Q1 (a) $f(t) = te^{-3t}\sin t$

$$\mathcal{L}[\sin t] = \frac{1}{s^2+1}, \quad \mathcal{L}[t\sin t] = -\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$\therefore \mathcal{L}[t\sin t] = -\frac{2s}{(s^2+1)^2}$$

$$\therefore \mathcal{L}[e^{-3t}. t\sin t] = -\frac{2(s+3)}{[(s+3)^2+1]^2} \quad \text{by (F.S.P).}$$

$$= \frac{-(2s+6)}{[s^2+6s+10]^2} \quad \underline{\text{Ans.}}$$

1(b) $f(x) = e^x \quad (-1, 1)$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x} \quad \text{where } C_n = \frac{1}{2}\int_{-1}^{1} f(x) e^{-in\pi x} dx$$

$$\therefore C_n = \frac{1}{2}\int_{-1}^{1} e^x . e^{-in\pi x} dx = \frac{1}{2}\int_{-1}^{1} e^{(1-in\pi)x} dx$$

$$= \frac{1}{2}\left[\frac{e^{(1-in\pi)x}}{1-in\pi}\right]_{-1}^{1} = \frac{1}{2}\left[\frac{e^{(1-in\pi)} - e^{-(1-in\pi)}}{(1-in\pi)}\right]$$

$$= \frac{1}{2}\left[\frac{(e^1 - e^{-1})(-1)^n}{1-in\pi}\right] \quad \because e^{-in\pi} = e^{in\pi} = (-1)^n$$

$$= \frac{(-1)^n . \sinh(1)}{1-in\pi} = \frac{(-1)^n \sinh(1)\,(1+in\pi)}{1+n^2\pi^2}$$

$$\therefore f(x) = \sinh(1) . \sum \frac{(-1)^n (1+in\pi)}{1+n^2\pi^2} e^{in\pi x}. \quad \underline{\text{Ans}}$$

1(c). $u = k(1+\cos\theta)$.

If u is a real part of an analytic function, it should be haromonic function because "real and imaginary parts of an analytic function are harmonic".

$u_r = 0, \quad u_{rr} = 0, \quad u_\theta = -k\sin\theta, \quad u_{\theta\theta} = -k\cos\theta.$

$\therefore$ Laplace eq. in polar form, $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \neq 0.$

i.e. $L.H.S. = \frac{-k}{r^2}\cos\theta \neq 0$ or $k \neq 0$.

for $\theta \neq \frac{(2n+1)\pi}{2}$ $\quad n = 0, 1, 2, ---$

For non-zero k, non-constant u.

Q.P Code : 25564

1 (d) Reg. line 1, $3x+2y=26 \Rightarrow 2y=-3x+26$.

$\Rightarrow y=-\frac{3}{2}x+13$.

$\therefore\ b_{yx}=-\frac{3}{2}$

Reg. line 2: $6x+y=31 \Rightarrow x=-\frac{1}{6}y+\frac{31}{6}$

$\therefore\ b_{xy}=-\frac{1}{6}$

Solving the given eqs.

$x=4,\quad y=7$

Hence (i) mean $\bar{x}=4,\quad \bar{y}=7$.

(ii) $r=-\sqrt{b_{xy}\cdot b_{yx}}=-\sqrt{\left(-\frac{1}{6}\right)\left(-\frac{3}{2}\right)}=-\frac{1}{2}=\boxed{-0.5}$

Ans

Q 2 (a) $f(t)=\sin 2t\cdot\cos 3t=\frac{1}{2}\left[\sin(2t-3t)+\sin(2t+3t)\right]$

$=\frac{1}{2}\left[\sin 5t-\sin t\right]$.

$\mathcal{L}[f(t)]=\frac{1}{2}\mathcal{L}[\sin 5t]-\frac{1}{2}\mathcal{L}[\sin t]$

$=\frac{5}{2(s^2+5^2)}-\frac{1}{2(s^2+1)}$

Putting $s=-1$

$\therefore\ \int_0^\infty e^{t}\sin 2t\cdot\cos 3t\,dt=\frac{1}{2}\left[\frac{5}{26}-\frac{1}{2}\right]$

