SE - Comp - Sem-IV, Nov-2017 (R-2012) ①

Solution QP code: 23022

Q.1 a) $\because r=1$, Put $z=e^{i\theta}$, $dz=ie^{i\theta}d\theta$, θ varies from 0 to 2π.

$$\therefore I=\int_0^{2\pi}(\log e^{i\theta})\, ie^{i\theta}\,d\theta = i\int_0^{2\pi} i\theta e^{i\theta}\,d\theta = -\int_0^{2\pi} e^{i\theta}\theta\,d\theta = 2\pi i.$$

b) Eigen values of Adj A $=\frac{|A|}{\lambda}$, where λ is an eigen value of A.

$\therefore$ eigen values of Adj A is 6, 3, 2.

c) Let the coefficient of regression be b_1 and b_2.

$\therefore \frac{b_1+b_2}{2}=p$ & $b_1-b_2=2q$

$\therefore b_1+b_2=2p$ & $b_1-b_2=2q$

$\therefore b_1=p+q$ & $b_2=p-q$

$\therefore$ Coefficient of correlation $= r=\sqrt{b_1b_2}=\sqrt{p^2-q^2}$.

d) The dual of the given problem is

Minimise $W=5y_1+6y_2+10y_3+12y_4$

Subject to $y_1+2y_2+y_3+4y_3\geq 2$

$2y_1-y_2+y_3\geq -1$

$-y_1+y_2+3y_3+y_4\geq 4$

$y_1, y_2, y_3, y_4\geq 0$

Q.2 a) we have $\int_C \frac{\cot z}{z}dz=\int_C\frac{\cos z}{z\sin z}dz$

The point $z=0$ lies inside the ellipse. Hence by Cauchy's integral formula

$$\int_C\frac{\cot z}{z}dz=\int_C\frac{\cos z}{z\sin z}dz=2\pi i\cos(0)=2\pi i.$$

b) The characteristic equation of A is

$$\begin{vmatrix}1-\lambda & 2 & 3\\ 2 & 3-\lambda & 4\\ 3 & 4 & 5-\lambda\end{vmatrix}=0$$

$\therefore \lambda^3-9\lambda^2-6\lambda=0$

$\therefore \lambda(\lambda^2-9\lambda-6)=0$

$\because$ all roots are distinct and the characteristic equation is satisfied by A.

The degree of minimal equation is equal to 3 and hence A is non-derogatory.

Q2 (c). We have $Z = \frac{X-m}{\sigma} = \frac{X-10}{4}$

(i) when $X=14$, $Z=1$

$\therefore P(|X-14| \le 1) = P(|Z| \le 1) =$ area between $(Z=-1$ & $Z=1)$

$= 2$(area between $Z=0$ & $Z=1$)

$= 2(0.3413) = 0.6826.$

(ii) when $X=5$, $Z=-1.25$, $X=18$, $Z=2$

$\therefore P(5 \le X \le 18) = P(-1.25 \le Z < 2)$

$=$ area between $Z=-1.25$ & $Z=2$

$=$ area between $Z=0$ & $Z=1.25$) + (area between $Z=0$ & $Z=2$)

$= 0.3944 + 0.4772 = 0.8716.$

(iii) when $X=12$, $Z=0.5$

$\therefore P(X \le 12) = P(Z \le 0.5) =$ area up to $Z = 0 \le 0.5$

$=$(area from $-\infty$ to $Z=0$) + (area from $Z=0$ to $Z=0.5$)

$= 0.5 + 0.1915 = 0.6915$

Q3 (a) We have the following probability distribution

X:	0	1	2	3	.....
$P(X=x)$	p	qp	q^2p	q^3p	.....

Since, we may get success in the first trial where the number of failures $X=0$ and the probability is p; we may get success in the second-trial when the number of failures $X=1$ and the probability is qp & so on.

$\therefore E(x) = \sum p_i x_i = P(0) + qP(1) + q^2P(2) + q^3P(3) + \cdots$

$= qp[1 + 2q + 3q^2 + \cdots]$

$= qp(1-q)^{-2} = \frac{qp}{p^2} = \frac{q}{p}.$

(b) We first express the given problem in standard form

$Z - 10x_1 - x_2 - x_3 + 0s_1 + 0s_2 = 0$

$x_1 + x_2 - 3x_3 + s_1 + 0s_2 = 10$

$4x_1 + x_2 + x_3 + 0s_1 + s_2 = 20$

Since there are two constraints, there will be two stack variables.

