256

P. G .D. O. R. M.
SEM - II

OPTIMISATION MODELS - II

Programme Co-ordinator : **Ms. Madhura Kulkarni**
Asst. Prof..-Cum. Asst. Director
University of Mumbai IDOL

Course Co-ordinator : **Ms. Megha Bansal**
IDOL, University of Mumbai

Course Writer : **Dr. C.V. Joshi (D.Literature)**
Visiting Faculty
Alkesh Dinesh Mody
University of Mumbai, Mumbai

April 2017, P.G.D.O.R.M. SEM - II, Optimisation Models - II

Published by : Incharge Director
Institute of Distance and Open Learning ,
University of Mumbai,
Vidyanagari, Mumbai - 400 098.

DTP Composed : Ashwini Arts
Gurukripa Chawl, M.C. Chagla Marg, Bamanwada,
Vile Parle (E), Mumbai - 400 099.

Printed by : ACME PACKS AND PRINTS (INDIA) PRIVATE LIMITED
A Wing, Gala No. 28, Ground Floor, Virwani Industrial Estate,
Vishweshwar Nagar Road, Goregaon (East), Mumbai 400 063.
Tel. : 91 - 22 - 4099 7676

CONTENTS

❖❖❖❖

P.G.D.O.R.M.

Semester - II

Optimization Models – II

UNIT – I

Assignment Techniques

Definition of Assignment Model, Mathematical Representation of Assignment Model, Hungarian Method of Solution of Assignment Model, Variation of the Assignment Model

UNIT – II

Transportation Techniques - I

Introduction to Transportation Model, Definition of Transportation Model, Matrix Terminology, Formulation and Solution of Transportation Model, Variants in Transportation Model

UNIT – III

Transportation Techniques - II

Variation in Transportation Problem, Trans Shipment Model, Time Minimization Problems

UNIT – IV

Network Analysis

Concept of Project Planning, Scheduling and Controlling, Work Break Down Structure, Basic Tools and Techniques of Project Management, Role of Network Technique in Project Management, Concept of Network or Arrow Diagram, Activity on Node Diagram, Critical Path Method, Concept of PERT, Concept of CPM, Cost Analysis and Crashing the Network

UNIT – V

Game Theory

Introduction to Theory of Games, Characteristics of Games, Game Models, Rules for Game Theory, Concept of Pure Game, Mixed Strategies – 2x2 Games, Mixed Strategies – 2xN or Mx2, Mixed Strategies – MxN Games

UNIT – VI

Markov Chains

Introduction to Markov Chains, Brand Switching Examples, Markov Process, Markov Analysis – Input and Output

REFERENCE / TEXT BOOKS

1. S.D. Sharma, Operations Research, Kedarnath Ramnath & Co. Meerut
2. P.K.Gupta & D.S.Hira, Operations Research, S.Chand & Co.
3. Kanti Swaroop, Operations Research, S.Chand & Sons.
4. Gillet, B.E. Introduction to operations Research - A Computer Algorithm Approach, McGraw Hill.
5. Introduction to operations Research, 7/e by Hillier. TMH.
6. Wagner, Operations Research Techniques
7. Taha H, Operations Research Techniques N
8. N.D. Vohra, Quantitative Techniques in Management, McGraw Hill
9. J.K. Sharma, Operations Research, MacMillan

❖❖❖❖

MODULE - I

ASSIGNMENT TECHNIQUES

1.1

ASSIGNMENT MODEL

Definition of Assignment Model, Mathematical Representation of Assignment Model, Examples

1.1.1 INTRODUCTION

In this Unit-I - Chapter 1.1, we shall discuss the concept of assignment model and formulation of the assignment model.

1.1.2 OBJECTIVES

At the end of this unit the learners will be able to

- Understand the definition of assignment model also called as the assignment problem
- Understand the mathematical representation of assignment model
- Understand the situations where the assignment model becomes important.

1.1.3 – DEFINITION OF ASSIGNMENT MODEL OR ASSIGNMENT PROBLEM (AP)

Assignment problem is a special type of linear programming problem which deals with the allocation of the various resources to the various activities on one to one basis. It does it in such a way that the cost or time involved in the process is minimum and profit or sale is maximum.

In a factory, a supervisor may have six workers available and six jobs to fire. He will have to take decision regarding which job should be given to which worker. Problem forms one to one basis. This is an assignment model also known as an assignment problem hereafter.

1.1.4 - Mathematical Representation of AP

As we know, the AP is a special type of Linear Programming Problem (LPP) where assignees are being assigned to perform task. For example, the assignees might be employees who need to be given work assignments. However, the assignees might not be people. They could be machines or vehicles or plants or even time slots to be assigned tasks. To fit the definition of an assignment problem, the problem need to formulate in a way that satisfies the following assumptions:

- The number of assignees and the number of tasks are the same
- Each assignees is to be assigned to exactly one task
- Each task is to performed by exactly one assignee
- There is a cost Cij associated with assignee i performing task j.
- The objective is to determine how all n assignments should be made to minimize the total cost.
- Any problem satisfying all these assumptions can be solved extremely efficiently by algorithm designed specifically for assignment problem.

Any problem satisfying all these assumptions can be solved extremely efficiently by algorithm designed specifically for assignment problem.

Mathematical form

The mathematical model for the assignment problem uses the following decision variables

Xij = 1 if assignee(worker) i perform task j

=0 if not.

Thus each xij is a binary variable (it has value 0 or 1). Lets Z denots the total cost, the assignment problem model is

$$\text{Min } Z = \sum_i \sum_j c_{ij} \, x_{ij}$$

Subject to

$$\sum_i x_{ij} = 1 \text{ for } i = 1 \ldots n$$

$$\sum_j x_{ij} = 1 \text{ for } j = 1 \ldots n$$

Xij ≥0 for all i and j.

The first set of functional constraints specifies that each assignee is to perform exactly one task, whereas the second set requires each task to be performed by exactly one assignee.

1.1.5 EXAMPLES OF AP

In many business situations, management needs to assign - personnel to jobs, - jobs to machines, - machines to job locations, or - salespersons to territories.

Consider the situation of assigning *n* jobs to *n* machines.

When a job i (=1,2,....,n) is assigned to machine j (=1,2, ...n) that incurs a cost Cij.

The objective is to assign the jobs to machines at the least possible total cost. This situation is as the *assignment problem*.

The job and the machine assignment cost per job is given in the following matrix called as the assignment cost matrix.

	Machine					
		1	2	…….	n	Source
Job	1	C11	C12	…….	C1n	1
	2	C21	C22	…….	C2n	1
	.	.	.		.	.
	.	.	.		.	.
	.	.	.		.	.
	n	Cn1	Cn2	…….	Cnn	1
Destination		1	1	……	1	

The assignment model can be expressed mathematically as follows:

Xij= 0, if the job j is not assigned to machine i

Xij= 1, if the job j is assigned to machine i.

$$\text{Min} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}$$

(Sum of assignments from a source should be exactly equal to 1):

$$\sum_{j=1}^{n} X_{ij} = 1 \qquad \text{For } i=1,2,\ldots,n$$

(Sum of assignments to a destination should be equal to the demanded quantity by that destination):

$$\sum_{i=1}^{n} X_{ij} = 1 \qquad \text{For } j=1,2,\ldots,n$$

(Quantities to be assigned can be either 0 or 1):

$$X_{ij} = 0 \text{ or } 1 \qquad \text{For all } i \text{ and } j.$$

Other examples of management assignment situations are as under:

- Ballston Electronics manufactures small electrical devices.
- Products are manufactured on five different assembly lines (1,2,3,4,5).
- When manufacturing is finished, products are transported from the assembly lines to one of the five different inspection areas (A,B,C,D,E).
- Transporting products from five assembly lines to five inspection areas requires different times (in minutes)

1.1.6 LET US SUM UP

In this UNIT 1 - Chapter 1.1, you have learnt the introduction and the definition of AP, mathematical representation of AP, assignment cost matrix formulation for an AP and examples of management assignment situations.

