SOLUTION. Q.P. Code 25569

Q 1(a). $u_r = \frac{1}{r} v_\theta$ and $u_\theta = -r v_r$

$f(z) = r^2 (\cos 2\theta + i \sin p\theta)$

using C-R eqs. $p = 2$.

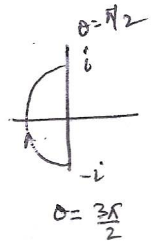
Q 1(b) $L[t e^{-3t} \sin t] = \frac{-(2s+6)}{[s^2+6s+10]^2}$

Q 1(c) $f(x) = \frac{\pi}{4}$ in $(0, \pi)$.

Half range sine series, $\frac{\pi}{4} = \frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots$

Put $x = \frac{\pi}{2}$, $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

Q 1(d).



$I = \int_C f(z) = \int_C (z - z^3)\,dz$, $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$

$I = \int_{3\pi/2}^{\pi/2} (e^{i\theta} - e^{3i\theta})\, i e^{i\theta} d\theta = i\int_{3\pi/2}^{\pi/2} (e^{2i\theta} - e^{4i\theta})\, d\theta$

$= i\left[\frac{e^{2i\theta}}{2i} - \frac{e^{4i\theta}}{4i}\right]_{3\pi/2}^{\pi/2} = \frac{1}{2}\left[e^{2i\theta} - \frac{e^{4i\theta}}{2}\right]_{\theta=3\pi/2}^{\pi/2}$

$= \frac{1}{2}\left[\left(e^{\pi i} - \frac{e^{2\pi i}}{2}\right) - \left(e^{3\pi i} - \frac{e^{6\pi i}}{2}\right)\right]$

$= \frac{1}{2}\left[\left(-1 - \frac{1}{2}\right) - \left(-1 - \frac{1}{2}\right)\right] = 0$

Q 2(a) $f(z) = \frac{2}{z-2} - \frac{1}{z-1}$

(i) $|z| < 1$, $f(z) = \sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$

(ii) $1 < |z| < 2$, $f(z) = -\frac{1}{z}\sum_{n=0}^{\infty} \frac{1}{z^n} - \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$

2(b) $f(x) = e^x$ $(-1, 1)$.

Complex form. $f(x) = \sinh(1) \sum \frac{(-1)^n (1 + in\pi)}{1 + n^2\pi^2} \times e^{in\pi x}$

2(c) $\frac{dx}{dt} + 2x = \cos \omega t$ $x(0) = 0$

$L\left[\frac{dx}{dt} + 2x\right] = L[\cos \omega t]$

$\Rightarrow s\bar{x} - x(0) + 2\bar{x} = \frac{s}{s^2+\omega^2}$ $\Rightarrow (s+2)\bar{x} = \frac{s}{s^2+\omega^2}$

$\bar{x} = \frac{s}{\ldots}$ $\Rightarrow x = L^{-1}\left[\frac{s}{\ldots}\right]$ $\ldots [2\cos\omega t - 2e^{-2t}$

Q3(a)

t \ x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	0	0.09	0.16	0.21	0.24	0.25	0.24	0.21	0.16	0.09	0
0.5	0	0.08	0.15	0.2	0.23	0.24	0.23	0.2	0.15	0.08	0
1.0	0	0.075	0.14	0.19	0.22	0.23	0.22	0.19	0.14	0.075	0
1.5	0	0.07	0.1325	0.18	0.21	0.22	0.21	0.18	0.1325	0.07	0

Q3(b). $w = \dfrac{-1-(1+2i)z}{i-z}$

Q3(c) $f(x) = \dfrac{\pi}{2} - \dfrac{4}{\pi}\sum_{n=1}^{\infty} \dfrac{1}{(2n+1)^2} \cos(2n+1)x$

use Parseval's id.

Q4(a) orthogonal Traj $2y - 3x^2y + y^3 = c$.

Q4(b) $f(x) = 1 - x^2$ in $(-1, 1)$

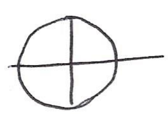
$f(x) = \dfrac{2}{3} - \dfrac{4}{\pi^2}\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} \cos n\pi x$

4(c) (i) $\mathcal{L}^{-1}\left[\dfrac{1}{s(s^2+9)}\right] = \dfrac{1-\cos 3t}{9}$

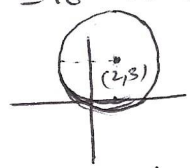
(ii) $\mathcal{L}^{-1}\left[\cot^{-1}(s+1)\right] = \dfrac{e^{-t}\sin t}{t}$

Q5(b) $x = u-2, \quad y = v-3$.