Table (1)

Iteration Number	Basic Variables	Coefficients of X_1	X_2	X_3	S_1	S_2	R.H.S solution	Ratio.
0	Z	-10	-1	-1	0	0	0	
	S_1	1	1	-3	1	0	10	$10/1 = 10$
S_2 leaves X_1 enters.	S_2	4*	1	3	0	1	20	$20/4 = 5$ ←

Table (2)

Iteration Number	Basic Variables	Coefficients of X_1	X_2	X_3	S_1	S_2	R.H.S Solution.
1	Z	0	$3/2$	$13/2$	0	$5/2$	50
	S_1	0	$3/4$	$-15/4$	1	$-1/4$	5
	X_1	1	$1/4$	$3/4$	0	$1/4$	5

Since all the coefficients in the objective equation in the row of Z are positive this is optimal solution. The values of the variables & of Z are given by the last column. Since x_2, x_2 do not appear in the second column, they are zero.

$\therefore X_1 = 5, \quad X_2 = 0, \quad X_3 = 0, \quad Z_{max} = 50.$

(c) Let $f(z) = \dfrac{1}{z(z+1)(z-2)} = \dfrac{a}{z} + \dfrac{b}{z+1} + \dfrac{c}{z-2}$

$\therefore 1 = a(z+1)(z-2) + bz(z-2) + cz(z+1)$

when $z=0$, $a = -1/2$, $z=-1$, $b = 1/3$, $z=2$, $c = 1/6$.

$\therefore f(z) = -\dfrac{1}{2z} + \dfrac{1}{3(z+1)} + \dfrac{1}{6(z-2)}$, Three cases arise $0<|z|<1$, $1<|z|<2$, $|z|>2$.

(i) when $0<|z|<1$, we write

$f(z) = -\dfrac{1}{2z} + \dfrac{1}{3(1+z)} - \dfrac{1}{12\left[1-\frac{z}{2}\right]}$, when $|z|<1$, $|z|<2$

$= \dfrac{1}{-2z} + \dfrac{1}{3}(1+z)^{-1} - \dfrac{1}{12}\left[1-\dfrac{z}{2}\right]^{-1}$

$= \dfrac{1}{-2z} + \dfrac{1}{3}\left[1 - z + z^2 - z^3 + z^4 \cdots\right] - \dfrac{1}{12}\left[1 + \left(\dfrac{z}{2}\right) + \left(\dfrac{z}{2}\right)^2 + \left(\dfrac{z}{2}\right)^3 + \cdots\right]$

(ii) when $1<|z|<2$, we write

$$f(z) = -\frac{1}{2z} + \frac{1}{3z\left(1+\frac{1}{z}\right)} - \frac{1}{12\left[1-\frac{z}{2}\right]}$$

$$= -\frac{1}{2z} + \frac{1}{3z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{12}\left(1-\frac{z}{2}\right)^{-1}$$

$$= -\frac{1}{2z} + \frac{1}{3z}\left[1-\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+\cdots\right] - \frac{1}{12}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{3}\right)^2+\cdots\right]$$

(iii) when $|z|>2$, we write

$$f(z) = -\frac{1}{2z} + \frac{1}{3z\left[1+\frac{1}{z}\right]} + \frac{1}{6z\left[1-\frac{2}{z}\right]}$$

$$= -\frac{1}{2z} + \frac{1}{3z}\left[1-\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3\cdots\right] + \frac{1}{6z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3\cdots\right]$$

The region of convergence is

$0<|z|<1$ $\quad$ $1<|z|<2$ $\quad$ $2<|z|<\infty$

Q4 (a) we have $E(X) = np = 2$ and $V(X) = npq = 4/3$

$$\therefore \frac{npq}{np} = \frac{4/3}{2} = \frac{2}{3}$$

$$\therefore q = \frac{2}{3}, \quad p = \frac{1}{3}, \quad n = 6$$

Hence, the distribution is

$$P(X=x) = {}^{n}C_{x}\, p^{x} q^{n-x} = {}^{6}C_{x}\left(\frac{1}{3}\right)^{x}\left(\frac{2}{3}\right)^{6-x}$$

Putting $x = 0, 1, 2 \cdots 6$, we get the following probability distribution of X.

X:	0	1	2	3	4	5	6
P(X=x):	$\frac{64}{729}$	$\frac{192}{729}$	$\frac{240}{729}$	$\frac{160}{729}$	$\frac{60}{729}$	$\frac{12}{729}$	$\frac{1}{729}$

(b) Calculation of R.