1.1.7. EXERCISES

1. Define an assignment problem and develop the assignment cost matrix for a hypothetical situation of your choice.
2. Give examples two examples of situations that resemble an assignment problem.
3. What do you understand by assignment cost matrix? Give an example of 2X2 assignment cost matrix.

1.1.8 SUGGESTED READINGS

Assignment problem section in any of the reference / text books

❖❖❖❖

1.2

HUNGARIAN METHOD

Hungarian Method of Solution of Assignment Model, Examples

Unit Structure

1.2.1 INTRODUCTION

In this Unit-I - Chapter 1.2, we shall discuss the Hungarian Method used to solve the assignment problem.

1.2.2 OBJECTIVES

At the end of this unit the learners will be able to

a. Understand the Hungarian Method for solving the assignment problem

1.2.3 HUNGARIAN METHOD

The Hungarian algorithm consists of the four steps. The first two steps are executed once, while Steps 3 and 4 are repeated until an optimal assignment is found. The input of the algorithm is an n by n square matrix with only nonnegative elements.

1. Subtract the smallest number in each row from every number in the row. This is called row reduction
2. Subtract the smallest number in each column of the new table from every number in the column. This is called column reduction.

3. Test whether an optimal assignment can be made. You do this by determining the minimum number of lines to cover all zeros. If the number of lines equals the number of rows, an optimal set of assignment is possible. Otherwise go on to step 4
4. If the number of lines is less than the number of rows, modify the table in the following way

   (a) Subtract the smallest uncovered number from every uncovered number in the table

   (b) Add the smallest uncovered number to the numbers at intersections of covering lines

   (c) Numbers crossed out but at the interactions of cross out lines carry over unchanged to the next table
5. Repeat step 3 and 4 until an optimal set of assignments is possible.
6. Make the assignments one at a time in positions that have zero elements. Begin with rows or columns that have only one zero. Since each row and each column needs to receive exactly one assignment, cross out both the row and the column involved after each assignment is made. Then move on to the rows and such row or column that are not yet crossed out to select the next assignment, with preference again given to any such row or column that has only one zero that is not crossed out. Continue until every row or column has exactly one assignment and so has been crossed out.

1.2.4 SOLVING AP USING HUNGARIAN METHOD

Hungarian Method – Example 1

We consider an example where four jobs (1, 2, and 3) need to be executed by four three machines (1, 2, and 3), one job per one machine. The matrix below shows the cost of assigning a job to a machine. The objective is to minimize the total cost of the assignment.

Step 1: Select the smallest value in each row.
Subtract this value from each value in that row

Step 2: Do the same for the columns that do not have any zero value

		Machine			
		1	2	3	
	1	5	7	9	1
Job	2	14	10	12	1
	3	15	13	16	1
		1	1	1	

		Machine		
		1	2	3
	1	**5**	7	9
Job	2	14	**10**	12
	3	15	**13**	16

		Machine		
		1	2	3
	1	0	2	4
Job	2	4	0	**2**
	3	2	0	3

	Machine		
	1	2	3
1	**0**	2	2
2	4	0	**0**
3	2	**0**	1

If not finished, continue with other columns.

Step 3: Assignments are made at zero values.
Therefore, we assign job 1 to machine 1; job 2 to machine 3, and job 3 to machine 2.Total cost is 5+12+13 = 30.

It is not always possible to obtain a feasible assignment as in here. Since the number of jobs and number of machines are all assigned, it is an optimal assignment with the order job1 → machine 1, job2 → to machine 2, job 3 → machine 3 and the minimum cost of assignment is 5+12+13 = 30 units

Hungarian Method – Example 2

	1	2	3	4
1	***1***	4	6	3
2	9	***7***	10	9
3	***4***	5	11	7
4	8	7	8	***5***

	1	2	**3**	4
1	0	3	**5**	2
2	2	0	**3**	2
3	0	1	**7**	3
4	3	2	**3**	0

	1	2	3	4
1	***0***	3	2	2
2	2	***0***	0	2
3	0	1	4	3
4	3	2	***0***	0

A feasible assignment is not possible at this moment.In such a case, the procedure is to draw a minimum number of lines through some of the rows and columns, Such that all zero values are crossed out.

	1	2	3	4
1	0	3	2	2
2	2	0	0	2
3	0	1	4	3
4	3	2	0	0

The next step is to select the smallest uncrossed out element. This element is *subtracted from every uncrossed out element* and *added to every element at the intersection* of two lines.

	1	2	3	4
1	***0***	2	1	1
2	3	0	***0***	2
3	0	***0***	3	2
4	4	2	0	***0***

We can now easily assign to the zero values. Solution is to assign (1 to 1), (2 to 3), (3 to 2) and (4 to 4).

If drawing lines do not provide an easy solution, then we should perform the task of drawing lines one more time.

Actually, we should continue drawing lines until a feasible assignment is possible.

Now since there is one to one assignment, optimal feasible solution has been reached and the minimum cost of assignment is 1+10+5+8 = 24.

Hungarian Method – Example 3

At the head office of a college there are five registration counters. Five persons are available for service.

	Person				
Counter	A	B	C	D	E
1	30	37	40	28	40
2	40	24	27	21	36
3	40	32	33	30	35
4	25	38	40	36	36
5	29	62	41	34	39

How should the counters be assigned to persons so as to maximize the profit?

Here, the highest value is 62. So we subtract each value from 62. The conversion is shown in the following table.

	Person				
Counter	A	B	C	D	E
1	32	25	22	34	22
2	22	38	35	41	26
3	22	30	29	32	27
4	37	24	22	26	26
5	33	0	21	28	23

Now this table must be used to solve the assignment problem using the Hungarian method.

The total cost of assignment = 1C + 2E + 3A + 4D + 5B. Substituting the values from the original table, the total cost is 40 + 36 + 40 + 36 + 62 = 214.

1.2.5 LET US SUM UP

In this INIT – I - Chapter 1.2, you have learnt the Hungarian Method to solve the AP.

1.2.6. EXERCISES

Question 1. Solve the assignment problem given below:

	J1	J2	J3	J4
W1	82	83	69	92
W2	77	37	49	92
W3	11	69	5	86
W4	8	9	98	23

Answer: Worker 1 should perform job 3, worker 2 job 2, worker 3 job 1, and worker 4 should perform job 4. The total cost of this optimal assignment is to 69 + 37 + 11 + 23 = 140.

Question 2. The Funny Toys Company has four men available for work on four separate jobs. Only one man can work on any one job. The cost of assigning each man to each job is given in the following table. The objective is to assign men to jobs in such a way that the total cost of assignment is minimum.

	Job			
Person	1	2	3	4
A	20	25	22	28
B	15	18	23	17
C	19	17	21	24
D	25	23	24	24

Answer: The total cost of assignment = A1 + B4 + C2 + D3.Substituting values from original table:20 + 17 + 17 + 24 = Rs. 78.

Question 3. Find an optimal solution to an assignment problem with the following cost matrix:

	M1	M2	M3	M4	M5
J1	13	5	20	5	6
J2	15	10	16	10	15
J3	6	12	14	10	13
J4	13	11	15	11	15
J5	15	6	16	10	6
J6	6	15	14	5	12

Answer: JI->M2, J2->M6, J3->M1, J4->M3, J5->M5, J6->M4 and the minimum cost= $ (5 + 0+ 6 +15 + 6 + 5) = $ 37.

1.2.7 SUGGESTED READINGS

Hungarian Method to solve the Assignment problem in any of the reference / text books

❖❖❖❖

1.3

VARIATION OF THE ASSIGNMENT MODEL

Variation of the Assignment Model

Unit Structure

1.3.1 INTRODUCTION

In this Unit-I - Chapter 1.3, we shall discuss the variations in the assignment model.

1.3.2 OBJECTIVES

At the end of this unit the learners will be able to

a. Understand the Hungarian Method for solving the unbalanced assignment problem.

