Revised **(3 Hours)** **[Total Marks: 80**

N.B.: (1) Attempt any **TWO** questions from each Section.
(2) Figures to the right indicate marks for respective subquestions.
(3) Answers to section I and section II should be written in the same answer book

SECTION - I

1. (a) Let A be a rectangle in $\mathbb{R}^n$. Prove that a bounded function $f : A \to \mathbb{R}$ is integrable if and only if for every $\varepsilon > 0$ there is a partition P of A such that $U(f,P) - L(f,p) < \varepsilon$. (6)

(b) Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

Show that f is not integrable. (4)

(c) Show that a subset A of $\mathbb{R}^n$ has measure zero if and only if for a given $\varepsilon > 0$ there is a countable collection of open rectangles $V_1, V_2, \cdots$ such that $A \subseteq \bigcup_i V_i$ and $\sum_i v(V_i) < \varepsilon$. (6)

(d) Change the order of integration and then evaluate (4)

$$\int_0^1 \int_0^y (x^2+y^2)dx\,dy \;+\; \int_1^2 \int_0^{2-y} (x^2+y^2)dx\,dy.$$

2. (a) If $\{E_j\}_{j\in J}$ is a countable collection of disjoint measurable sets then prove that $\bigcup_{j\in J} E_j$ is measurable and $m\left(\bigcup_{j\in J} E_j\right) = \sum_{j\in J} m(E_j)$. (10)

(b) Show that there is a non-measurable subset in $\mathbb{R}$. (10)

3. (a) Show that the function χ_A is measurable if and only if the set A is measurable. (5)

(b) Define the integral of simple function ϕ that has a canonical representation. Let ϕ and ψ be simple functions defined on a set of finite measure E. Then prove that $\int_E (\alpha\phi + \beta\psi) = \alpha \int_E \phi + \beta \int_E \psi$ for any α and β. (5)

(c) Let f be a bounded measurable function on a set E of finite measure. If $\int_E f = 0$ then prove that $f = 0$ a.e. (5)

(d) Show by an example that Lebesgue integrable function may not be Riemann integrable. (5)

4. (a) State and prove Fatau's lemma. Show by an example that the inequality in Fatau's lemma may be a strict inequality. (10)

(b) Let f_n be a sequence of measurable functions on E. Suppose there is a function g that is integrable over E and dominates f_n on E in the sense that $|f_n| \leqslant g$ on E for all n. If $f_n \to f$ pointwise a.e. on E then prove that f is integrable over E and $\lim_{n\to\infty} \int_E f_n = \int_E f$. Also show by an example that the condition g is integrable cannot be dropped. (10)

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SECTION - II

5. (a) Define Fejer Kernel $F_N(x)$. Show that the N-th Fejer kernel $F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}$. (4)

   (b) Let f be an integrable function on the circle which is differentiable at a point x_0. Show $S_N(f)(x_0) \to f(x_0)$ as $N \to \infty$, where $S_N(f)(x) = \sum_{k=-N}^{N} \widehat{f}(n)e^{inx}$. (6)

   (c) Find the Fourier coefficient and hence find the Fourier series of the function $f(x) = \pi - x$, where $-\pi \leq x \leq \pi$. (6)

   (d) Define Dirichlet's kernel $D_N(x)$. Show that $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)dx = 1$. (4)

6. (a) Let $\{e_k\}_{k=1}^{\infty}$ is an orthonormal set in a Hilbert space H. Show that the following statements are equivalent: (12)

   (1) Finite linear combinations of elements in $\{e_k\}$ are dense in H.

   (2) If $x \in H$ and $\langle x, e_i\rangle = 0$ for all i, then $x = 0$.

   (3) If $x \in H$ and $S_N(x) = \sum_{k=1}^{N} a_k e_k$, where $a_k = \langle x, e_k\rangle$, then $S_N(x) \to x$ as $N \to \infty$ in the norm.

   (4) If $a_k = \langle x, e_k\rangle$, then $\|x\|^2 = \sum_{k=1}^{\infty} |a_k|^2$.

   (b) If S is a closed subspace of a Hilbert space H, then show that $H = S \oplus S^{\perp}$. (4)

   (c) Let H be a Hilbert space over $\mathbb{C}$. Prove that for any $x, y \in H$, $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$. (4)

7. (a) Let $f \in L^2([-\pi, \pi])$. Then for any collection of complex numbers $\{c_k\}_{k=-N}^{N}$, show that

$$\left\| f - \sum_{k=-N}^{N} \widehat{f}(k)e^{ikx} \right\|_2 \leq \left\| f - \sum_{k=-N}^{N} c_k e^{ikx} \right\|_2 .$$

   Equality holds if and only if $c_k = \widehat{f}(k)$ for $-N \leq k \leq N$. (8)

   (b) Show that $L^2([-\pi, \pi])$ is unitarily isomorphic to $\ell^2(\mathbb{Z})$. (6)

   (c) If $f \in L^2([-\pi, \pi])$, then show that $\sum_{-\infty}^{\infty} |\widehat{f}(n)|^2 \leq \|f\|^2$. (6)

8. (a) Define Poisson kernel $P_r(x)$. Prove that if $0 \leq r < 1$, then $P_r(x) = \frac{1-r^2}{1-2r\cos x + r^2}$. Further show that $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(x)dx = 1$. (6)

   (b) Prove that the Fourier series of an integrable function on the circle is Abel summable to f at every point of continuity of f. (8)

   (c) Find the solution of the Dirichlet's problem $\Delta u = 0$ in the unit disc with boundary condition $u(1, \theta) = \sin^2\theta + \sin\theta$. (6)

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