-\mu}\mu}{1!} = e^{-0.89}(0.89) = 0.3655$ $\qquad f(1) = 109.646 \simeq 110$

$p(2) = \frac{e^{-\mu}\mu^2}{2!} = \frac{e^{-0.89}(0.89)^2}{2} = 0.1626$ $\qquad f(2) = 48.79 \simeq 49$

$p(3) = \frac{e^{-\mu}\mu^3}{3!} = \frac{e^{-0.89}(0.89)^3}{3!} = 0.048$ $\qquad f(3) = 14.475 \simeq 14$

$p(4) = \frac{e^{-\mu}\mu^4}{4!} = \frac{e^{-0.89}(0.89)^4}{4!} = 0.0107$ $\qquad f(4) = 3.22 \simeq 3$

$p(5) = \frac{e^{-\mu}\mu^5}{5!} = \frac{e^{-0.89}(0.89)^5}{5!} = 1.910 \times 10^{-3}$ $\qquad f(5) = 0.57 \simeq 1$

$\overline{\qquad 300 \qquad}$

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Q5) a) Moment generating moment $= E(e^{tx})$

$$M_0(t) = \sum e^{tx_i} p_i = e^{t\times 0}\cdot\frac{1}{6} + e^{t\times 1}\times\frac{1}{3} + e^{2t}\times\frac{1}{3} + e^{3t}\times\frac{1}{6}$$

$$= \frac{1}{6} + \frac{e^t}{3} + \frac{e^{2t}}{3} + \frac{e^{3t}}{6}.$$

$$\mu_1' = \frac{d}{dt}[M_0(t)]/_{t=0} = \left[\frac{e^t}{3} + \frac{2}{3}e^{2t} + \frac{3e^{3t}}{6}\right]/_{t=0}$$

$$= \left[\frac{1}{3} + \frac{2}{3} + \frac{1}{2}\right] = \frac{6+3}{6} = \frac{3}{2}$$

$$\mu_2' = \frac{d^2}{dt^2}[M_0(t)]/_{t=0} = \left[\frac{e^t}{3} + \frac{4}{3}e^{2t} + \frac{9e^{3t}}{6}\right]/_{t=0}$$

$$= \frac{1}{3} + \frac{4}{3} + \frac{3}{2} = \frac{5}{3} + \frac{3}{2} = \frac{19}{6}.$$

$$\text{Variance} = \mu_2' - (\mu_1')^2 = \frac{19}{6} - (3/2)^2 = \frac{19}{6} - \frac{9}{4} = \frac{38-27}{12} = \frac{11}{12}$$

b) characteristic eqn of A is

$$\lambda^3 - (2+3+2)\lambda^2 + 11\lambda - 5 = 0$$

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$\lambda = 5, 1$

$\therefore$ the eigen values of $A^3 - 2A + I$

are $\lambda^3 - 2\lambda + 1$

for $\lambda = 5$ the eigen value is $5^3 - 2\times 5 + 1 = 116$.

$\lambda = 1$ ——— " ——— $1^3 - 2\times 1 + 1 = 0$.

To find eigen vector of A.

for $\lambda = 1$ $(A - \lambda I)X = 0$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 + x_3 = 0.$

put $x_1 = s$, $x_2 = t$

$x_3 = -x_1 - 2x_2$

$= -s - 2t$

$$\therefore X = s\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

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for $\lambda = 5$

$(A-\lambda I)X = 0$

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$-3x_1 + 2x_2 + x_3 = 0$

$x_1 - 2x_2 + x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x_1}{4} = \frac{-x_2}{-4} = \frac{x_3}{4}.$$

$$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ The eigen vectors of $A^3 - 2A + I$ is also

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

or the linear combination is also a eigen vector.

c) singularities are $z^2 - 2z - 3 = 0$

$(z-3)(z+1) = 0$

$z = 3, -1$

these singularities divide the z plane into ~~the~~ analytic regions $|z| < 1$, $1 < |z| < 3$, $|z| > 3$.

~~when $|z| < 1$~~ consider $\frac{z-1}{z^2-2z-3} = \frac{A}{(z-3)} + \frac{B}{z+1}$

$$= \frac{1}{2}\frac{1}{z-3} + \frac{1}{2}\frac{1}{z+1}$$

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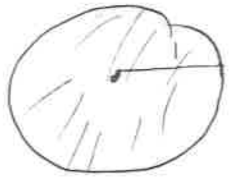
$$= \frac{1}{2}\left[\frac{1}{z-3}\right] + \frac{1}{2}\left[\frac{1}{z+1}\right]$$

$|z| < 1 < 3$

$|z/3| < 1$

$$= \frac{1}{2}\left[\frac{1}{-3(1-z/3)}\right] + \frac{1}{2}\left[\frac{1}{z+1}\right]$$



$$= -\frac{1}{6}(1-z/3)^{-1} + \frac{1}{2}(1+z)^{-1}$$

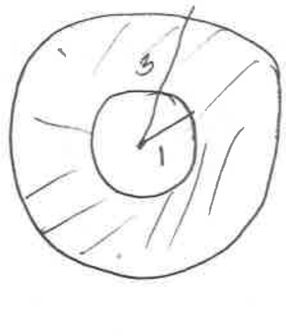
$$= -\frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \cdots\right) + \frac{1}{2}\left(1 - z + z^2 - \cdots\right)$$

$$= \frac{1}{3} - \frac{5}{9}z + \frac{13}{27}z^2 - \cdots$$

This is a Taylor's series.