X	R_1	Y	R_2	D^2 $(R_1-R_2)^2$
40	3	46	1	4.00
42	2	43	3.5	2.25
45	1	44	2	1.00
35	6	39	6	0.00
36	5	40	5	0.00
39	4	43	3.5	0.25
N = 6				$\Sigma D^2 = 7.50$

$$R = 1 - \frac{6\left[\Sigma D^2 + \frac{1}{12}(2^3-2)\right]}{N^3-N}$$

$$= 1 - \frac{6(7.5+0.5)}{216-6}$$

$$= 1 - \frac{48}{210} = 1 - 0.229$$

$$= 0.771$$

$$\therefore \boxed{R = 0.771}$$

(c) The characteristic equation of A is

$(1-\lambda)(\lambda^2-4\lambda+3)=0$

$\therefore \lambda = 1, 1, 3$

For $\lambda=3$, $(A-\lambda I)X=0$ gives, Algebraic multiplicity $=1=$ geometric multiplicity.

$X_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$

For $\lambda=1$, $(A-\lambda I)X=0$ gives, Algebraic multiplicity $=2$ = geometric multiplicity.

$X_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$, $X_3 = \begin{bmatrix}0\\0\\-1\end{bmatrix}$

$\therefore$ A.M & G.M of each eigen ~~vectors~~ values are equal, the matrix is diagonalisable. The diagonalising matrix is

$M = [X_1, X_2, X_3] = \begin{bmatrix}1 & 1 & 0\\1 & -1 & 0\\0 & 0 & -1\end{bmatrix}$ & diagonal matrix $D = \begin{bmatrix}3 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$.

Q.5 (a) Null Hypothesis $H_0: \mu = 5.4$

Alternative Hypothesis $H_a: \mu \neq 5.4$

Test statistic: Since the population S.D is unknown but sample S.D. s is known and since sample is large.

$$Z = \frac{\bar{x}-\mu}{s/\sqrt{n}} = \frac{6.2-5.4}{10.24/\sqrt{50}} = 1.74,$$

$\therefore |Z| = 1.74$

Level of significance $\alpha = 0.05$

critical value: The value of Z_α at 5% level of significance from the table = 1.96.

Decision: Since the computed value of $|Z| = 1.74$ is less than the critical value $Z_\alpha = 1.96$, the null hypothesis is accepted.

∴ The sample is drawn from the population with mean 5.4.

(b) consider the contour consisting of a semi-circle & diameter on the real axis with centre at the origin.

Now, $z \cdot f(z) = z \cdot \dfrac{1}{(z^2+a^2)^3} \to 0$ as $|z| \to \infty$

The poles are given by, $z^2+a^2=0$ ∴ $z = ai, -ai$ of these $z=ai$ lies in the upper half of the z-plane. It is a pole of order 3.

∴ Residue (at $z=ai$) $= \lim_{z\to ai} \dfrac{1}{2!} \dfrac{d^2}{dz^2} (z-ai)^3 \dfrac{1}{(z-ai)^3 (z+ai)^3} = \dfrac{3}{16a^5 i}$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^3} = 2\pi i \left(\frac{3}{16a^5 i}\right) = \frac{3\pi}{8a^5}$$

$$\therefore \int_{0}^{\infty} \frac{dx}{(x^2+a^2)^3} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^3} = \frac{3\pi}{16a^5}.$$

(c) we rewrite the problem as

$f(x_1, x_2) = 8x_1 + 10x_2 - x_1^2 - x_2^2$

& $h(x_1, x_2) = 3x_1 + 2x_2 - 6$

Now, Kuhn-Tucker conditions are

$\dfrac{\partial f}{\partial x_1} - \lambda \dfrac{\partial h}{\partial x_1} = 0$, $\dfrac{\partial f}{\partial x_2} - \lambda \dfrac{\partial h}{\partial x_2} = 0$, $h(x_1, x_2) \le 0$

$\lambda h(x_1, x_2) = 0$, $\lambda \ge 0$

∴ we get

$8 - 2x_1 - 3\lambda = 0$ — (1)

$10 - 2x_2 - 2\lambda = 0$ — (2)

$\lambda(3x_1 + 2x_2 - 6) = 0$ — (3)

$3x_1 + 2x_2 - 6 \le 0$ — (4)

$x_1, x_2, \lambda \ge 0$ — (5)

From (3), we get $\lambda = 0$ or $3x_1 + 2x_2 - 6 = 0$

Case (1): If $\lambda = 0$, from (1) & (2)

$8-2x_1=0$ & $10-2x_2=0$ $\therefore x_1=4, x_2=5$

But then for $x_1=4, x_2=5$, condition (3) is not satisfied.

$\therefore$ Hence, $\lambda=0$ does not yield a feasible solution.

