Discrete Mathematics Solution

Subject code - T366

Q.1 (a)

(i) Basis for induction:

For $n=1$,

$1.1! = 1$ and $(1+1)! - 1 = 2! - 1 = 1$

(ii) Induction step:

Assume for $n=k$,

$1.1! + 2.2! + \cdots + k.k! = (k+1)! - 1$

For $n = k+1$,

$1.1! + 2.2! + \cdots + k.k! + (k+1).(k+1)!$

$= (k+1)! - 1 + (k+1).(k+1)!$ (by induction hypothesis)

$= (k+1)! + (k+1).(k+1)! - 1$

$= (k+1)!(1+k+1) - 1$

$= (k+1)!(k+2) - 1$

$= (k+2)! - 1$

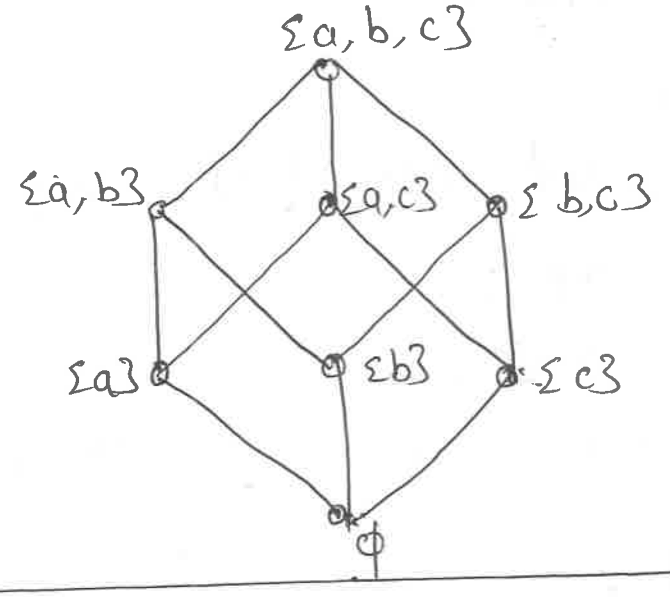
Hence the result is proved.

(b) Let $P(A) = \{A, \phi, \{a,b\}, \{b,c\}, \{a,c\}, \{a\}, \{b\}, \{c\}\}$

Set containment $\subseteq$ is always a partial order since for any subset B of A, $B \subseteq B$, i.e. $\subseteq$ is reflexive. If $B \subseteq C$ and $C \subseteq B$, $B = C$

(antisymmetry). If $B \subseteq C$ and $C \subseteq D$ then $B \subseteq D$ (transitivity)

Hasse diagram



(d)

Consider function $f: N \rightarrow N$ where N is set of natural numbers including zero.

$$f(j) = j^2 + 2$$

Let us verify ~~th~~ can $f(j) = f(i)$?

$$j^2 + 2 = i^2 + 2$$
$$j^2 = i^2$$
$$j = i$$

That

Hence no two members of N will have same image with function f.

Hence f is one to one.

As $f(j) = j^2 + 2$, the range of f is set containing elements greater than 2.

Hence the member of N (range) 0 & 1 are not images of any member of N. In other words, there exists no i such that $f(i) = 0$ or $f(i) = 1$.

Hence f is not onto function.

Q.2 (a) First without applying the restriction, the number of ways to distribute Rs. 601 among his three sons is $C(601+3-1, 2) = C(603, 2)$.

• Let the sons receive x, y, z rupees resp. Suppose the first son receives more than the combined total of the other two it will follow that $x+y+z = 601$ means $2(y+z) < 601$ which means that $y+z = 300$, in which case x must be 301 atleast.

Hence the total number of ways in which the first son will receive more money than the combined total of the other two is $C(300+3-1, 2) = C(302, 2)$.

The same result is applicable when the second son receives more than the combined total of the first & third sons or when the third son receives more than the combined total of the first & second.

