Q.P.Code: 011474

Time: 3 Hours [Total Marks: 100]

N.B. 1) Attempt any **five** questions out of **eight**.
2) All questions carry equal marks.

Q. 1. (a) State the class equation for a finite group. Prove it by explaining clearly all the notation used. **(10)**

(b) Let G be a group and let p be a prime dividing the order of G. Prove that any two Sylow p-subgroups of G are conjugate to each other. **(10)**

Q. 2. (a) Let G be a group and let H be a normal subgroup of G. Prove that G is solvable if and only if both H and G/H are solvable. **(10)**

(b) (i) Prove that the center of a group G is a normal subgroup of G. Determine the center of the group of quaternions. **(5)**

(ii) Classify (upto isomorphism) all groups of order 6 with correct justification. **(5)**

Q. 3. (a) State and prove the first isomorphism theorem for modules over a commutative ring R with unity. **(10)**

(b) (i) State (without proof) the Hilbert basis theorem. Define the terms: Noetherian ring, Noetherian module. **(5)**

(ii) Prove that any Artinian ring has finitely many maximal ideals. **(5)**

Q. 4. (a) Construct a finite field of order 9 with correct justification. **(10)**

(b) (i) State (without proof) the structure theorem for modules over a principal ideal domain. Give an example of a ring which is not a principal ideal domain. **(5)**

(ii) Define the terms: free module, torsion module. Give one example of each with correct justification. **(5)**

Q. 5. (a) (i) Determine the degree of the field extension $\mathbb{Q}(\sqrt{2}+\sqrt{3})$ over $\mathbb{Q}$ with correct justification. **(5)**

(ii) State (without proof) primitive element theorem. Give an example of an extension of $\mathbb{Q}$ which is not normal with correct justification. **(5)**

(b) State and prove the fundamental theorem of Galois theory. **(10)**

Q. 6. (a) (i) Determine with correct justification whether the cubic equation $X^3 - 1$ is solvable by radicals over $\mathbb{Q}$. **(5)**

(ii) Is the polynomial $X_1^2 + X_2^2$ symmetric in the variables X_1, X_2? If yes, express it in terms of the elementary symmetric polynomials. **(5)**

(b) Prove that an angle θ is constructible by straightedge and compass if and only if $\cos\theta$ is constructible by straight edge and compass. **(10)**

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Q. 7. (a) (i) Prove that the map $a + b\sqrt{3} \mapsto a - b\sqrt{3}$ is an automorphism of $\mathbb{Q}(\sqrt{3})$. Find the fixed field of this automorphism. **(5)**

(ii) Determine the degree of the splitting field of X^2+X+1 over $\mathbb{Q}$ with correct justification. **(5)**

(b) Let R be a commutative ring with unity and let M be an R-module. Prove that M is a Noetherian R-module if and only if every submodule of M is finitely generated. **(10)**

Q. 8. (a) (i) Define the term: normal extension. Let k be a field and let K be a degree two extension of k. Prove that K is a normal extension of k. **(5)**

(ii) Let S_3 act on itself by left-multiplication. Find the orbit of $(1\ 2), (1\ 2\ 3)$ under this action. **(5)**

(b) Let k be a field. Prove that there exists an algebraically closed field containing k as a subfield. **(10)**
