Q.P. Code 24735

Duration - 3 hrs. Marks: 80

Solution

① ⓐ $f(x) = a_0 + \sum a_n \cos nx + \sum b_n \sin nx$ $f(x) = x$

$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x\,dx = \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = \frac{1}{2\pi} \times \frac{4\pi^2}{2} = \pi$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$

$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{2\pi} = \frac{1}{\pi}\left[0 + \frac{1}{n^2} - \frac{1}{n^2}\right]$

$= 0$

$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx = \frac{1}{\pi}\left[x\frac{-\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{2\pi}$

$= \frac{1}{\pi}\left[-\frac{2\pi}{n} - 0\right] = -\frac{2}{n}$

$$\boxed{f(x) = \pi - 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}.}$$

ⓑ $f(z) = e^x(\cos y - i\sin y) = u + iv$

$u = e^x\cos y$ $v = -e^x\sin y$

$u_x = e^x\cos y$ $v_x = -e^x\sin y$

$u_y = -e^x\sin y$ $v_y = -e^x\cos y$

C-R eqn. $u_x = v_y$ $u_y = -v_x$ not satisfy

Given fn. is not analytic.

ⓒ $L(\sin^5 t) = L\left[\frac{e^{it} - e^{-it}}{2i}\right]^5 = L\,\frac{1}{16}\left[\sin 5t - 5\sin 3t + 10\sin t\right]$

$= \frac{1}{16}\left[\frac{5}{s^2+5^2} - 5\frac{3}{s^2+3^2} + 10\frac{1}{s^2+1}\right]$

(d) $\nabla\phi = 4(x^3 i + y^3 j + z^3 k) = 4(i - 8j + k)$ at $(1,-2,1)$

$\overline{AB} = \overline{OB} - \overline{OA} = (2-1)i + (6+2)j + (-1-1)k$

$= i + 8j - 2k$

Directional Derivative at A in the direction of $\overline{AB} = -\dfrac{260}{\sqrt{69}}$

Q.2 (a) $\vec{F} = (x^2 - yz)i + (y^2 - zx)j + (x^2 - zy)k$

For irrotational vector $\quad curl\ \bar{F} = 0$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$$

Scalar Potential ; $\bar{F} = \nabla\phi$

$\therefore \phi = \dfrac{x^3}{3} + \dfrac{y^3}{3} + \dfrac{z^3}{3} - xyz$

(b) $J_{1/2}(x) = \sum_{m=0}^{\infty} \dfrac{(-1)^m (x/2)^{2m+1/2}}{m!\, \Gamma(m+3/2)}$

$= \sqrt{\dfrac{x}{2}} \left[ \dfrac{1}{1/2\ \sqrt{\pi}} - \dfrac{1}{(3/2)^2} \dfrac{1}{\sqrt{\pi}} \dfrac{x^2}{2^2} + \cdots \right]$

$= \sqrt{\dfrac{2}{\pi x}} \left[ x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots \right]$

$= \sqrt{\dfrac{2}{\pi x}} \cdot \sin x$

(c) $f(x) = 1 - x^2 \qquad (-1,1)$ $f(x)$ is even fn.

$\because f(-x) = f(x)$

so $b_n = 0$

$a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \frac{2}{3}$ $\quad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$

$a_n = 2\int_0^1 (1-x^2)\cos n\pi x\,dx = -\frac{4(-1)^n}{n^2\pi^2}$

So. $f(x) = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x.$

Q.3 (a) $f(z) = (x^2+2axy+by^2) + i(cx^2+2dxy+y^2)$

$u = x^2+2axy+by^2 \qquad v = cx^2+2dxy+y^2$

$u_x = 2x+2ay \qquad u_y = 2ax+2by$

$v_x = 2cx+2dy \qquad v_y = 2dx+2y$

from C-R eqn. $\quad u_x = v_y \qquad u_y = -v_x$

$a = 1 \quad b = -1 \quad c = -1 \quad d = 1$

(b) $\int_{-1}^1 f_1 f_2\,dx = \int_{-1}^1 1\times x\,dx = 0$ $\qquad f_1 = 1,\ f_2 = x,\ f_3 = -1+ax+bx^2$

