QP Code : 79052

(3 Hours) [Total Marks: 100

N.B.: (1) All questions are compulsory.
(2) Attempt any two subquestions from a), b) c) in each question
(3) Figures to the right indicate marks for respective subquestions.

Q1. (a) (i) If W is a subspace of a vector space V then show that for any $x, y \in V$, the cosets $W+x$, $W+y$ are either identical or disjoint. (5)

(ii) Let $A_{n\times n}$ be a real matrix. Let $\lambda \in \mathbb{R}$ is an eigen value of A and X is corresponding eigen vector. Show that for any $k \in \mathbb{N}$, λ^k is an eigen value of A^k. (5)

(b) (i) Let G be a group, $a \in G$ and $N(a) = \{x \in G \ : \ ax = xa\}$. Show that $N(a)$ is a subgroup of G. (5)

(ii) Let G, G' be groups and $f : G \to G'$ be an isomorphism of groups. Show that, (5)

(p) $f(a^k) = (f(a))^k$ for each $a \in G$ and for each $k \in \mathbb{Z}$.

(q) $o(a) = o(f(a))$ for each $a \in G$.

(c) (i) Define an ideal. If I, J are ideals of a ring R, then prove that $I \cap J$ is also an ideal of R. (5)

(ii) Prove that finite integral domain is a field. (5)

Q2. (a) State and prove the Cayley Hamilton theorem. (10)

(b) (i) Let V be a finite dimensional inner product space and $f : V \to V$ is map such that i) $f(0) = 0$ and ii) $\|f(x) - f(y)\| = \|x - y\|$ for any $x, y \in V$ then prove that f is an orthogonal linear transformation. (6)

(ii) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation given by $T(x, y, z) = (x - y, x - z, 2x - y - z)$. Find Ker T. Write basis of Ker$T$ and a basis of $\mathbb{R}^3/KerT$. (4)

(c) (i) Show that every quadratic form $Q(X)$ can be reduced to a standard form $\sum_{i=1}^{n} \lambda_i y_i^2$ by an orthogonal change of variable $X = PY$ for $X^t = (x_1, x_2, ..., x_n)^t$ and $Y = (y_1, y_2, ..., y_n)^t \in \mathbb{R}^n$. (6)

(i) Let $A_{n\times n}$ real matrix with n distinct eigen values. Show that A is diagonalizable. (4)

[P.T.O.]

Q3. (a) State and prove Lagrange's Theorem for groups. (10)

(b) (i) Show that every subgroup of a cyclic group is cyclic. (6)

(ii) If order of a in the group G is n then prove that order of $a^m = \dfrac{n}{gcd(n,m)}$. (4)

(c) (i) Let G be an infinite cyclic group generated by a. Show that a and a^{-1} are the only generators of G. (6)

(ii) Let $G\{\overline{5}, \overline{15}, \overline{25}, \overline{35}\}$ under multiplication of residue classes mod 40. Form composition table of G. (4)

Q4. (a) State and prove the first isomorphism theorem for groups. (10)

(b) (i) Let G_1 and G_2 be abelian groups. Show that their product $G_1 \times G_2$ is also abelian. (6)

(ii) Show that there are only two non-isomorphic groups of order 4. (4)

(c) (i) Prove that every subgroup of a group of index 2 is a normal subgroup and hence or otherwise prove that A_n is a normal subgroup of S_n. (6)

(ii) Show that groups $(\mathbb{Q}, +)$ and $(\mathbb{Q}^*, \cdot)$ are not isomorphic. (4)

Q5. (a) Define maximal ideal of a ring. Show that an ideal M of a commutative ring R is a maximal ideal if and only if R/M is field. (10)

(b) (i) Define characteristic of a ring. Show that characteristic of an integral domain is either 0 or a prime integer. (6)

(ii) Find the units of $\mathbb{Z}[i]$. (4)

(c) (i) Show that every Euclidean domain is a Principal ideal domain. (6)

(ii) Define irreducible elements. Show that $1 + \sqrt{-5}$ is irreducible in $\mathbb{Z}[\sqrt{-5}]$. (4).

Con. 1361-17.