(REVISED COURSE)

(3 Hours) **[Total Marks: 100]**

N.B. 1) All questions are **compulsory**.
2) Attempt any **two** subquestios from **part(a), part(b) and part(c)**.
3) Figures to the right indicate marks for respective subquestions.

1. (a) Show that a nontrivial graph is bipartite if and only if it contains no odd cycle. Hence prove that the vertex chromatic number of a graph with at least one edge and no odd cycle is 2 (10)

(b) i. Prove that if B is a board of darkened squares that decomposes into the two disjoint sub boards B_1 and B_2, then prove that $R(x, B) = R(x, B_1)R(x, B_2)$ where $R(x, B)$ is a rook polynomial for board B. (5)

ii. Show that every planar graph is 6-vertex colorable. (5)

(c) i. Show that a connected graph on n vertices is tree if and only if the chromatic polynomial of G is $k(k-1)^{n-1}$ (5)

ii. State and prove Euler's formula for planar graph (5)

2. (a) Let G be a graph with p vertices and $p-1$ edges. Prove that following statements are equivalent. (10)

i. G is connected.
ii. G is acyclic.
iii. G is a tree.

(b) i. If G is a connected graph with at least 3 vertices and G contains a cut edge then prove that it contains a cut vertex. (6)

ii. State $Havel - Hakimi$ theorem for degree sequence of a graph. Using the theorem check whether the sequence $5, 4, 3, 3, 2, 2, 2, 1$ is graphical and if so, draw the graph with this degree sequence. (4)

(c) i. Let A denote the adjacency matrix of a connected graph G with $V(G) = \{v_1, v_2, \ldots v_n\}$, then show that the distance between v_i and v_j is the smallest integer $n \geq 0$ such that $(A^n)_{ij} \neq 0$. (6)

ii. If G is self complementary graph of order p, show that $p \equiv 0$ or $1 (mod 4)$ (4)

3. (a) If G is a graph on p vertices with $p \geq 3$ such that $deg(u) + deg(v) \geq p$ for every pair of non adjacent vertices u and v in G, then prove that G is Hamiltonian. (10)

(b) i. Prove that the cube graph Q_k is connected bipartite $k-$regular graph with 2^k vertices. (6)

ii. Show that complete bipartite graph $K_{n,n}$ is Hamiltonian. (4)

(c) i. Let G be a simple graph with p vertices and q edges with $p \geq 3$. If $q \geq \frac{p^2-3p+6}{2}$ then prove that G is Hamiltonian. (6)

ii. Define closure of a graph $C(G)$ and show that if the closure of graph G is complete then G is Hamiltonian. (4)

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4. (a) Show that every planar graph is 5-vertex colorable. (10)
   (b) i. For a simple graph G of order p and size q, prove that $\pi_k(G)$, the chromatic polynomial of the graph G, is monic polynomial of degree p in k with integer coefficients and constant term zero. (6)
   ii. Let G be a graph, prove that $\chi'(G) \geq \Delta(G)$ where $\chi'(G)$ represents edge chromatic number of a graph G and $\Delta(G)$ denotes the maximum degree of G. Give an example of graph for which $\chi'(G) > \Delta(G)$ (4)
   (c) i. Define $val(f)$, value of f and $cap(K)$, capacity of cut K. If f is any flow and K be any cut in a network N with $val(f) = cap(K)$ then show that f is maximum flow and K is minimum cut. (6)
   ii. If G is simple planar graph on $p \geq 11$ vertices, then show that complement $\overline{G}$ is planar. (4)

5. (a) Find the recurrence relation for a_n, the number of ways to place parantheses to multiply the n numbers k_1 x k_2 x ... x k_n on a calculator and prove using generating function that the solution to this recurrence relation is a Catalan Number. (10)
   (b) i. Show that the number of different system of distinct representatives for the family $A_i = \{1, 2, \ldots, n\} - \{i\}$, $1 \leq i \leq n$ is D_n, the number of derangements on n symbols. (6)
   ii. If h_n denote the number of nonnegative integral solutions of the equation $2e_1 + 5e_2 + 10e_3 + 3e_4 + 4e_5 = n$, find the generating function of h_n. (4)
   (c) i. Let B denotes a forbidden chess board in which a special square $*$ has been identified. Let D denote the board obtained from the original board by deleting the row and column containing the special square and E denote the board obtained from the original board where only the special square $*$ is removed from the board, then prove that $R(x, B) = xR(x, D) + R(x, E)$. (6)
   ii. Find the number h_n of bags of fruits that can be made out of apples, bananas, oranges and pears, where, in each bag, the number of apples is even, the number of bananas is multiple of 5, the number of oranges is at most 4, and the number of pears is 0 or 1. (4)

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