1.3.3 VARIATION OF THE ASSIGNMENT MODEL – UNBALANCED ASSIGNMENTS

Unbalanced Assignment Problem

Whenever the cost matrix of an assignment problem is not a square matrix, that is, whenever the number of sources (rows) is not equal to the number of destinations (columns), the assignment problem is called an unbalanced assignment problem. In such problems, dummy rows (or columns) are added in the matrix so as to complete it to form a square matrix. The dummy rows or columns will contain all costs elements as zeroes. The Hungarian method may be used to solve the problem.

Example 1. A company has 4 machines on which to do 3 jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table:

		Machines			
		W	X	Y	Z
Jobs	A	18	24	28	32
	B	8	13	17	19
	C	10	15	19	22

Solution to Example 1:

18	24	28	32
8	13	17	19
10	15	19	22
0	0	0	0

Row Reduced matrix

0	6	10	14
0	5	9	11
0	5	9	12
0	0	0	0

I Modified Matrix

0	6	10	14
0	5	9	11
0	5	9	12
0	0	0	0

N < n i.e. 2 < 4

II Modified Matrix

0	1	5	9
0	0	4	6
0	0	4	7
5	0	0	0

N < n i.e. 3 < 4

III Modified Matrix

0	1	1	5
0	0	0	2
0	0	0	3
9	4	0	0

N = n

Zero assignment

Multiple assignments exists

Solution -I

[0]	1	1	5
~~0~~	[0]	~~0~~	2
~~0~~	~~0~~	[0]	3
9	4	~~0~~	[0]

Optimal assignment	W – A	X – B	Y – C
Cost	18	13	19

Minimum cost = 18 + 13 + 19 = Rs 50

Solution -II

[0]	1	1	5
~~0~~	~~0~~	[0]	2
~~0~~	[0]	~~0~~	3
9	4	~~0~~	[0]

Optimal assignment	W – A	X – C	Y – B
Cost	18	17	15

Minimum cost = 18 + 17 + 15 = Rs 50

Restricted Assignment Problem

A restricted assignment problem is the one in which one or more allocations are prohibited or not possible. For such allocations we assign "M", which is infinitely high cost. No allocation is given in M.

Travelling Salesman Problem

The traveling salesman problem consists of a salesman and a set of cities. The salesman has to visit each one of the cities starting from a certain one (e.g. the hometown) and returning to the same city. The challenge of the problem is that the traveling salesman wants to minimize the total length of the trip.

The method to solve the travelling salesman problem is out of the present course.

1.3.4 LET US SUM UP

In this Unit I - Chapter 1.3, you have learnt the Hungarian Method to solve the AP and the variations in AP.

1.3.5. EXERCISES

Question 1: A company has 5 jobs to be done. The following matrix shows the return in terms of rupees on assigning ith(i = 1, 2, 3, 4, 5) machine to the jth job (j = A, B, C, D, E). Assign the five jobs to the five machines so as to maximize the total expected profit.

		Jobs				
		A	B	C	D	E
Machines	1	5	11	10	12	4
	2	2	4	6	3	5
	3	3	12	5	14	6
	4	6	14	4	11	7
	5	7	9	8	12	5

Answer: Optimal assignment 1 – C 2 – E 3 – D 4 – B 5 – A

Maximum profit = 10 + 5 + 14 + 14 + 7 = Rs. 50

Question 2:
Solve the given unbalanced assignment problem.

	Job			
Person	1	2	3	4
A	20	25	22	28
B	15	18	23	17
C	19	17	21	24

Answer: Minimum cost is 54**.**

Question 3:

You work as a sales manager for a toy manufacturer, and you currently have three salespeople on the road meeting buyers. Your salespeople are in Austin, TX; Boston, MA; and Chicago, IL. You want them to fly to three other cities: Denver, CO; Edmonton, Alberta; and Fargo, ND. The table below shows the cost of airplane tickets in dollars between these cities.

From \ To	Denver	Edmonton	Fargo
Austin	250	400	350
Boston	400	600	350
Chicago	200	400	250

Where should you send each of your salespeople in order to minimize airfare?

Answer: Optimal assignment is:

$$\begin{bmatrix} 250 & \boxed{400} & 350 \\ 400 & 600 & \boxed{350} \\ \boxed{200} & 400 & 250 \end{bmatrix}$$

Total cost of assignment is: $400 + $350 + $200 = $950.

1.3.6 SUGGESTED READINGS

Hungarian Method to solve the unbalanced Assignment problem in any of the reference / text books

❖❖❖❖

MODULE - II

TRANSPORTATION TECHNIQUES – I

2.1

TRANSPORTATION MODEL

Introduction to Transportation Model, Definition of Transportation Model, Matrix Terminology

Unit Structure

2.1.1 INTRODUCTION

In this Unit-II - Chapter 2.1, we shall discuss the concept of transportation model also known as the transportation problem (TP) and formulation of the transportation problem.

2.1.2 OBJECTIVES

At the end of this unit the learners will be able to

- Understand the definition of transportation model also called as the transportation problem (TP).
- Understand the matrix representation of transportation problem
- Understand the situations where the transportation problem becomes important.

2.1.3 DEFINITION OF TRANSPORTATION MODEL OR TRANSPORTATION PROBLEM (TP)

The transportation problem is a special type of linear programming problem where the objective is to minimize the cost of distributing a product from a number of **sources** or **origins** to a number of **destinations.** Because of its special structure the usual simplex method is not suitable for solving transportation problems. These problems require a special method of solution. The **origin** of a transportation problem is the location from which shipments are dispatched.

The **destination** of a transportation problem is the location to which shipments are transported.

2.1.4 MATRIX REPRESENTATION OF TP

To From	E	F	G	H	Factory Supply
A	10	30	25	15 14	14
B	20	15 9	20	10 1	10
C	10 10	30	20 5	20	15
D	30	40 5	35 7	45	12
Dummy	0	0	0 1	0	1
Destination Requirements	10	15	12	15	52 52

The **unit transportation cost** is the cost of transporting one unit of the consignment from an origin to a destination.

2.1.5 EXAMPLES OF TP

Example 1: Balanced transportation model.
Consider the following problem with 2 factories and 3 warehouses:

	Warehouse 1	Warehouse 2	Warehouse 3	Supply
Factory 1	c_{11}	c_{12}	c_{13}	20
Factory 2	c_{21}	c_{22}	c_{23}	10
Demand	7	10	13	

$$\begin{aligned} \text{Total supply} &= 20 + 10 = 30 \\ \text{Total demand} &= 7 + 10 + 13 = 30 \\ &= \text{Total supply} \end{aligned}$$

Since Total supply = Total demand, the problem is balanced.

Example 2: Unbalanced transportation model

There are two cases to consider, namely excess demand and excess supply.

Suppose the demand at warehouse 1 above is 9 units. Then the total supply and total demand are unequal, and the problem is unbalanced. In this case it is not possible to satisfy all the demand at each destination simultaneously. We reformulate the model as follows: since demand exceeds supply by 2 units, we introduce a dummy source (i.e. a fictitious factory) which has a capacity of 2. The amount shipped from this dummy source to a destination represents the shortage quantity at that destination.

It is necessary to specify the costs associated with the dummy source. There are two situations to consider.

(a) Since the source does not exist, no shipping from the source will occur, so the unit transportation costs can be set to zero.

(b) Alternatively, if a penalty cost, P, is incurred for every unit of unsatisfied demand, then the unit transportation costs should be set equal to the unit penalty costs.

	Warehouse 1	Warehouse 2	Warehouse 3	Supply
Factory 1	c_{11}	c_{12}	c_{13}	20
Factory 2	c_{21}	c_{22}	c_{23}	10
dummy	P	P	P	2
Demand	7	10	13	

In effect we are allocating the shortage to different destinations.

2. If supply exceeds demand then a dummy destination is added which absorbs the surplus units. Any units shipped from a source to a dummy destination represent a surplus at that source. Again, there are two cases to consider for how the unit transportation costs should be determined. (a) Since no shipping takes place, the unit transportation costs can be set to zero. (b) If there is a cost for storing, S, the surplus production then the unit transportation costs should be set equal to the unit storage costs.