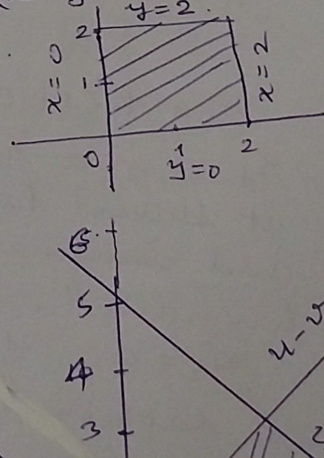
$=\frac{1}{2}\left[\frac{5-13}{26}\right]=\frac{1}{2}\times\left(\frac{-8}{26}\right)$

$=-\frac{2}{13}$

2 (b). $w=(1+i)z+2-i \Rightarrow u+iv=(1+i)(x+iy)+2-i$

$=(x-y+2)+i(x+y-1)$

$\therefore u=x-y+2$ and $v=x+y-1$.



z-plane

Line $x=0$ gives.

$u=-y+2$ and $v=y-1$

$\therefore\ \boxed{u+v=1}$.

Line $x=2$, gives

$u=4-y,\quad v=1+y$.

$\therefore\ \boxed{u+v=5}$

Line $y=0$ gives.

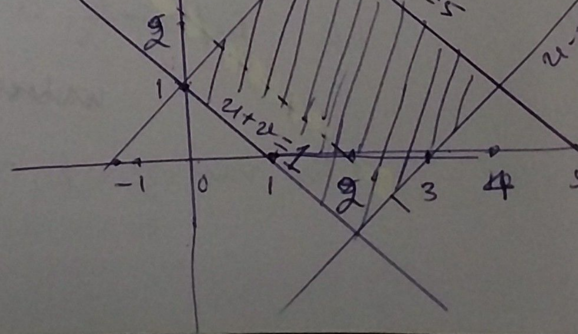
$u=x+2$ & $v=x-1$

$\therefore\ \boxed{u-v=3}$.

Line $y=2$ gives,

$u=x,\quad v=x+1$

$\therefore\ \boxed{u-v=-1}$



Q2(c). $f(x) = |x|$ in $(-\pi, \pi)$.

Since $f(x) = |x|$ is an even functi.

b_n's $= 0$, $\therefore$ F.S. is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx = \frac{2}{\pi} \left[x \cdot \frac{\sin nx}{n} \Big|_0^{\pi} + 1 \cdot \frac{\cos nx}{n^2} \Big|_0^{\pi} \right]$$

$$= \frac{2}{\pi} \left[0 + \frac{\cos n\pi - \cos 0}{n^2} \right] = \frac{2}{\pi} \left[\frac{(-1)^n - 1}{n^2} \right]$$

$$= \begin{cases} -\frac{4}{n^2\pi} & \text{if } n \text{ is odd.} \\ 0 & \text{if } n \text{ is even} \end{cases} \qquad a_0 = \frac{2}{\pi} \int_0^{\pi} x \, dx$$

$\therefore$ F.S. is given by.

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots \right)$$

Put $x = 0$.

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right).$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

Q3(a). $F(s) = \dfrac{s}{(s^2+9)(s^2+4)}$

Takes $G(s) = \dfrac{1}{s^2+4}$, $g(t) = L^{-1}[G(s)]$

$$= L^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2} \sin 2t.$$

$H(s) = \dfrac{s}{s^2+9} \Rightarrow h(t) = L^{-1}[H(s)] = \cos 3t.$

$\therefore$ By Convolution Th.,

$$L^{-1}[G(s) \cdot H(s)] = (g * h)(t) = \int_0^t g(u) \, h(t-u) \, du$$

$$= \frac{1}{2} \int_0^t \sin 2u \cos 3(t-u) \, du.$$

$$= \frac{1}{4} \int_0^t \left[\sin(2u - 3t + 3u) + \sin(2u + 3t - 3u) \right] du$$

$$\therefore \mathcal{L}^{-1}[F(s)] = \frac{1}{4}\int_0^t [\sin(5u-3t) + \sin(3t-u)]\,du.$$

$$= \frac{1}{4}\left[-\frac{\cos(5u-3t)}{5} + \frac{\cos(3t-u)}{1}\right]_0^t$$

$$= \frac{1}{4}\left[\left\{\cos(3t-t) - \frac{1}{5}\cos(5t-3t)\right\} - \left\{\cos(3t) - \frac{1}{5}\cos(-3t)\right\}\right]$$

$$= \frac{1}{4}\left[\left(\cos 2t - \frac{1}{5}\cos 2t\right) - \left(\cos 3t - \frac{1}{5}\cos 3t\right)\right]$$

$$= \frac{1}{4}\left[\frac{4}{5}\cos 2t - \frac{4}{5}\cos 3t\right] = \boxed{\frac{1}{5}(\cos 2t - \cos 3t)}$$

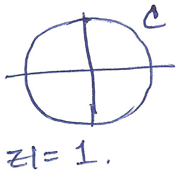
Ans.