$\therefore$ we reject these values.

case (2): If $\lambda \neq 0$, $3x_1+2x_2-6=0$ — (6)

from above equation. $x_1=4/13$, $x_2=33/13$, $\lambda>0$.

These values satisfy all necessary condition.

$\therefore$ The optimal solution is $x_1=\frac{4}{13}$, $x_2=\frac{33}{13}$

$\therefore Z_{max}=21.3$

Q.6 (a). Null Hypothesis H_0: Accidents are equally distributed over all the days of a week.

Alternative Hypothesis H_a: Accidents do not occur equally.

Calculation of test statistic: If the accidents occur equally on all days of a week, there will be $(84)/7=12$ accidents per day.

$$\therefore \chi^2 = \Sigma \frac{(O-E)^2}{E} = \frac{(13-12)^2}{12} + \frac{(15-12)^2}{12} + \cdots \frac{(14-12)^2}{12} = \frac{1}{2}[1+9+9+1+0+4+4]$$

$$= \frac{28}{12} = 2.33.$$

Level of significance: $\alpha = 0.05$

Degree of freedom $= n-1 = 7-1=6$.

Critical value: For 6 degree of freedom at 5% level of significance table value of χ^2 is 12.59.

Decision: Since the calculated value of χ^2 is less than the table value. The hypothesis is accepted.

$\therefore$ The accident occur equally on all working days.

(b) when population standard deviation σ_1 & σ_2 are known, we can assume $\overline{x_1}-\overline{x_2}$ to be normal with mean zero and $S.E. = \sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$ & hence, use Z-distribution.

∴ Null Hypothesis $H_0 : \mu_1 = \mu_2$

Alternative Hypothesis : $\mu_1 \neq \mu_2$

Calculation of test statistic:

$$S.E = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = \sqrt{\frac{75^2}{15} + \frac{80^2}{8}} = 34.28$$

$$\therefore Z = \frac{\bar{x}_1 - \bar{x}_2}{S.E.} = \frac{980 - 1012}{34.28} = -0.93$$

$\therefore |Z| = 0.93$

Level of significance : $\alpha = 0.05$

Critical value : The table value of z at $\alpha = 0.05$ is $Z_\alpha = 1.96$.

Decision: Since the computed value $|z| = 0.93$ is less than the table value 1.96, the hypothesis is accepted.

∴ The population mean are equal $\mu_1 = \mu_2$.

(c) we have Maximise $z' = -z = -2x_1 - x_2 - 0s_2 - 0s_3 - MA_1 - MA_2$ —(1)

Subject to $3x_1 + x_2 + 0s_2 + 0s_3 + A_1 + 0A_2 = 3$ —(2)

$4x_1 + 3x_2 - s_2 + 0s_3 + 0A_1 + A_2 = 6$ —(3)

$x_1 + 2x_2 + 0s_2 + s_3 + 0A_1 + 0A_2 = 3$ —(4)

Multiply (2) & (3) by M & add to (1).

$z' = (-2+7M)x_1 + (-1+4M)x_2 - Ms_2 + 0s_3 - A_1 - 0A_2 - 9M$

∴ Maximize $z' = (-2+7M)x_1 + (-1+4M)x_2 - Ms_2 + 0s_3 - A_1 - 0A_2 - 9M$

$\therefore z' + (2-7M)x_1 + (1-4M)x_2 + Ms_2 + 0s_3 + 0A_1 + 0A_2 = -9M$

Iteration number	Basic Var.	x_1	x_2	s_2	s_3	A_1	A_2	R.H.S Soln	Ratio.
0	z'	$2-7M$	$1-4M$	M	0	0	0	$-9M$	
A_1 leaves	A_1	3^*	1	0	0	1	0	3	1 ←
x_2 enters.	A_2	4	3	-1	0	0	1	6	1.5
	S_3	1	2	0	1	0	0	3	3
1	z'	0	$\frac{1}{3} - \frac{5M}{3}$	M	0		0	$-2-2M$.	
	x_1	1	$1/3$	0	0		0	1	3
A_2 leaves	A_2	0	$5/3^*$	-1	0		1	2	6/5 ←
x_2 enters.	S_3	0	$5/3$	0	1		0	2	6/5

2	z'	0	0	1/5	0			−12/5
	x_1	1	0	1/5	0			3/5
	x_2	0	1	−2/5	0			6/5
	s_3	0	0	1	1			0

$\therefore x_1 = \frac{3}{5}$, $x_2 = \frac{6}{5}$, $z'_{max} = \frac{-12}{5}$ $\therefore$ $Z_{min} = \frac{12}{5}$