	Warehouse 1	Warehouse 2	Warehouse 3	dummy	Supply
Factory 1	c_{11}	c_{12}	c_{13}	S	20
Factory 2	c_{21}	c_{22}	c_{23}	S	10
Demand	7	10	13	4	

Here we are allocating the excess supply to the different destinations. From now on, we will discuss balanced transportation problems only, as any unbalanced problem can always be balanced according to the above constructions

2.1.6 LET US SUM UP

In this unit we have seen the definition of transportation model also called as the transportation problem (TP) and we have understood the matrix representation of transportation problem and the situations where the transportation problem becomes important.

2.1.7. EXERCISES

Question 1: Explain the concept of transportation problem (TP) and give one example.

Question 2: What do you understand by transportation cost matrix?

2.1.8 SUGGESTED READINGS

Transportation problem section in any of the reference / text books

❖❖❖❖

2.2

TRANSPORTATION MODEL

Formulation and Solution of Transportation Model

Unit Structure

2.2.1 INTRODUCTION

In this Unit-II - Chapter 2.2, we shall discuss formulation and solution of transportation problem (TP) and method of optimizing the TP.

2.2.2 OBJECTIVES

At the end of this unit the learners will be able to

- Understand formulation and solution of transportation problem (TP)
- Understand the North – West Corner Rule, Least Cost Method and Vogel's Approximation or Penalty Method to find the initial basic feasible solution to the transportation problem
- Understand the Vogel's Approximation or Penalty Method to optimize the transportation problem
- Understand the Post – Optimality Analysis of the transportation problem

2.2.3 FORMULATION AND SOLUTION OF TRANSPORTATION MODEL

In general, a transportation problem is specified by the following information:

1 A set of *m supply points* from which a good is shipped. Supply point i can supply at most s_i units. In the Powerco example, $m = 3$, $s_1 = 35$, $s_2 = 50$, and $s_3 = 40$.

2 A set of *n demand points* to which the good is shipped. Demand point j must receive at least d_j units of the shipped good. In the Powerco example, $n = 4$, $d_1 = 45$, $d_2 = 20$, $d_3 = 30$, and $d_4 = 30$.

3 Each unit produced at supply point i and shipped to demand point j incurs a *variable cost* of c_{ij}. In the Powerco example, $c_{12} = 6$.

Let

$$x_{ij} = \text{number of units shipped from supply point } i \text{ to demand point } j$$

then the general formulation of a transportation problem is

$$\min \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} c_{ij}x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{j=n} x_{ij} \le s_i \quad (i = 1, 2, \ldots, m) \qquad \text{(Supply constraints)}$$

$$\sum_{i=1}^{i=m} x_{ij} \ge d_j \quad (j = 1, 2, \ldots, n) \qquad \text{(Demand constraints)}$$

$$x_{ij} \ge 0 \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$$

If a problem has the constraints given in (1) and is a *maximization* problem, then it is still a transportation problem (see Problem 7 at the end of this section). If

$$\sum_{i=1}^{i=m} s_i = \sum_{j=1}^{j=n} d_j$$

then total supply equals total demand, and the problem is said to be a **balanced transportation problem.**

Powerco has three electric power plants that supply the needs of four cities. Each power plant can supply the following numbers of kilowatt-hours (kwh) of electricity: plant 1— 35 million; plant 2—50 million; plant 3—40 million. The peak power demands in these cities, which occur at the same time (2 P.M.), are as follows (in kwh): city 1—45 million; city 2—20 million; city 3—30 million; city 4—30 million. The costs of sending 1 million kwh of electricity from plant to city depend on the distance the electricity must travel. Formulate the problem as a transportation problem to minimize the cost of meeting each city's peak power demand.

A Transportation Tableau

					Supply
	c_{11}	c_{12}	. . .	c_{1n}	s_1
	c_{21}	c_{22}	. . .	c_{2n}	s_2
	⋮	⋮		⋮	⋮
	c_{m1}	c_{m2}	. . .	c_{mn}	s_m
Demand	d_1	d_2	. . .	d_n	

Transportation Tableau for Powerco

	City 1	City 2	City 3	City 4	Supply
Plant 1	[8]	10 [6]	25 [10]	[9]	35
Plant 2	45 [9]	[12]	5 [13]	[7]	50
Plant 3	[14]	10 [9]	[16]	30 [5]	40
Demand	45	20	30	30	

Shipping Costs, Supply, and Demand for Powerco

	To				
From	City 1	City 2	City 3	City 4	Supply (million kwh)
Plant 1	$8	$6	$10	$9	35
Plant 2	$9	$12	$13	$7	50
Plant 3	$14	$9	$16	$5	40
Demand (million kwh)	45	20	30	30	

2.2.4 NORTH – WEST CORNER RULE

(ai and bj denote supply and demand respectively)

Make the transportation problem as minimization and balanced problem and then follow the following steps.

Step1: Select the upper left (north-west) cell of the transportation matrix and allocate the maximum possible value to X11 which is equal to min(a1,b1).

Step2:
If allocation made is equal to the supply available at the first source (a1 in first row), then move vertically down to the cell (2,1).
If allocation made is equal to demand of the first destination (b1 in first column), then move horizontally to the cell (1,2).
If a1=b1 , then allocate X11= a1 or b1 and move to cell (2,2).

Step3: Continue the process until an allocation is made in the south-east corner cell of the transportation table.

Example: Solve the Transportation Table to find Initial Basic Feasible Solution using North-West Corner Method.

	D_1	D_2	D_3	D_4	Supply
S_1	19 (5)	30 (2)	50	10	7
S_2	70	30 (6)	40 (3)	60	9
S_3	40	8	70 (4)	20 (14)	18
Demand	5	8	7	14	34

Total Cost =19*5+30*2+30*6+40*3+70*4+20*14 = Rs. 1015

2.2.5 LEAST COST METHOD

Step1: Select the cell having lowest unit cost in the entire table and allocate the minimum of supply or demand values in that cell.

Step2: Then eliminate the row or column in which supply or demand is exhausted. If both the supply and demand values are same, you can eliminate either of the row or column.

In case, the smallest unit cost is not unique, then select the cell where maximum allocation can be made.

Step3: Repeat the process with next lowest unit cost and continue until the entire available supply at various sources and demand at various destinations is satisfied.

	D1	D2	D3	D4	Supply
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8 (8)	70	20	18
Demand	5	8	7	14	34

	D1	D3	D4	Supply
S1	19	50	10 (7)	7
S2	70	40	60	9
S3	40	70	20	10
Demand	5	7	14	34

	D1	D3	D4	Supply
S2	70	40	60	9
S3	40	70	20 (7)	10
Demand	5	7	7	34

	D1	D3	Supply
S2	70	40 (7)	9
S3	40	70	3
Demand	5	7	34

	D1	Supply
S2	70 (2)	2
S3	40 (3)	3
Demand	5	34

The total transportation cost obtained by this method
= 8*8+10*7+20*7+40*7+70*2+40*3
= Rs.814

Here, we can see that the Least Cost Method involves a lower cost than the North-West Corner Method.

2.2.6 VOGEL'S APPROXIMATION OR PENALTY METHOD

Step1: Calculate penalty for each row and column by taking the difference between the two smallest unit costs. This penalty or extra cost has to be paid if one fails to allocate the minimum unit transportation cost.

Step2: Select the row or column with the highest penalty and select the minimum unit cost of that row or column. Then, allocate the minimum of supply or demand values in that cell. If there is a tie, then select the cell where maximum allocation could be made.

Step3: Adjust the supply and demand and eliminate the satisfied row or column. If a row and column are satisfied simultaneously, only of them is eliminated and the other one is assigned a zero value. Any row or column having zero supply or demand cannot be used in calculating future penalties.

Step4: Repeat the process until all the supply sources and demand destinations are satisfied.

