QP Code : 75688

Revised] (3 Hours) **[Total Marks:80**

Instructions:

- Attempt **any two** questions from **each section**
- **All questions** carry **equal marks**
- **Answers to Section I and Section II** should be written in the **same answer book.**

SECTION I (Attempt **any two** questions)

1. (a) Define convergence of a sequence in a metric space (X, d). Define Cauchy sequence in a metric space (X, d). Prove that every convergent sequence in (X, d) is Cauchy. Does the converse of the above statement hold?

   (b) State and prove Lebesgue covering lemma.

2. (a) Let (X, d_1), (Y, d_2) be metric spaces. Let $f : X \to Y$ be a function. Prove that f is continuous on X if and only if inverse image of an open set in Y is an open set in X.

   (b) Define compact set. Define uniform continuity of a function $f : (X, d_1) \to (Y, d_2)$, where (X, d_1), (Y, d_2) are metric spaces. Prove that if K is a compact subset of X, and f is continuous on K then f is uniformly continuous on K.

3. (a) Define partial derivative. Find partial derivatives of all possible orders for the function $f(x, y, z) = (x^2y^2, 3xy^3z, xz^3)$.

   (b) State (without proof) chain rule. Write the matrices for f', g', $(f \circ g)'$ for the following functions and evaluate them at the point $(2, 5)$: $f(x, y) = (x + y, x^2 + y^2, 2x + 3y)$ and $g(u, v) = (u^2, v^3)$.

4. (a) State and prove mean value theorem.

   (b) State implicit function theorem. Examine whether the function $f(x, y) = x^2 + y^2 - 4$ can be expressed as a function $y = g(x)$ in a neighbourhood of the point $(0, -2)$.

SECTION II (Attempt **any two** questions)

5. (a) Define base of a topological space. Define product topology. Prove that if $\mathcal{B}$ is a basis for the topology on X and $\mathcal{C}$ is a basis for the topology on Y then prove that the collection $\mathcal{D} = \{B \times C \mid B \in \mathcal{B}, C \in \mathcal{C}\}$ is a basis for the topology on $X \times Y$.

   (b) Let (X, τ) be a topological space and $A \neq \emptyset$ be a subset of X. Define interior of A. Prove that if $A \subset B$, then $i(A) \subset i(B)$ and for all subsets A, B of X, $i(A \cap B) = i(A) \cap i(B)$.

6. (a) Define first countable topological space. Define second countable topological space. Prove that a second countable topological space is first countable.

   (b) Define T_1 topological space. Prove that a topological space is a T_1 space if and only if every one point subset of it is a closed subset.

[TURN OVER

PA-Con.1158-17.

7. (a) State (without proof) tube lemma. Let f be a continuous real-valued function on $[a, b]$. Prove that the set $\{(x, f(x)) \mid x \in [a, b]\}$ is a compact subset of $\mathbb{R}^2$.

   (b) Define local compactness of a topological space (X, τ). Define regular topological space. Prove that if X is a regular space such that X is locally compact at $x \in X$, then x has a local base of compact neighbourhoods in X.

8. (a) Define complete metric space (X, d). Assume that for each $n \in \mathbb{N}$, F_n's are closed and bounded subsets of X such that $F_1 \supset F_2 \supset \cdots \supset F_n \supset F_{n+1} \supset \cdots$ and $\text{diam}(F_n) \to 0$ and $n \to \infty$. Prove that $\bigcap_{n=1}^{\infty} F_n$ contains precisely one point.

   (b) Define total boundedness of a metric space (X, d). Prove that if (X, d) is a compact metric space, then (X, d) is complete and totally bounded.