Q 3(b) $\quad \dfrac{\partial^2 u}{\partial x^2} - 100\dfrac{\partial u}{\partial t} = 0, \quad u(0,t)=0, \; u(1,t)=0$

$u(x,0) = x(1-x).$

t \ x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	0	0.09	0.16	0.21	0.24	0.25	0.24	0.21	0.16	0.09	0.0
0.5	0	0.08	0.15	0.2	0.23	0.24	0.23	0.2	0.15	0.08	0
1.6	0	0.075	0.14	0.19	0.22	0.23	0.22	0.19	0.14	0.075	0
1.5	0	0.07	0.1325	0.18	0.21	0.22	0.21	0.18	0.1325	0.07	0

3(c) (i) $\quad I = \displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta},$



$C: |z| = 1.$

Put $z = e^{i\theta}$ $\quad C: |z| = 1.$

$$\therefore \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{z^2+1}{2z}, \quad d\theta = \frac{dz}{iz}$$

$$\therefore I = \int_C \frac{dz/iz}{5+4\left(\frac{z^2+1}{2z}\right)} = \frac{1}{i}\int_C \frac{dz}{5z+2z^2+2} = \frac{1}{2i}\int_C \frac{dz}{(z+2)(z+\frac{1}{2})}$$

$f(z) = \dfrac{1}{2i(z+2)(z+\frac{1}{2})}$ has simple poles at $z=-2$ (outside) and $z = -\frac{1}{2}$ (inside)

$$\therefore \operatorname*{Res}_{z=-\frac{1}{2}} f(z) = \lim_{z\to -\frac{1}{2}}\left[\left(z+\tfrac{1}{2}\right)f(z)\right] = \lim_{z\to -\frac{1}{2}}\left[\frac{1}{2i(z+2)}\right]$$

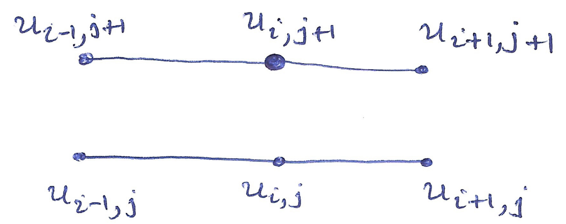
$$= \frac{1}{3i}$$

$\therefore$ By Residue Theorem.

$$\int_C f(z)\,dz = 2\pi i \cdot \operatorname*{Res}_{z=-\frac{1}{2}} f(z) = 2\pi i \times \frac{1}{3i} = \boxed{\frac{2\pi}{3}}$$

Ans.

Q 4 (a)



Simplified Crank - Nicholson scheme,

$$u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i-1,j+1} + u_{i+1,j+1}\right]$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0, \quad 0 \le x \le 1.$$

$u(0,t) = 0, \quad u(5,t) = 100, \quad u(x,0) = 20x$

$a = 1, \quad k = ah^2 = 1.$

t \ x	0	1	2	3	4	5
0	0	20	40	60	80	100
1	0	u_{11}	u_{21}	u_{31}	u_{41}	100

At $j=0$, $u_{i1} = \frac{1}{4}\left[u_{i-1,0} + u_{i+1,0} + u_{i-1,1} + u_{i+1,1}\right]$

for $i=1$

$\therefore \quad u_{11} = \frac{1}{4}\left[u_{0,0} + u_{20} + u_{01} + u_{21}\right]$

$\Rightarrow \quad u_{11} = \frac{1}{4}\left[0 + 40 + 0 + u_{21}\right]$

$\Rightarrow \quad 4u_{11} - u_{21} = 40 \quad$ — (1)

for $i=2$, $\quad u_{21} = \frac{1}{4}\left[u_{10} + u_{30} + u_{11} + u_{31}\right]$

$= \frac{1}{4}\left[20 + 60 + u_{11} + u_{31}\right].$

$\Rightarrow \quad 4u_{21} = u_{11} + u_{31} + 80$

$\Rightarrow \quad u_{11} - 4u_{21} + u_{31} = -80 \quad$ —(2).