	D1	D2	D3	D4	Supply	Row Diff.
S1	19	30	50	10	7	**9**
S2	70	30	40	60	9	**10**
S3	40	8 (8)	70	20	18	**12**
Demand	5	8	7	14	34	
Col.Diff.	**21**	**22**	**10**	**10**		

	D1	D3	D4	Supply	Row Diff.
S1	19 (5)	50	10	7	**9**
S2	70	40	60	9	**20**
S3	40	70	20	10	**20**
Demand	5	7	14	34	
Col.Diff	**21**	**10**	**10**		

	D3	D4	Supply	Row Diff.
S1	50	10	2	**40**
S2	40	60	9	**20**
S3	70	20 (10)	10	**50**
Demand	7	14	34	
Col.Diff.	**10**	**10**		

	D3	D4	Supply	Row Diff.
S1	50	10 (2)	2	**40**
S2	40	60	9	**20**
Demand	7	4	34	
Col.Diff.	**10**	**50**		

	D3	D4	Supply	Row Diff.
S2	40 (7)	60 (2)	9	**20**
Demand	7	2	34	
Col.Diff.				

The total transportation cost obtained by this method
= 8*8+19*5+20*10+10*2+40*7+60*2 = = Rs.779

Here, we can see that Vogel's Approximation Method involves the lowest cost than North-West Corner Method and Least Cost Method and hence is the most preferred method of finding initial basic feasible solution.

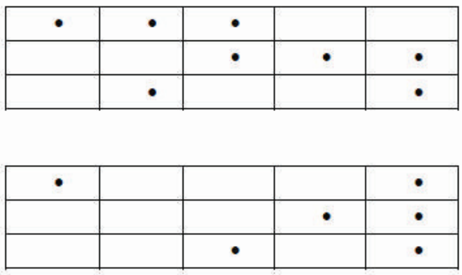
2.2.7 POST – OPTIMALITY ANALYSIS – MODI METHOD

Examining the Initial Basic Feasible Solution for Non - Degeneracy

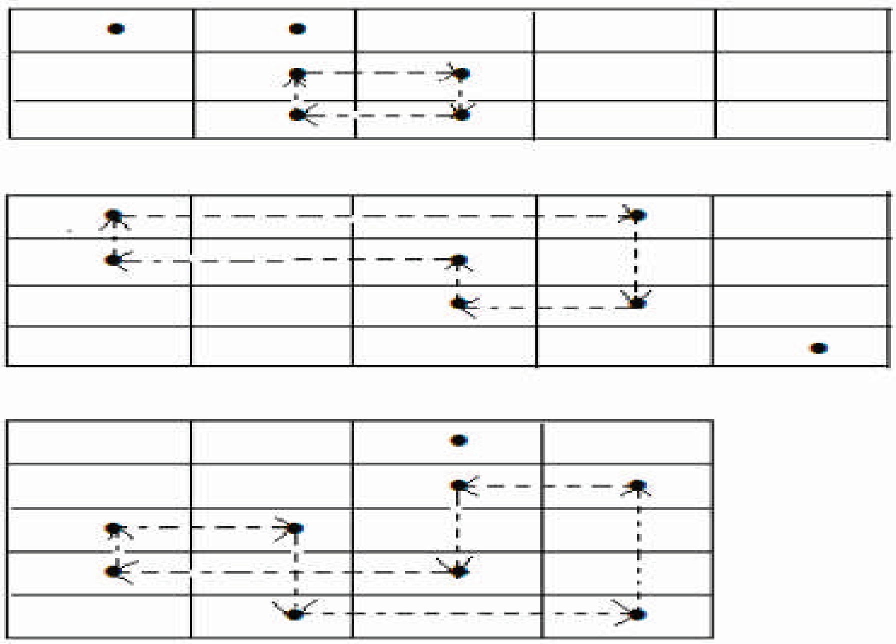
Examine the initial basic feasible solution for non-degeneracy. If it is said to be non-degenerate then it has the following two properties

- Initial basic feasible solution must contain exactly m + n – 1 number of individual allocations.
- These allocations must be in independent positions

Independent Positions



Non-Independent Positions



If the number of allocations is < (m+n-1) then put a very small hypothetical allocation € in the non-allocated cell with least cost and use the MODI method to further optimize the transportation solution.

Transportation Algorithm for Minimization Problem (MODI Method)

Step1: Construct the transportation table entering the origin capacities ai, the destination requirement bjand the cost cij

Step2: Find an initial basic feasible solution by vogel's method or by any of the given method.

Step3: For all the basic variables xij, solve the system of equations ui + vj = cij, for all i, j for which cell(i, j) is in the basis, starting initially with some ui = 0, calculate the values of ui and vj on the transportation table

Step4: Compute the cost differences dij = cij – (ui + vj) for all the non-basic cells

Step5: Apply optimality test by examining the sign of each dij

- If all dij≥ 0, the current basic feasible solution is optimal
- If at least one dij< 0, select the variable xrs (most negative) to enter the basis.
- Solution under test is not optimal if any dij is negative and further improvement is required by repeating the above process.

Step6: Let the variable Xrs enter the basis. Allocate an unknown quantity Ө to the cell (r, s). Then construct a loop that starts and ends at the cell (r, s) and connects some of the basic cells. The amount Ө is added to and subtracted from the transition cells of the loop in such a manner that the availabilities and requirements remain satisfied.

Step7: Assign the largest possible value to the Ө in such a way that the value of at least one basic variable becomes zero and the other basic variables remain non-negative. The basic cell whose allocation has been made zero will leave the basis.

Step8: Now, return to step 3 and repeat the process until an optimal solution is obtained.

Example

- **Find an optimal solution**

	W_1	W_2	W_3	W_4	Availability
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Requirement	5	8	7	14	

- **Find the initial basic feasible solution using Vogel's approximation method.**

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	**2**(10)	X	X
F_2	(70)	(30)	**7**(40)	**2**(60)	X	X
F_3	(40)	**8**(8)	(70)	**10**(20)	X	X
Requirement	X	X	X	X		
Penalty	X	X	X	X		

- **Check for Non-degeneracy**

The initial basic feasible solution has m + n – 1 i.e. 3 + 4 – 1 = 6 allocations in independent positions. Hence optimality test is satisfied.

- **Calculation of uiand vj: - ui+ vj= cij**

					u_i
	• (19)			• (10)	$u_1 = -10$
			• (40)	• (60)	$u_2 = 40$
		• (8)		• (20)	$u_3 = 0$
v_j	$v_1 = 29$	$v_2 = 8$	$v_3 = 0$	$v_4 = 20$	

Assign a 'u' value to zero. (Convenient rule is to select the ui, which has the largest number of allocations in its row)

Let u3 = 0, then

u3 + v4= 20 which implies 0 + v4 = 20, so v4 = 20

u2 + v4= 60 which implies u2 + 20 = 60, so u2 = 40

u1 + v4= 10 which implies u1 + 20 = 10, so u1 = -10

u2 + v3= 40 which implies 40 + v3 = 40, so v3 = 0

u3 + v2= 8 which implies 0 + v2 = 8, so v2 = 8

u1 + v1= 19 which implies -10 + v1= 19, so v1 = 29

- **Calculation of cost differences for non-basic cells dij= cij– (ui+ vj)**

c_{ij}

•	(30)	(50)	•
(70)	(30)	•	•
(40)	•	(70)	•

$u_i + v_j$

•	-2	-10	•
69	48	•	•
29	•	0	•

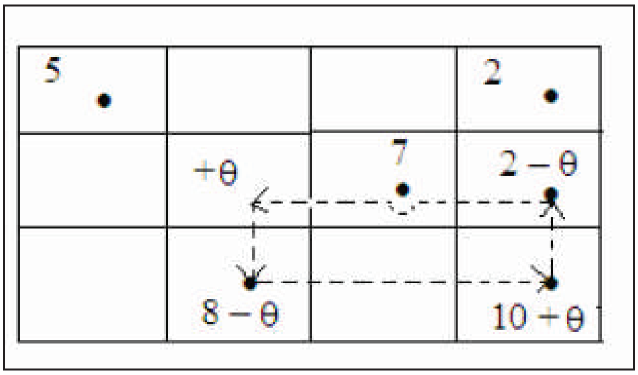
$d_{ij} = c_{ij} - (u_i + v_j)$

•	32	60	•
1	**-18**	•	•
11	•	70	•

- **Optimality test**

dij< 0 i.e. d22 = -18 so X22is entering the basis

- **Construction of loop and allocation of unknown quantity Ө**



We allocate Ө to the cell (2, 2). Reallocation is done by transferring the maximum possible amount Ө in the marked cell. The value of Ө is obtained by equating to zero to the corners of the closed loop. i.e. min(8-Ө, 2-Ө) = 0 which gives Ө = 2. Therefore x24 is outgoing as it becomes zero.

