Q.P.Code:11426

Time: 3 Hours [Total Marks: 80]

N.B. 1) Attempt any **two** questions from each section.
2) All questions carry equal marks.
(3) Answers to Section-I and Section-II should be written in the same answer book.

SECTION-I

Q. 1. (a) Define the term: simple group and give one example other than that of A_5. Prove that the group A_5 is simple. **(10)**

(b) Define the term: semi-direct product. Prove that the dihedral group D_4 of order 8 is a semi-direct product of $\mathbb{Z}_2$ and $\mathbb{Z}_4$. **(10)**

Q. 2. (a) State and prove Maschke's theorem. **(10)**

(b) Write down the character table for S_3 with correct justification. Verify the orthogonality relations amongst any two distinct characters. **(10)**

Q. 3. (a) Prove that if M is a submodule over a principal ideal domain R, then every R-submodule of M is free. **(10)**

(b) State and prove the first isomorphism theorem for modules over a commutative ring R with unity. **(10)**

Q. 4. (a) (i) State (without proof) the structure theorem for finitely generated modules over a principal ideal domain. Explain clearly all the notation used. **(5)**

(ii) Let M be a finitely generated R-module, where R is a principal ideal domain. Prove that M is torsion free if and only if M is free. **(5)**

(b) (i) Find the rational canonical form over $\mathbb{Q}$ of the matrix $\begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{pmatrix}$. **(5)**

(i) Find the Jordan canonical form over $\mathbb{Q}$ of the same matrix as above i.e., $\begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{pmatrix}$. **(5)**

SECTION-II

Q. 1. (a) Let E/F and K/E be algebraic field extensions. Prove that degree of K/F is the product of the degrees of E/F and K/E. **(10)**

(b) Define the term: splitting field. Find the degree of the splitting field of $\mathbb{Q}(\sqrt[3]{2})$ over $\mathbb{Q}$ with correct justification. **(10)**

Q. 2. (a) Prove that separable extensions form a distinguished class. **(10)**

(b) Define the term: normal extension. Give an example of a field extension which is not normal with correct justification. Prove that if E/F is an extension of fields of degree 2, prove that E is a normal extension of F. **(10)**

Q. 3. (a) State and prove the fundamental theorem of Galois theory. **(10)**

(b) Prove that the set of all complex numbers is algebraically closed. **(10)**

Q. 4. (a) Determine the solvability of the quintic $X^5 - 4X + 2$ over $\mathbb{Q}$ with correct justification. **(10)**

(b) Define the term: constructible number. Prove that if α, β are constructible so is $\alpha\beta$. **(10)**

—— PAPER ENDS ——