Hence the number of ways so that no son receives more than the combined total of the other two is $C(603, 2) - 3C(302, 2)$

$$= \frac{603 \times 602}{2} - 3\left(\frac{302 \times 301}{2}\right)$$

$$= 181503 - 136353$$

$$= \boxed{45150}$$

(b)

$$M_R^* = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(c)

$$a_n = \frac{2}{9} 5^n + \frac{1}{5} n 5^n + \frac{2^{n+2}}{9}$$

(d)

$$\begin{aligned} g \circ f &= g(f(x)) \\ &= g(x^3) \\ &= 4(x^3)^2 + 1 \\ &= 4x^6 + 1 \end{aligned}$$

$$\begin{aligned} f \circ g &= f(g(x)) \\ &= f(4x^2+1) \\ &= (4x^2+1)^3 \\ &= 64x^6 + 48x^4 + 12x^2 + 1 \end{aligned}$$

$$\begin{aligned} f^2 = f \circ f &= f(f(x)) \\ &= f(x^3) \\ &= (x^3)^3 \\ &= x^9 \end{aligned}$$

$$\begin{aligned} g^2 = g \circ g &= g(g(x)) \\ &= g(4x^2+1) \\ &= 4(4x^2+1) + 1 \\ &= 16x^2 + 5 \end{aligned}$$

Q.3 (a) Let A denote the event of the student having prepared for the exam, & B denote the event of the student passing in the entrance exam.

Given, $P(B/A) = 0.99$, $P(B/\bar{A}) = 0.05$

$P(A) = 0.7$

We have to find $P(\bar{A}/\bar{B})$.

$$P(\bar{A}/\bar{B}) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{B})}$$

Now $P(A \cap B) = P(B/A) \cdot P(A)$

$= (0.99)(0.7) = 0.693$

Also $P(\bar{A} \cap B) = P(B/\bar{A}) \cdot P(\bar{A})$

$= (0.05)(0.3) = 0.015$

$P(B) = P(A \cap B) + P(\bar{A} \cap B)$

$= 0.693 + 0.015 = 0.708$

Hence, $P(\bar{B}) = 1 - 0.708 = 0.292$

Now, $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$

$= 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - 0.7 - 0.708 + 0.693$

$= 0.285$

Hence $P(\bar{A}/\bar{B}) = \frac{0.285}{0.292} = \boxed{0.976}$

(b) We have

$T =$ the set of triangles in a plane.

& $R = \{ (a,b) \mid a, b \in T, a$ is congruent to $b \}$

(i) R is reflexive, since for any $a \in T$, a is congruent to a.

(ii) R is symmetric, since for any $a, b \in T$ if a is congruent to b then b is congruent to a.

(iii) R is transitive, since if $(a,b) \in R$ and $(b,c) \in R$ then a is congruent to b and b is congruent to c then a is congruent to c.

(c) Let $x \in A$ and $y = f(x) = 2x^3 - 1$.

Then $\frac{1}{2}(y+1) = x^3$;

$\therefore x = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}} = g(y) = g(f(x))$
$= (g \circ f)(x)$

Thus $g \circ f = I_A$. Similarly, $f \circ g = I_B$

so both f & g are bijections.

(d) (i) $Z_4 = \{0, 1, 2, 3\}$

Table

$\odot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

(ii) Let $a \odot b = r$, where
$ab = pn + r$ — ①

Then $(a \odot b) \odot c = r \odot c$
$= s$, where $rc = qn + s$ ②

Let $b \odot c = t$, where $bc = ln + t$ — ③

$a \odot (b \odot c) = a \odot t = k$, where
$at = mn + k$ — ④

we have to prove $s = k$.

Now $a(bc) = aln + at = aln + mn + k$ — ⑤
$(ab)c = (pn + r)c = pnc + rc$ — 5
$= pnc + qn + s$ — ⑥

Now since equations (5) & (6) are equal, it follows that $k = s$.

Hence $(a \odot b) \odot c = a \odot (b \odot c)$

Hence $(Z_n, \odot)$ is a semigroup for any n.