Case ii $\quad 1 < |z| < 3$

$\frac{1}{|z|} < 1 \qquad \frac{|z|}{3} < 1$



$$f(z) = \frac{1}{2}\frac{1}{z-3} + \frac{1}{2}\frac{1}{z+1}$$

$$= \frac{1}{2}\frac{1}{(-3(1-z/3))} + \frac{1}{2}\frac{1}{z(1+1/z)}$$

$$= -\frac{1}{6}(1-z/3)^{-1} + \frac{1}{2z}(1+\frac{1}{z})^{-1}$$

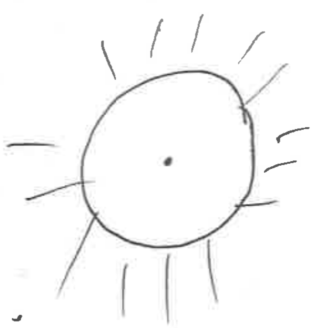
$$= -\frac{1}{6}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \cdots\right) + \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \cdots\right)$$

$$= \frac{1}{2}\left[\frac{1}{z} - \frac{1}{z^2} + \cdots\right] - \frac{1}{6}\left[1 + \frac{z}{3} + \cdots\right]$$

This is a Laurent's series

Case iii when $|z| > 3$ clearly $|z| > 1$

$|3/z| < 1 \qquad \frac{1}{|z|} < 1$



$$f(z) = \frac{1}{2z}\left(\frac{1}{1+\frac{1}{z}}\right) + \frac{1}{2}\frac{1}{z(1-3/z)}$$

$$= \frac{1}{z} + \frac{1}{z^2} + \frac{5}{z^3} + \frac{13}{z^4} + \cdots$$

This is a Laurent's series

Q6) a) assume the solⁿ as $\bar{y}(x) = c_0 + c_1 x + c_2 x^2$

$y(0) = 0 \Rightarrow \cancel{c_0 + c_2 = 0}$ $c_0 = 0$

$y(1) = 0$ $\therefore$ $c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$\therefore$ $\bar{y}(x) = c_1 x - c_1 x^2 = c_1(x - x^2)$

$\bar{y}'(x) = c_1(1-2x)$

$I = \int_0^1 \left\{ x\left[c_1(x-x^2)\right] + \frac{1}{2} c_1^2 (1-2x)^2 \right\} dx$

$= \frac{c_1}{12} + \frac{c_1^2}{6}$

Its stationery values are given by

$\frac{dI}{dc_1} = 0$

$c_1 = -1/4 \Rightarrow c_2 = \frac{1}{4}$

$\bar{y}(x = \frac{1}{4} x(x-1)$

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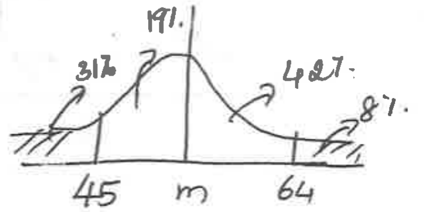
b)



as 31% of items are under 45

50 − 31 = 19% are b/w 45 & m

and as 8% items are over 64

50 − 8 = 42% are b/w m and 64.

standard normal variate is given by

$z = \frac{x - m}{\sigma}$

at $x = 45$, $z = z_1$ is −ve

and at $x=64$ $Z=Z_2$ is +ve.

$z = \frac{x-m}{\sigma}$, 19% area correspond to 0.5

$x=45$ $Z_1 = \frac{45-m}{\sigma}$

$\therefore -0.5 = \frac{45-m}{\sigma} \longrightarrow (1)$.

$x=64$, $Z_2 = \frac{64-m}{\sigma}$.

$1.41 = \frac{64-m}{\sigma} \longrightarrow (2)$.

from (1) & (2) $m \simeq 50$ $\sigma \simeq 10$

c) The characteristic eqn is

$\lambda^3 - \lambda^2 - \lambda + 1 = 0$

By Cayley Hamilton thm satisfied by A

$A^3 - A^2 - A + I = 0 \longrightarrow (1)$

$\therefore A^3 = A + A^2 - I$ . .

we prove the required result by the method of induction.

Let the ~~prove~~ assume result is true for $n=k$.

i.e., suppose $A^k = A^{k-2} + A^2 - I$ be true.

$\times^{ly}$ A, $A^{k+1} = A^{k-1} + A^3 - A$.

by (1) $A^3 - A = A^2 - I$

$\therefore A^{k+1} = A^{k-1} + A^2 - I = A^{(k+1)-2} + A^2 - I$

Hence the result is true for $n=k+1$

To find A^{50} we put successively $n = 2, 4, \ldots 50$

$$A^2 = I + A^2 - I.$$

$$A^4 = A^2 + A^2 - I$$

$$A^6 = A^4 + A^2 - I$$

$$\vdots$$

$$A^{48} = A^{46} + A^2 - I$$

$$A^{50} = A^{48} + A^2 - I.$$

$$\therefore A^{50} = 25A^2 - 24I$$

$$= 25\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} - 24\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$