5 (19)			2 (10)
	2 (30)	7 (40)	
	6 (8)		12 (20)

Minimum transportation cost is 5 (19) + 2 (10) + 2 (30) + 7 (40) + 6 (8) + 12 (20) = Rs. 743

- **Improved Solution**

				u_i
• (19)			• (10)	$u_1 = -10$
	• (30)	• (40)		$u_2 = 22$
	• (8)		• (20)	$u_3 = 0$
v_j: $v_1 = 29$	$v_2 = 8$	$v_3 = 18$	$v_4 = 20$	

c_{ij}

•	(30)	(50)	•
(70)	•	•	(60)
(40)	•	(70)	•

$u_i + v_j$

•	-2	8	•
51	•	•	42
29	•	18	•

$d_{ij} = c_{ij} - (u_i + v_j)$

•	32	42	•
19	•	•	18
11	•	52	•

Since dij> 0, an optimal solution is obtained with minimal cost Rs.743

2.2.8 LET US SUM UP

In this unit we have seen the formulation of transportation model, North – West Corner Rule, Least Cost Method, Vogel's Approximation or Penalty Method and Post – Optimality Analysis.

2.2.9. EXERCISES

Question 1: Explain the method of formulation of transportation problem (TP) and give an example.

Question 2: Solve by North West corner, least cost and Vogel's method obtain an optimal solution for the following problem.

				Available
From	50	30	220	1
	90	45	170	3
	250	200	50	4
Required	4	2	2	

Question 2 Answer:

1(50)		
3(90)	0(45)	
	2(200)	2(50)

Minimum transportation cost is 1 (50) + 3 (90) + 2 (200) + 2 (50) = Rs. 820

Question 3:

When the evaluation of any empty cell yields the same cost as the existing allocation, ***an alternate optimal solution*** exists (see Stepping Stone Method – alternate solutions). Assume that all other cells are optimally assigned. In such cases, management has additional flexibility and can invoke no transportation cost factors in deciding on a final shipping schedule.

Degeneracy exists in a transportation problem when the number of filled cells is less than the number of rows plus the number of columns minus one (m + n - 1). Degeneracy may be observed either during the initial allocation when the first entry in a row or column satisfies both the row and column requirements or during VAM method application, when the added and subtracted values are equal. Degeneracy requires some adjustment in the matrix to evaluate the solution achieved. The form of this adjustment involves inserting some value in an empty cell so a closed path can be developed to evaluate other empty cells. This value may be thought of as an infinitely small amount, having no direct bearing on the cost of the solution.

A fictive corporation A has a contract to supply motors for all tractors produced by a fictive corporation B. Corporation B manufactures the tractors at four locations around Central Europe: Prague, Warsaw, Budapest and Vienna. Plans call for the following numbers of tractors to be produced at each location:

- Prague 9000
- Warsaw 12000
- Budapest 9000

Corporation A has three plants that can produce the motors. The plants and production capacities are

- Hamburg 8000
- Munich 7000
- Leipzig 10000
- Dresden 5000

Due to varying production and transportation costs, the profit earns on each motor depends on where they were produced and where they were shipped. The following transportation table gives the accounting department estimates of the euro profit per unit (motor).

Shipped to / Produced at	Prague	Warsaw	Budapest	Source Capacity
Hamburg	70	90	130	8 000
Munich	80	130	60	7 000
Leipzig	65	110	100	10 000
Dresden	95	80	35	5 000
Destination Capacity	9 000	12 000	9 000	30 000 30 000

"The Euro Profit Per One Shipped Motor"

Convert the profit matrix into cost matrix, by subtracting every cell element from the maximum element and then solve the following matrix for initial basic feasible solution:

Shipped to **Produced at**	Prague	Warsaw	Budapest	Source Capacity
Hamburg	60	40	0	8000
Munich	50	0	70	7000
Leipzig	65	20	30	10000
Dresden	35	50	95	5000
Destination Capacity	9000	12000	9000	30000

Answer: (Cell Hamburg - Budapest was assigned first, Munich - Warsaw second, Leipzig - Warsaw third, Leipzig – Budapest fourth, Dresden – Prague fifth and Leipzig – Prague sixth.) Total profit: 3335000 euro.

2.2.10 SUGGESTED READINGS

Transportation problem section in any of the reference / text books

❖❖❖❖

2.3

TRANSPORTATION MODEL

Variants in Transportation Model

Unit Structure

2.3.1 INTRODUCTION

This chapter introduces you to variations in transportation problems like degeneracy and alternate optimal solution

2.3.2 OBJECTIVES

After studying this unit you will be able to understand

- Variation in Transportation Problem.

2.3.3 VARIANTS IN TRANSPORTATION MODEL

If the basic feasible solution of a transportation problem with m origins and n destinations has fewer than m + n – 1 positive xij (occupied cells), the problem is said to be a degenerate transportation problem. Degeneracy can occur at two stages:

- At the initial solution
- During the testing of the optimal solution.

A degenerate basic feasible solution in a transportation problem exists if and only if some partial sum of availability's (row(s)) is equal to a partial sum of requirements (column(s)).

To resolve degeneracy, we make use of an artificial quantity (d). The quantity d is assigned to that unoccupied cell, which has the minimum transportation cost.

➢ Degeneracy in Transportation Problem Example

Factory	Dealer				Supply
	1	2	3	4	
A	2	2	2	4	1000
B	4	6	4	3	700
C	3	2	1	0	900
Requirement	900	800	500	400	

Initial Basic Feasible Solution

Factory	Dealer				Supply
	1	2	3	4	
A	2 (900)	2 (100)	2	4	1000
B	4	6 (700)	4	3	700
C	3	2	1 (500)	0 (400)	900
Requirement	900	800	500	400	

Number of basic variables = m + n – 1 = 3 + 4 – 1 = 6
Since number of basic variables is less than 6, therefore, it is a degenerate transportation problem.

To resolve *degeneracy*, we make use of an artificial quantity(d). The quantity d is assigned to that unoccupied cell, which has the minimum transportation cost.

The quantity d is so small that it does not affect the supply and demand constraints.

In the above table, there is a tie in selecting the smallest unoccupied cell. In this situation, you can choose any cell arbitrarily. We select the cell C2 as shown in the following table.

Factory	Dealer 1	Dealer 2	Dealer 3	Dealer 4	Supply
A	2 (900)	2 (100)	2	4	1000
B	4	6 (700)	4	3	700
C	3	2 (d)	1 (500)	0 (400)	900 + d
Requirement	900	800 + d	500	400	2600 + d

Using the MODI method the final optimal solution will be as under:

Factory	Dealer 1	Dealer 2	Dealer 3	Dealer 4	Supply
A	2 (200)	2 (800)	2	4	1000
B	4 (700)	6	4	3	700
C	3	2 (d)	1 (500)	0 (400)	900
Requirement	900	800	500	400	2600

The optimal solution is
2 X 200 + 2 X 800 + 4 X 700 + 2 X d + 1 X 500 + 0 X 400 = 5300 + 2d.

Notice that *d* is a very small quantity so it can be neglected in the **optimal solution**. Thus, the net transportation cost is Rs. 5300.