for $i=3$, $\quad u_{31} = \frac{1}{4}\left[u_{20} + u_{40} + u_{21} + u_{41}\right].$

$\Rightarrow 4u_{31} = 40 + 80 + u_{21} + u_{41}$

$\Rightarrow \quad 4u_{31} - u_{21} - u_{41} = 120 \quad$ —(3).

for $i=4$, $\quad u_{41} = \frac{1}{4}\left[u_{30} + u_{50} + u_{31} + u_{51}\right]$

$= \frac{1}{4}\left[60 + 100 + u_{31} + u_{51}\right]$

$\Rightarrow \quad 4u_{41} = u_{31} + 100 + 160.$

$\Rightarrow \quad 4u_{41} - u_{31} = 260 \quad$ —— (4).

Solving equations (1), (2), (3) and (4).

$u_{11} = 20, \quad u_{21} = 40, \quad u_{31} = 60, \quad u_{41} = 80.$

Remark: Misprint in I.C. $u(x,0) = 20x$, 1

If students have written correct Cranck-Nicholson scheme, value of k, made the table, full 06 marks may be awarded.

4(b) $f(z) = \dfrac{z}{(z-1)(z-2)} = \dfrac{2}{(z-2)(z-1)}$

(i) $|z| < 1$,

$f(z) = (1-z)^{-1} - \left(1 - \frac{z}{2}\right)^{-1} = \sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$ 2

(ii) $1 < |z| < 2$.

$f(z) = -\frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} - \left(1 - \frac{z}{2}\right)^{-1}$

$= -\frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} - \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n.$

4(c). $\mathcal{L}[y'' - 3y' + 2y] = \mathcal{L}[4e^{2t}]$.

$\Rightarrow [s^2 - 3s + 2]\, Y(s) + 3s - 14 = \dfrac{4}{s-2}$

$Y(s) = \dfrac{-3s^2 + 20s - 24}{(s-2)^2 (s-1)} = -\dfrac{7}{s-1} + \dfrac{4}{s-2} + \dfrac{4}{(s-2)^2}$

$\therefore\ y(t) = \mathcal{L}^{-1}[Y(s)]$

$= -7e^{t} + 4e^{2t} + 4te^{2t}$

5(a) $u(x,y) = e^{-x}\{(x^2-y^2)\cos y + 2xy\sin y\}$

$u_x = -e^{-x}\{(x^2-y^2)\cos y + 2xy\sin y\} + e^{-x}\{2x\cos y + 2y\sin y\}$.

$= e^{-x}\{-(x^2-y^2)\cos y - 2xy\sin y + 2x\cos y + 2y\sin y\}$.

$u_y = e^{-x}\{(x^2-y^2)(-\sin y) - 2y\cos y + 2x\sin y + 2xy\cos y\}$.

$u_x(z,0) = e^{-z}\{-z^2+2z\}$. , $u_y(z,0) = 0$

$\therefore\ f(z) = \int [u_x(z,0) - i\,u_y(z,0)]\,dz + c$

$= \int e^{-z}(-z^2+2z)\,dz + c \quad = z^2 e^{-z} + c$

5(b). $\mathcal{L}[t\sqrt{1+\sin t}] = \mathcal{L}[\sin(\frac{t}{2}) + \cos(\frac{t}{2})] = \frac{1}{2[s^2+(\frac{1}{2})^2]} + \frac{s}{s^2+(\frac{1}{2})^2}$

$= \frac{1}{2}\frac{4}{(4s^2+1)} + \frac{4s}{(4s^2+1)} = \frac{2(2s+1)}{4s^2+1}$

$\therefore\ \mathcal{L}[t\sqrt{1+\sin t}] = -\frac{d}{ds}\left[\frac{2(2s+1)}{4s^2+1}\right] = \frac{4(4s^2+4s-1)}{(4s^2+1)^2}$

5(c) $f(x) = x, \quad 0 < x < 2.$

Fourier half range cosine series, $(l=2)$.

$x = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{2}$,

where $a_n = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx = \left[x\cdot\frac{\sin(n\pi x/2)}{n\pi/2} + \frac{\cos(n\pi x/2)}{n^2\pi^2/2^2}\right]_0^2$

$= 0 + \frac{4}{n^2\pi^2}\cos n\pi - 0 - \frac{4}{n^2\pi^2} = \frac{4}{n^2\pi^2}[(-1)^n - 1]$

$\therefore\ a_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{8}{n^2\pi^2} & \text{if } n \text{ is odd.} \end{cases}$ and $a_0 = \frac{2}{2}\int_0^2 x\,dx = 2$.

