TOTAL MARKS: 60;
TOTAL TIME: $2\frac{1}{2}$ HRS.

N.B. 1) All **questions** are **compulsory**.
2) All questions carry **equal** marks.
3) In the first four questions attempt **any two** subquestions from (a), (b), (c).
4) In the fifth question attempt **any four** subquestions out of **six**.

1. (a) Let E be an algebraic extension of k and let $\sigma : E \to E$ be an embedding of E into itself over k. Prove that σ is an automorphism of E. **(6)**
   (b) (i) Let $E = F(\alpha)$, where $\alpha \in E$ is algebraic over F of degree 2017. Show that $E = F(\alpha^2)$. **(3)**
   (ii) Find the degree over $\mathbb{Q}$ of the splitting field of $X^5 - 7$ with correct justification. **(3)**
   (c) (i) Find the degree of $\sqrt{1+\sqrt{3}}$ over $\mathbb{Q}$ with correct justification. **(3)**
   (ii) Define the term: algebraically closed field. Prove that if k is an algebraically closed field and E is a finite algebraic extension of k, then $E = k$. **(3)**

2. (a) Let F be a field of characteristic zero and let K/F be a finite separable extension. Show that $K = F(\alpha)$ for some $\alpha \in K$. **(6)**
   (b) (i) With correct justification, construct a tower of field extensions $k \subset K \subset F$ with F/k normal but K/k not necessarily normal. **(3)**
   (ii) Define the term: separable closure. Determine the separable closure of $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$. **(3)**
   (c) Prove that all finite, separable extensions form a distinguished class. **(6)**

3. (a) Let H be a finite group of automorphisms of a field K. Then prove that K is a Galois extension of its fixed field K^H. Further show that H is the Galios group of K/K^H. **(6)**
   (b) Determine with correct justification the Galois group of the extension $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$. **(6)**
   (c) Prove that the field of complex numbers is algebraically closed. **(6)**

4. (a) Let F be a field of characteristic zero such that F does not contain any primitive n-th root of unity. Let $a \in F$. If K is the splitting field of $X^n - a$ over F, then prove that the Galois group of K over F is solvable. **(6)**
   (b) (i) Prove that if α, β are positive constructible numbers, then so is $\alpha - \beta$. **(3)**
   (ii) Prove that trisection of the angle $\frac{\pi}{3}$ is impossible. **(3)**
   (c) Let A, B be lines or circles defined by linear or quadratic equations respectively, that have coefficients in a subfield F of the real numbers. Then prove that the points of intersection of A and B have coordinates in F, or in a real quadratic field extension F' of F. **(6)**

——————-TURN OVER——————-

5. (a) Let τ be the map $\tau : \mathbb{Q}(\sqrt{3}) \to \mathbb{Q}(\sqrt{3})$ defined by $\tau(a + b\sqrt{3}) = a - b\sqrt{3}$. Prove that τ is an automorphism of $\mathbb{Q}(\sqrt{3})$. Find the fixed field of τ. **(3)**

(b) Let K be an extension of a field F of finite degree n. Let $\alpha \in K$. Prove that α is algebraic over F and its degree divides n. **(3)**

(c) Prove that every constructible real number is algebraic. State clearly all results used. **(3)**

(d) Define the terms: normal extension, Galois extension. Is the extension $\mathbb{Q}(\sqrt[4]{2})$ a Galois extension of $\mathbb{Q}$? Justify. **(3)**

(e) Determine the degree of the splitting field of the polynomial $X^4 - 1$ over $\mathbb{Q}$ with correct justification. **(3)**

(f) Let $\mathbb{F}_p$ denote the finite field of characteristic p, where p is a prime. Prove that for any polynomial $f(X) \in \mathbb{F}_p[X]$, $f(X)^p = f(X^p)$. **(3)**

————PAPER-ENDS————————-