- **What do you think happens when the problem is maximization instead of a minimization problem?**

a) Identify the largest value in the tableau and subtract all the other cell “profits” from that value.

b) Then replace the original cell profits with the resulting values. These values reflect the opportunity costs that would be incurred by using routes with unit profits that are less than the largest unit profit.

c) Then solve the tableau in the usual way for the minimum cost solution. Minimizing lost opportunity costs is the same as maximizing the total profit. The optimal solution would have to be transformed to the original “profits” so as to find the optimal value in the original problem.

- **A multiple optimal solution**

This problem occurs when there is more than one optimal solution. This would be indicated when more than one unoccupied cell have zero value for the net cost change in the optimal solution.

Thus a reallocation to cell having a net cost change equal to zero will have no effect on transportation cost. This reallocation will provide another solution with same transportation cost, but the route employed will be different from those for the original optimal solution. This is important because they provide management with added flexibility in decision making.

- **A prohibited route** is assigned a large cost (M) so that it will never receive an allocation.

2.3.4 LET US SUM UP

In this chapter you have learnt different variants of the transportation problem.

2.3.5 EXERCISES

Question 1: Solve the following unbalanced transportation problem by yourself:

Warehouses				
Plant	**W1**	**W2**	**W3**	**Supply**
A	28	17	26	500
B	19	12	16	300
Demand	250	250	500	

Question 2: Solve the following degenerate transportation problem by yourself:

Dealers					
Factory	**1**	**2**	**3**	**4**	**Supply**
A	2	2	2	4	1000
B	4	6	4	3	700
C	3	2	1	0	900
Requirement	900	800	500	400	

Question 2 Solution…

Here, S1 = 1000, S2 = 700, S3 = 900 R1 = 900, R2 = 800, R3 = 500, R4 = 400

Since R3 + R4 = S3 so the given problem is a degeneracy problem. Now we will solve the transportation problem by Matrix Minimum Method.

To resolve degeneracy, we make use of an artificial quantity (d). The quantity d is so small that it does not affect the supply and demand constraints.

Degeneracy can be avoided if we ensure that no partial sum of s_i (supply) and r_j (requirement) are the same. We set up a new problem where:
$s_i = s_i + d$ i = 1, 2, ..., m $r_j = r_j$ $r_n = r_n + md$

Dealers					
Factory	1	2	3	4	Supply
A	2 900	2 100+d	2	4	1000 +d
B	4	6 700–d	4 2d	3	700 + d
C	3	2	1 500 –2d	0 400+3d	900 +d
Requirement	900	800	500	400 + 3d	

Substituting d = 0.

Dealers					
Factory	1	2	3	4	Supply
A	2 900	2 100	2	4	1000
B	4	6 700	4 0	3	700
C	3	2	1 500	0 400	900
Requirement	900	800	500	400 + 3d	

Initial basic feasible solution:
2 * 900 + 2 * 100 + 6 * 700 + 4 * 0 + 1 * 500 + 0 * 400 = 6700.

Now degeneracy has been removed. Use MODI method to optimize this solution.

Question 3: Solve the following transportation problem by yourself and find the optimal solution.

	W_1	W_2	W_3	W_4	Supply
F_1	1	2	4	4	6
F_2	4	3	2	0	8
F_3	0	2	2	1	10
Demand	4	5	8	6	

Cost Matrix

$W_j \rightarrow$ Warehouse, $F_i \rightarrow$ Factory and cell entries are unit/costs.

2.3.6 SUGGESTED READINGS

Transportation problem section in any of the reference / text books

REFERENCE / TEXT BOOKS

❖❖❖❖

MODULE - III

TRANSPORTATION TECHNIQUES –II

3.1

VARIATION IN TRANSPORTATION PROBLEM

Variation in Transportation Problem, Trans Shipment Model, Time Minimization Problems

Unit Structure :

3.1.1 INTRODUCTION

In this Unit-III - Chapter 3.1, we shall discuss the variations in transportation problem (TP).

3.1.2 OBJECTIVES

At the end of this unit the learners will be able to

- Understand types of variations in the transportation problem.

3.1.3 VARIATIONS IN TRANSPORTATION MODEL OR TRANSPORTATION PROBLEM (TP)

A very common supply chain involves the shipment of goods from suppliers at one set of locations to customers at another set of locations.

The classic transportation model is characterized by a set of supply sources (each with known capacities), a set of demand locations (each with known requirements) and the unit costs of transportation between supply-demand pairs.

The transportation model has two kinds of constraints:

- Less-than capacity constraints and
- Greater-than demand constraints

If total capacity equals total demand, both capacity and demand constraints are “=”.

If capacity exceeds demand, the capacity constraints are “<” and the demand constraints are “>”.

If demand exceeds capacity, the capacity constraints are “>” and the demand constraints are “<”.

In the transportation model, we have supply and demand constraints. The solution to the model provides shadow prices on each. The shadow price on a demand constraint tells us how much it costs to ship the marginal unit to the corresponding location.

The variations in the transportation problem can be understood through Sensitivity or Post-Optimality analysis of the transportation problem.

In this section we discuss the following three aspects of sensitivity analysis for the transportation problem:

a) Changing the objective function coefficient of a non-basic variable.
b) Changing the objective function coefficient of a basic variable.
c) Increasing a single supply by Δ and a single demand by Δ.

Let us consider the Powerco Example problem with the following initial transportation cost matrix:

From	*To*				
	City 1	*City 2*	*City 3*	*City 4*	*Supply (Million kwh)*
Plant 1	$8	S6	$10	$9	35
Plant 2	$9	$12	$13	$7	50
Plant 3	$14	S9	$16	$5	40
Demand (Million kwh)	45	20	30	30	

Decision Variable

Since we have to determine how much electricity is sent from each plant to each city;

X_{ij}= Amount of electricity produced at plant i and sent to city j

X_{14} = Amount of electricity produced at plant 1 and sent to city 4.

Objective Function

Since we want to minimize the total cost of shipping from plants to cities;

Minimize $Z = 8X_{11}+6X_{12}+10X_{13}+9X_{14}+9X_{21}+12X_{22}+13X_{23}+7X_{24}$
$+14X_{31}+9X_{32}+16X_{33}+5X_{34}$

Supply Constraints

Since each supply point has a limited production capacity;
$X_{11}+X_{12}+X_{13}+X_{14} <= 35$
$X_{21}+X_{22}+X_{23}+X_{24} <= 50$
$X_{31}+X_{32}+X_{33}+X_{34} <= 40$

Demand Constraints

Since each supply point has a limited production capacity;
$X_{11}+X_{21}+X_{31} >= 45$
$X_{12}+X_{22}+X_{32} >= 20$
$X_{13}+X_{23}+X_{33} >= 30$
$X_{14}+X_{24}+X_{34} >= 30$

Sign Constraints

Since a negative amount of electricity can not be shipped all Xij's must be non negative; $X_{ij} >= 0$ (i= 1,2,3; j= 1,2,3,4)

LP Formulation of Powerco's Problem

Min $Z = 8X_{11}+6X_{12}+10X_{13}+9X_{14}+9X_{21}+12X_{22}+13X_{23}+7X_{24}$
$+14X_{31}+9X_{32}+16X_{33}+5X_{34}$
S.T.: $X_{11}+X_{12}+X_{13}+X_{14} <= 35$ (Supply Constraints)
$X_{21}+X_{22}+X_{23}+X_{24} <= 50$
$X_{31}+X_{32}+X_{33}+X_{34} <= 40$
$X_{11}+X_{21}+X_{31} >= 45$ (Demand Constraints)
$X_{12}+X_{22}+X_{32} >= 20$
$X_{13}+X_{23}+X_{33} >= 30$
$X_{14}+X_{24}+X_{34} >= 30$
$X_{ij} >= 0$ (i= 1,2,3; j= 1,2,3,4)

How to Pivot a Transportation Problem

Based on the transportation tableau, the following steps should be performed.

Step 1. Determine (by a criterion to be developed shortly, for example northwest corner method) the variable that should enter the basis.