$\therefore\ x = 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right]$.

Using Parseval's identity;

$\frac{1}{l}\int_0^l [f(x)]^2 dx = \frac{1}{2}\left[\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + \cdots\right]$.

$\Rightarrow \frac{4}{3} = 1 + \frac{32}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right]$

$\Rightarrow \frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$

6(a) $f(a) = \oint_c \frac{3z^2+7z+1}{z-a} dz$, $c: x^2+y^2=4$

c is a circle with centre (0,0) radius 2.

If $g(z) = 3z^2+7z+1$, then $g(z)$ is analytic.

$\therefore f(a) = \oint_c \frac{g(z)}{z-a} dz$.

For $a=3$, outside c, $f(3) = \oint_c \frac{g(z)}{z-3} dz = 0$ (by Cauchy integral Theorem).

For $a = 1-i$ lies inside c.

$\therefore f(a) = 2\pi i\, g(a)$ by Cauchy's integral formula

$\therefore f'(1-i) = 2\pi i\,[6(1-i)+7] = 26\pi i + 12\pi = 2[13\pi i + 6]$.

and $f''(1-i) = 12\pi i$

6(b).

x	y	$X = x-68$	$Y = y-68$	X^2	Y^2	XY
65	67	-3	-1	9	1	3
66	68	-2	0	4	0	0
67	64	-1	-4	1	16	4
68	68	0	0	0	0	0
69	72	1	4	1	16	4
71	69	3	1	9	1	3
73	70	5	2	25	4	10
				$\Sigma X^2 = 49$	$\Sigma Y^2 = 38$	$\Sigma XY = 24$.

Taking $\bar{x} = 68$ and $\bar{y} = 68$, $r = \frac{\Sigma XY}{\sqrt{\Sigma X^2 . \Sigma Y^2}}$

$$r = \frac{24}{\sqrt{49 \times 38}} = \frac{24}{7\sqrt{38}} = 0.556.$$

Remark: Since x is increasing by a fixed no. but y is not increasing/decreasing by a fixed no. if the data is written in ascending order, rank correlation coefficient will not be equal to Karl pearson coefficient, so here, Karl Pearson correlation should be calculated.

Ans 6 (c) $y = (C_1 \cos mx + C_2 \sin mx)(C_3 \cos mct + C_4 \sin mct)$

(i) $x = 0, y = 0 \Rightarrow C_1 = 0$.

$\therefore$ $y = C_2 \sin mx (C_3 \cos mct + C_4 \sin mct)$

(ii) initially at rest $\frac{\partial y}{\partial t} = 0$ at $t = 0$. gives,

$C_2 C_4 \cdot mc = 0$, For non-trivial solution $C_4 = 0$ $(C_2 \neq 0)$

$\therefore$ $y = C_2 C_3 \sin mx \cdot \cos mct$.

$= A \sin mx \cdot \cos mct$ (where $A = C_2 C_3$ a constant).

(iii) . $y = 0$ at $x = l$ $\forall t$.

$A \sin ml \cdot \cos mct = 0$

$\therefore$ For non-trivial soln. $A \neq 0$ $\therefore$ $\sin ml = 0$.

$\Rightarrow$ $m = \frac{n\pi}{l}$, $n = 1, 2, 3, ---$

Hence,

$$y = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi ct}{l}$$

(iv) At $t = 0$, $y = kx(l - x)$, gives

$$kx(l-x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

A, Half range Fourier sine series, with Fourier Coefficient,

$$A_n = \frac{2}{l} \int_0^l kx(l-x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2k}{l} \left[x(l-x)\left(-\frac{l}{n\pi} \cos \frac{n\pi x}{l}\right) - (l-2x)\left(-\frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3} \cos \frac{n\pi x}{l}\right) \right]_0^l$$

$$= \frac{4kl^2}{n^3\pi^3}(1 - \cos n\pi) = \frac{4kl^2}{n^3\pi^3}\left[1 - (-1)^n\right]$$

$$\therefore \quad y(x,t) = \frac{8kl^2}{\pi^3} \left[\frac{1}{1^3} \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} + \frac{1}{3^3} \sin \frac{3\pi x}{l} \cdot \cos \frac{3\pi ct}{l} + --- \right]$$

— o —