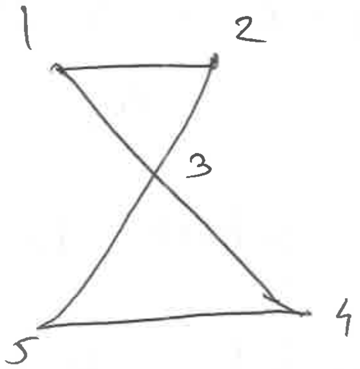
Q.4 (a) (i) — 14

(ii) — The set of students who got an A in the first examination is 20

- The set of students who got an A in the second examination only is 20.

Q.4 (b) (i) 01 (ii) 11 (iii) 10

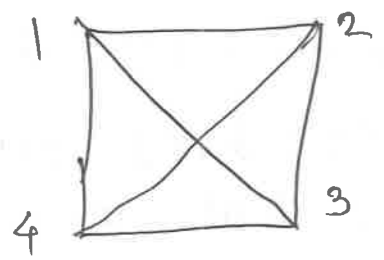
Q.4 (c) (i) No



This graph is Eulerian but not Hamiltonian.

(ii) No

A complete graph of even no. of vertices ≥ 3 is hamiltonian but not Eulerian graph.



This K_4 graph is hamiltonian but not Eulerian.

Q.4 (d) Consider the columns $h_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, h_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

& $h_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ of H. The sum of these 5

three columns is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Hence, the minimum weight of the code is 3 which is equal to its minimum distance.

Now, the code can detect k errors or less if its minimum distance is k+1. Therefore, the code generated by H can detect 2 errors or less.

Also it can correct k errors if the minimum distance is 2k+1. In this case the code can correct only single error.

Q.5 (a) Suppose there are n people, then, since people shake hands only once, the labels on the pigeonhole will go from 0 to (n-1).

That is we have n people & n holes.

But it is not possible say for 0th & $(n-1)^{th}$ holes both to be occupied.

Thus, we have atmost (n-1) holes occupied at any one time.

Hence, by the principle atleast one of the holes has two occupants, which shows that there are atleast two people who have shaken hands the same times.

Q. 5 (b) (i) Yes, it is a lattice

(ii) not a lattice

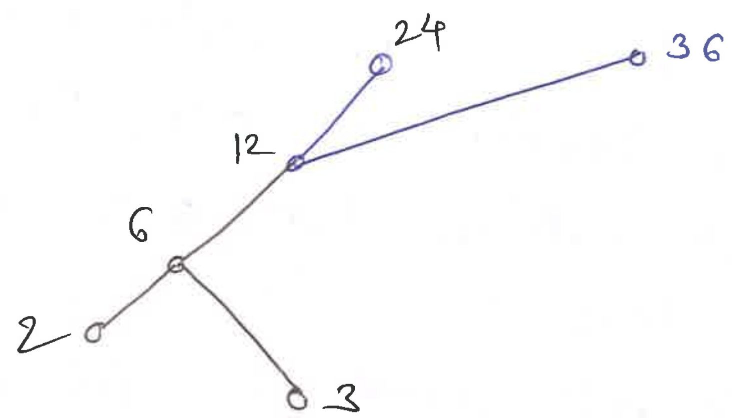
Q. 5 (c)

$R_1 = \{ (a,b), (b,a), (a,c), (c,d), (b,d), (c,c) \}$ and

$R_2 = \{ (a,a), (b,b), (c,c), (c,a), (b,c), (a,b) \}$

R_1 is not an equivalence relation since R_1 is not reflexive. R_2 is also not an equivalence relation since R_2 is not symmetric, as $(a,b) \in R$, $(b,a) \notin R$.

Q. 5 (d)



(i) Maximal elements are 24, 36
Minimal elements are 2, 3

(ii) chain : $\{2, 6, 12, 24\}$.
Antichain : $\{2, 3\}$ or $\{24, 36\}$

(iii) The poset is not a lattice since the set $\{2,3\}$ has no greatest lower bound. & set $\{24,36\}$ has no least upper bound.

Q.6 (a)

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

If A is given set then

1. if $a, b \in A$ then $a \times_7 b \in A$ (closure property)
2. $(a \times_7 b) \times_7 c = a \times_7 (b \times_7 c)$ in above table $\forall\, a, b, c \in A$
3. 1 is identity element.
4. every element has inverse as shown in table.

$\therefore$ it is group.