$\int_{-1}^1 f_1 f_1\,dx = 2 \neq 0$ & $\int_{-1}^1 f_2 f_2\,dx = \frac{2}{3} \neq 0$

$\therefore$ $f_1$ & $f_2$ are orthogonal

Now. if $f_3$ is orthogonal to both $f_1$ & $f_2$ then

$\int_{-1}^1 f_1 f_3\,dx = 0 \qquad -2+\frac{2b}{3} = 0 \qquad \boxed{b=3}$

$\int_{-1}^1 f_2 f_3\,dx = 0 \qquad \boxed{a=0}$

$\boxed{f_3(x) = 3x^2-1}$

$\therefore \int_{-1}^1 f_3\cdot f_3\,dx = 9/5 \neq 0$

(c) $L^{-1}\left[\frac{s^2}{(s^2+1^2)(s^2+2^2)}\right]$ $\quad \phi_1(s) = \frac{s}{s^2+1^2}$ ; $\phi_2(s) = \frac{s}{s^2+2^2}$

$L^{-1}[\phi_1(s)] = \cos t$ $\qquad L^{-1}\phi_2(s) = \cos 2t$

Convolution th'm.

$$L^{-1}\phi(s) = \int_0^t \cos u \cos 2(t-u)\, du$$

$$= \frac{\sin t - 2\sin 2t}{1^2 - 2^2}$$

$$= -\frac{1}{3}(\sin t - 2\sin 2t)$$

Q.4 (a) Green's. th'm $\int_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$\int_C \bar{F}\cdot d\bar{r} = \int_C (-x^2 y\, i + xy^2 j)\cdot(dx\, i + dy\, j)$

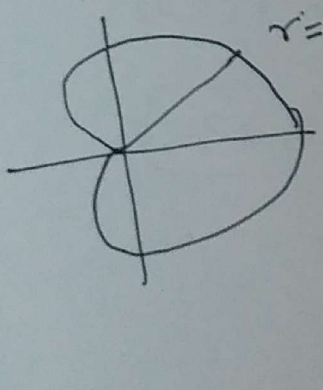
$= \int_C (-x^2 y\, dx + xy^2\, dy)$

By comparison $P = -x^2 y$, $Q = xy^2$

$\frac{\partial Q}{\partial x} = y^2$ $\quad \frac{\partial P}{\partial y} = -x^2$ $\quad \int_C \bar{F}\cdot d\bar{r} = \iint_R (y^2 + x^2)\, dx\,dy$ —(1)

Solve by using double integration

$r = a(1+\cos\theta)$



$$\boxed{I = \frac{35\pi}{16} a^4}$$

4. (b) $v = e^x(x\sin y + y\cos y)$

$v_x = e^x(x\sin y + y\cos y) + e^x \sin y$

$v_{xx} = e^x(x\sin y + y\cos y) + e^x \sin y + e^x \sin y$

$v_y = e^x(x\cos y + \cos y - y\sin y)$

$v_{yy} = e^x(-x\sin y - \sin y - \sin y - y\cos y)$

$v_{xx} + v_{yy} = 0$ $\therefore$ v satisfying Laplace eqn.

$v_x = e^x(x\sin y + y\cos y + \sin y)$ $\quad \psi_2(z,0) = 0$

$v_y = e^x(x\cos y + \cos y - y\sin y)$ $\quad \psi_1(z,0) = e^z(z+1)$

$f'(z) = \psi_1 + i\psi_2 = e^z(z+1)$

$f(z) = \int e^z(z+1)\,dz$

$\boxed{f(z) = ze^z}$ → analytic fn.

