(3 Hours) **[Total Marks: 100]**

N.B.: (1) All questions are compulsory.
(2) Attempt any two subquestions from part(a), part(b) and part(c).
(3) Figures to the right indicate marks for respective subquestions.
(4)Use of non programmable calculators allowed

1 (a) (i) Prove that the variance σ_v^2 of a portfolio cannot be greater than the variance σ_1^2 and σ_2^2 of the components, if short sales are not allowed. (5)

(ii) Let $\Omega = \{2,3\}$ and $F = \{\phi, \Omega, \{2\}, \{3\}\}$. Check if $X(\omega) = 2$ is a random variable w.r.t. the given σ field. (5)

(b) (i) Calculate the price of a 9 month European put option on a non-dividend paying stock with a strike price of 30, the risk free interest rate is 6% p.a. and the volatility is 8% p.a. (5)

(ii) State and prove Total Probability theorem. (5)

(c) (i) Define a field, an atom in the field. Show that different atoms in a field must be disjoint. (5)

(ii) Define finitely additive probability measure, countably additive probability measure sub additive property of a probability measure. (5)

2 (a) Let $\tilde{F}$ be a σ- field that satisfies the condition (10)

$A \subset \tilde{F}$

If F is a σ-field and $A \subset F$ then $\tilde{F} \subset F$. Show that $\mathcal{F}_A = \tilde{F}$

(b) (i) Find $\lim_{n\to\infty} sup A_n$ and $\lim_{n\to\infty} \inf A_n$ (6)
where $A_{2n} = \{2, 4, 6, \ldots\}$; $A_{2n-1} = \{1, 3, 7, \ldots\}, n \in \mathbb{N}$

(ii) Show that $f(x) = \begin{cases} x - [x] \; for \; x \in [0, 2] \\ \quad otherwise \end{cases}$ is a density function. Let P be an absolutely continuous probability measure with density given by the above function f. Find $P(A)$ where $A = [2,4]$. (4)

(c) (i) Let $\mathcal{A}$ be the family of finite subsets of Ω. Determine the σ- field generated by $\mathcal{A}$. (6)

(ii) If $A_1 \subset A_2 \subset A_3 \subset \cdots$, then show that $P(\cup_{k=1}^{\infty} A_k) = \lim_{k\to\infty} P(A_k)$. (4)

3 (a) Define a random variable.Let X be a random variable. Show that the map $P_X(B)=P(\{X \in B\})$ is a probability measure on the σ-field B of Borel subsets of $\mathbb{R}$. Further, if X and Y and random variables defined on the same probability space, Show that the distribution P_Y of Y can be obtained from the joint distribution as $P_Y(B) = P_{X,Y}(\mathbb{R} \times B)$ for any Borel set B in $\mathbb{R}$. (10)

(b) (i) State and prove Jensen Inequality. (6)

(ii) Define distribution function, F_X, of a random variable X. Show that F_X is non-decreasing. (4)

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(c) (i) Define conditional expectation of X given B where X is random variable and B is any event. Further define conditional expectation of X given Y where X is an arbitrary random variable and let Y is a discrete random variable. (6)

(ii) Find the expectation of the random variable with Binomial distribution. (4)

4 (a) State and prove weak law of large numbers. (10)

(b) (i) What is meant by the following assumptions used in the mathematical model of a financial security- Liquidity, Short selling, No Arbitrage Principle. (6)

(ii) Explain risk-neutral valuation approach to value a European call option using a one step binomial tree. (4)

(c) (i) Exhibit an arbitrage opportunity if the forward price $F(0,T)$ of a stock paying no dividends is greater than $S(0)^{e^{rT}}$. Assume continuous compounding with risk free interest rate r. (6)

(ii) What is meant by a long forward position, an European call option. (4)

5 (a) Prove that a binomial tree model admits no arbitrage if and only if $\mathrm{d} < \mathrm{r} < \mathrm{u}$. Assume 1-step tree model. (10)

(b) (i) Show that price of an American call option $(C_A) =$ price of a European call option (C_E), whenever the strike price X and expiry time T are the same for both options. (6)

(ii) State Black-Scholes-Merton formula for an European call option price. List 4 factors that affect the stock option price (4)

(c) (i) An investor buys a European put on a share for Rs.10. The stock price is 55 and the strike price is 50. Under what circumstances does the investor make a profit? Under what circumstances will the option be exercised? (6)

(ii) Show that price of an European put option (P_E) on a stock paying no dividends satisfies the following inequality: (4)

$\max\{0, -S(0) + Xe^{-rT}\} \leq P_E \leq Xe^{-rT}$