$\therefore \quad |z| = 4 \Rightarrow x^2+y^2 = 16 \Rightarrow (u-2)^2 + (v-3)^2 = 16$



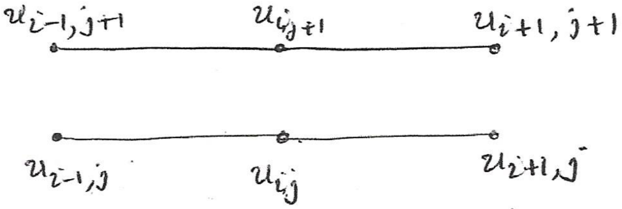
z-plane



w-plane

5(c). $v_{xx} + v_{yy} = 0$, $f(z) = z^3 + 3z^2 + \text{const}$.

5(a)



Simplified Crank - Nicholson scheme,

$$u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i-1,j+1} + u_{i+1,j+1}\right] \quad -(1)$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0, \quad 0 \le x \le 5$$

$u(0,t)=0$, $u(5,t)=100$, $u(x,0)=20x$

$a=1$, $k = a h^2 = 1$.

$j=1$	$t=1$	0 (u_{01})	u_{11}	u_{21}	u_{31}	u_{41}	100 (u_{51})
$j=0$	$t=0$	0 (u_{00})	20 (u_{10})	40 (u_{20})	60 (u_{30})	80 (u_{40})	100 (u_{50})
	$\uparrow t$ / $x \rightarrow$	0	1	2	3	4	5
		$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$

For $j=1$, eq. (1) gives.

$$u_{i1} = \frac{1}{4}\left[u_{i-1,0} + u_{i+1,0} + u_{i-1,1} + u_{i+1,1}\right] \quad -(2).$$

$\therefore$ For $i=1$, this gives.

$$u_{11} = \frac{1}{4}\left[u_{0,0} + u_{20} + u_{01} + u_{21}\right].$$

$$\Rightarrow u_{11} = \frac{1}{4}\left[0 + 40 + 0 + u_{21}\right]$$

$$\Rightarrow 4u_{11} - u_{21} = 40 \quad —— (3).$$

For $i=2$, eq. (2) gives;

$$u_{21} = \frac{1}{4}\left[u_{10} + u_{30} + u_{11} + u_{31}\right] = \frac{1}{4}\left[20 + 60 + u_{11} + u_{31}\right]$$

$$\Rightarrow 4u_{21} = u_{11} + u_{31} + 80 \quad —(4).$$

For $i=3$, eq. (2) gives,

$$u_{31} = \frac{1}{4}\left[u_{20} + u_{40} + u_{21} + u_{41}\right] = \frac{1}{4}\left[40 + 80 + u_{21} + u_{41}\right]$$

$$\Rightarrow 4u_{31} - u_{21} - u_{41} = 120 \quad —(5).$$

For $i=4$, eq. (2) gives,

$$u_{41} = \frac{1}{4}\left[u_{30} + u_{50} + u_{31} + u_{51}\right] = \frac{1}{4}\left[60 + 100 + u_{31} + \underbrace{u_{51}}_{=100}\right]$$

$$\Rightarrow 4u_{41} - u_{31} = 260 \quad —— (6).$$

Contd...

Solving $u_{11} = 20$, $u_{21} = 40$, $u_{31} = 60$, $u_{41} = 80$

2. 6(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = \frac{\pi}{2}$

8

04/04

6(b). $\displaystyle\int_0^{\infty} e^{-t}(1+3t+t^2)H(t-2)\,dt = \frac{20}{e^2} = 2.706$.

Ans 6(c) $y = (C_1 \cos mx + C_2 \sin mx)(C_3 \cos mct + C_4 \sin mct)$

(i) $x=0,\ y=0 \Rightarrow C_1 = 0$.

$\therefore$ $y = C_2 \sin mx\,(C_3 \cos mct + C_4 \sin mct)$

(ii) initially at rest $\frac{\partial y}{\partial t} = 0$ at $t=0$. gives,

$C_2 C_4\, mc = 0$, For non-trivial solution $C_4 = 0$ $(C_2 \neq 0)$

$\therefore$ $y = C_2 C_3 \sin mx \cdot \cos mct$.

$= A \sin mx \cdot \cos mct$ (where $A = C_2 C_3$ a constant).

(iii) $y = 0$ at $x = l$ $\forall t$.

$A \sin ml \cdot \cos mct = 0$

$\therefore$ For non-trivial soln. $A \neq 0$ $\therefore \sin ml = 0$.

$\Rightarrow$ $m = \frac{n\pi}{l}$, $n = 1, 2, 3, \ldots$

Hence, $$y = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} \cdot \cos\frac{n\pi ct}{l}$$

(iv) At $t=0$, $y = kx(l-x)$, gives

$$kx(l-x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$$

A Half range Fourier Sine series, with Fourier Coefficient,

$$A_n = \frac{2}{l}\int_0^l kx(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2k}{l}\left[x(l-x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l-2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{4kl^2}{n^3\pi^3}(1-\cos n\pi) = \frac{4kl^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$\therefore\ y(x,t) = \frac{8kl^2}{\pi^3}\left[\frac{1}{1^3}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} + \frac{1}{3^3}\sin\frac{3\pi x}{l}\cdot\cos\frac{3\pi ct}{l} + \cdots\right]$$