Q.6 (b)

(i) $G(x) = \dfrac{1}{3-6x}$

$= \dfrac{1}{3}\left(\dfrac{1}{1-2x}\right)$

$= \dfrac{1}{3}\left[1 + 2x + (2x)^2 + (2x)^3 + \cdots\right]$

$= \dfrac{1}{3}\left[1 + 2x + 4x^2 + 8x^3 + \cdots\right]$

$= \left[\dfrac{1}{3} + \dfrac{2}{3}x + \dfrac{4}{3}x^2 + \dfrac{8}{3}x^3 + \cdots\right]$

$\therefore$ The sequence is

$(a_n) \Rightarrow \dfrac{1}{3}, \dfrac{2}{3}, \dfrac{4}{3}, \dfrac{8}{3}, \cdots$

OR

$$a_n = \frac{1}{3}(2)^n$$

(ii)

$$G(x) = \frac{x}{1-5x+6x^2}$$

$$= \frac{x}{(1-2x)(1-3x)}$$

By partial fraction,

$$\frac{x}{(1-2x)(1-3x)} = \frac{A}{1-2x} + \frac{B}{1-3x}$$

$$\therefore x = A(1-3x) + B(1-2x)$$

put $x = \frac{1}{3}$

$$\therefore \frac{1}{3} = B\left(1-\frac{2}{3}\right)$$

$$\therefore B = 1$$

put $x = \frac{1}{2}$

$$\therefore \frac{1}{2} = A\left(1-\frac{3}{2}\right)$$

$$A = -1$$

$$\therefore G(x) = \frac{-1}{1-2x} + \frac{1}{1-3x}$$

$$= -(1+2x+4x^2+8x^3+\cdots) + (1+3x+9x^2+27x^3+\cdots)$$

$$a_n = -2^n + 3^n$$

$$= 3^n - 2^n$$

Q. 6 (c) Here both the graphs G_1 & G_2 contain 8 vertices & 10 edges. The number of vertices of degree 2 in both the graphs are 4. Also the number of vertices of degree 3 in both the graphs are 4.

For adjacency, consider the vertex 1 of degree 3 in G_1, It is adjacent to 2 vertices of degree 3 & 1 vertex of degree 2. But in G_2 there does not exist any vertex of degree 3 which is adjacent to two vertices of degree 3 & 1 vertex of degree 2.

Hence adjacency is not preserved. Hence given graphs are not isomorphic.

Q6 (d)

$$= (A \cap B) \cup [B \cap ((C \cap D) \cup (C \cap \bar{D}))]$$

$$= (A \cap B) \cup [B \cap (C \cap (D \cup \bar{D}))]$$ -- Distributive Laws

$$= (A \cap B) \cup [B \cap (C \cap U)]$$ -- Universal set $D \cup \bar{D} = U$

$$= (A \cap B) \cup (B \cap C)$$ -- Identity law

$$= (B \cap A) \cup (B \cap C)$$ -- commutative law

$$= B \cap (A \cup C)$$ -- Distributive law

$$= \text{RHS.}$$

Q 6 (d)

Let $(a,x) \in A \times (X \cap Y)$ - - - I

By defn of the Cartesian product

$a \in A$ and $x \in X \cap Y$

Since $x \in X \cap Y$

$\therefore$ $x \in X$ and $x \in Y$

$(a,x) \in A \times X$ and $(a,x) \in A \times Y$

$(a,x) \in (A \times Y) \cap (A \times Y)$

$\therefore A \times (X \cap Y) \subseteq (A \times X) \cap (A \times Y)$ - - - II

Let $(a,x) \in (A \times X) \cap (A \times Y)$

$(a,x) \in (A \times X)$ and $(a,x) \in A \times Y$

$a \in A$, $x \in X$ and $x \in Y$

$\therefore$ $a \in A$ and $x \in X \cap Y$

$(a,x) \in A \times (X \cap Y)$

$\therefore$ $(A \times X) \cap (A \times Y) \subseteq A \times (X \cap Y)$ - - III

From eqn I & II we have

$$\boxed{A \times (X \cap Y) = (A \times X) \cap (A \times Y)}$$