$u = e^x(x\cos y - y\sin y)$ → harmonic Conjugate

(c) $\int J_3(x) = -J_2(x) - \frac{2}{x}J_1(x)$

$J_n(x) = \frac{x}{2n}\left[J_{n+1}(x) + J_{n-1}(x)\right]$

$n=1$ $\quad J_1(x) = \frac{x}{2}\left[J_2(x) + J_0(x)\right]$

$J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$

$\Rightarrow \int J_3(x)\,dx = J_0(x) - \frac{4}{x}J_1(x)$

Q.5 (a) $f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}$ $\quad f(x) = e^{ax}$ $\quad (-\pi, \pi)$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}\cdot e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x}\, dx = \frac{1}{2\pi}\left[\frac{e^{(a-in)x}}{a-in}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi}e^{-in\pi} - e^{-a\pi}e^{in\pi}\right]$$

$$e^{\pm in\pi} = (-1)^n$$

$$\therefore\ C_n = \frac{(-1)^n}{\pi(a-in)}\left[\frac{e^{a\pi}-e^{-a\pi}}{2}\right] = \frac{(-1)^n \sinh a\pi\,(a+in)}{\pi(a^2+n^2)}$$

$$f(x) = e^{ax} = \sum_{-\infty}^{\infty} (-1)^n \frac{\sinh a\pi\,(a+in)}{\pi(a^2+n^2)}\, e^{inx}$$

(b) $3\frac{dy}{dt} + 2y = e^{3t}$ $\quad y(0) = 1$

$$3L(y') + 2L(y) = Le^{3t}$$

$$3[s\bar{y} - y(0)] + 2\bar{y} = \frac{1}{s-3} \qquad y(0) = 1$$

$$3[s\bar{y} - 1] + 2\bar{y} = \frac{1}{s-3}$$

$$(3s+2)\bar{y} = \frac{1}{s-3} + 3 = \frac{3s-8}{s-3}$$

$$\bar{y} = \frac{3s-8}{(s-3)(3s+2)}$$

$$\bar{y} = \frac{30}{11}\,\frac{1}{3s+2} + \frac{1}{11}\,\frac{1}{s-3}$$

$$\boxed{y = \frac{10}{11}e^{-2/3\,t} + \frac{1}{11}e^{3t}}$$

(c) $f(z) = u + iv \quad i f(z) = iu - v \quad (1+i) f(z) = (u-v) + i(u+v)$
$= U + iV$

$U = u - v = (x-y)(x^2+4xy+y^2) = x^3 + 3x^2y - 3xy^2 - y^3$

$\frac{\partial U}{\partial x} = 3x^2 + 6xy - 3y^2 = \phi_1(x,y)$

$\frac{\partial U}{\partial y} = 3x^2 - 6xy - 3y^2 = \phi_2(x,y)$

$\phi_1(z,0) = 3z^2 \quad \phi_2(z,0) = 3z^2$

$(1+i) f'(z) = \phi_1 - i\phi_2 = 3z^2 - i3z^2$

$(1+i) f(z) = -3i \int z^2 dz - 3i \int z^2 dz$

$$\boxed{f(z) = -iz^3 + c}$$

Q.6 (a) G.S. $y = \sqrt{x}\left[A J_{1/4}(x^2) + B J_{-1/4}(x^2)\right]$

(b) $w = \frac{az+b}{cz+d} = \frac{3z+2i}{zi+6}$

(c) Half range sine Series

$f(x) = \sum bn \sin \frac{n\pi x}{l}$

$bn = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$

$= \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx\right]$

$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi-x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]^{\pi}$

$= \frac{4}{\pi} \frac{\sin(n\pi/2)}{n^2} \qquad b_1 = \frac{4}{\pi}\frac{1}{1^2} \quad b_2 = 0 \quad b_3 = \frac{4}{\pi}\frac{1}{3^2}$

$\therefore f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \cdots\right]$