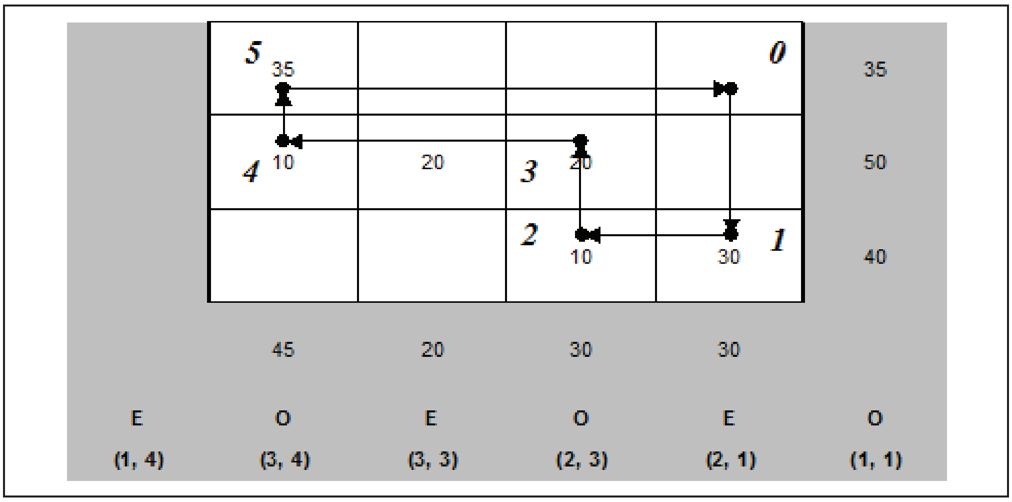
Step 2. Find the loop (it can be shown that there is only one loop) involving the entering variable and some of the basic variables.

Step 3. Counting the cells in the loop, label them as even cells or odd cells.

Step 4. Find the odd cells whose variable assumes the smallest value. Call this value θ. The variable corresponding to this odd cell will leave the basis. To perform the pivot, decrease the value of each odd cell by θ and increase the value of each even cell by θ. The variables that are not in the loop remain unchanged. The pivot is now complete. If θ=0, the entering variable will equal 0, and an odd variable that has a current value of 0 will leave the basis. In this case a degenerate bfs existed before and will result after the pivot. If more than one odd cell in the loop equals θ, you may arbitrarily choose one of these odd cells to leave the basis; again a degenerate bfs will result

Illustration of pivoting procedure on the Powerco example

We want to find the bfs that would result if X14 were entered into the basis, the North-West Corner bfs and the loop is shown in the following table.



New bfs after X_{14} is pivoted into basis. Since There is no loop involving the cells (1,1), (1,4), (2,1), (2,2), (3,3) and (3, 4) the new solution is a bfs.

35-20			0+20	35
10+20	20	20-20 (nonbasic)		50
		10+20	30-20	40
45	20	30	30	

After the pivot the new bfs is X11=15, X14=20, X21=30, X22=20, X33=30 and X34=10

In the pivoting procedure:

- Since each row has as many +20s as –20s, the new solution will satisfy each supply and demand constraint.
- By choosing the smallest odd variable (X_{23}) to leave the basis, we ensured that all variables will remain nonnegative.

Using the MODI method, the optimal solution for Powerco is X11=10, X13=25, X21=45, X23=5, X32=10 and X34=30.

As a result of this solution the objective function value becomes:
Z=6(10)+10(25)+9(45)+13(5)+9(10)+5(30)=$1020

a) Changing the objective function coefficient of a non-basic variable.

Let's try to answer the following question about Powerco as an example:

For what range of values of the cost of shipping 1 million kwh of electricity from plant 1 to city 1 will the current basis remain optimal?

We will need to use the MODI method and the final optimal solution having u_i and v_j values.

Suppose we change c_{11} from 8 to 8+ Δ.
Now $\check{c}_{11} = u_1+v_1-c_{11}=0+6-(8+\Delta)= -2-\Delta$.

Thus the current basis remains optimal for –2- Δ<=0, or Δ>=-2, and c_{11}>=8-2=6.

b) Changing the objective function coefficient of a basic variable.

For what range of values of the cost of shipping 1 million kwh of electricity from plant 1 to city 3 will the current basis remain optimal?

Suppose we change c_{13} from 10 to 10+ Δ.

Now $\check{c}_{13}=0$ changes from $u_1+v_3=10$ to $u_1+v_3=10+\Delta$.

Thus, to find the u_i's and v_j's we must solve the following equations:

$u_1=0$ $u_1+v_2=6$ $u_2+v_1=9$ $u_2+v_3=13$
$u_3+v_2=9$ $u_1+v_3=10+\Delta$ $u_3+v_4=5$

Solving these equations, we obtain $u_1=0, v_2=6, v_3=10+\Delta, v_1=6+\Delta$, $u_2=3-\Delta$, $u_3=3$, and $v_4=2$.

We now price out each non-basic variable. The current basis will remain optimal as long as each non-basic variable has a non-positive coefficient in row 0.

$\check{c}_{11} = u_1+v_1-8=\Delta-2<=0$ for $\Delta<=2$
$\check{c}_{14} = u_1+v_4-9=-7$
$\check{c}_{22} = u_2+v_2-12=-3-\Delta<=0$ for $\Delta>=-3$
$\check{c}_{24} = u_2+v_4-7=-2-\Delta<=0$ for $\Delta>=-2$
$\check{c}_{31} = u_3+v_1-14=-5+\Delta<=0$ for $\Delta<=5$
$\check{c}_{33} = u_3+v_3-16=\Delta-3<=0$ for $\Delta<=3$

Thus, the current basis remains optimal for $-2<=\Delta<=2$, or $8=10-2<=c_{13}<=10+2=12$

c) Increasing a single supply by Δ and a single demand by Δ.

Changing both supply and demand by the same amount will maintain the balance of the transportation problem. Since ui's and vj's may be thought of as the negative of each constraint's shadow price, we know that if the current basis remains optimal,
New Z value = old Z value$+\Delta u_i+\Delta v_j$

For example if we increase plant 1,s supply and city 2's demand by 1 unit, then

New cost=1020+1(0)+1(6)=$1026

We can also find the new values of the decision variables as follows:

- If Xij is a basic variable in the optimal solution, increase Xij by Δ.
- If Xij is a nonbasic variable in the optimal solution, find the loop involving Xij and some of the basic variables. Find an odd cell in the loop that is in row I. Increase the value of this odd cell by Δ

and go around the loop, alternately increasing and then decreasing current basic variables in the loop by Δ.

3.1.4 LET US SUM UP

In this unit we have seen different variations and have defined the changes in the transportation matrix using the sensitivity analysis.

3.1.5. EXERCISES

Question 1: Please work out any two examples from the *Transportation problem sensitivity analysis section in any of the reference / text books.*

Question 2: Building Brick Company (BBC) has orders for 80 tons of bricks at three suburban locations as follows: Northwood -- 25 tons, Westwood -- 45 tons, and Eastwood -- 10 tons. BBC has two plants, each of which can produce 50 tons per week. How should end of week shipments be made to fill the above orders given the following delivery cost per ton:

	Northwood	Westwood	Eastwood
Plant 1	24	30	40
Plant 2	30	40	42

Answer:

From	To	Amount	Cost
Plant 1	Northwood	5	120
Plant 1	Westwood	45	1,350
Plant 2	Northwood	20	600
Plant 2	Eastwood	10	420

Total Cost = $2,490

Question 3:A company has two plants producing a certain product that is to be shipped to three distribution centers. The unit production costs are the same at the two plants, and the shipping cost per unit is shown below:

		Distribution Center		
		1	2	3
Plant	A	$4	$6	$4
	B	$6	$5	$2

Shipments are made once per week. During each week, each plant produces at most 60 units and each distribution center needs at least 40 units. How many units should be shipped from each distribution center to each distribution center, so as to minimize cost?

Find the optimal solution to this transportation problem.

3.1.6 SUGGESTED READINGS

Transportation problem sensitivity analysis section in any of the reference / text books

❖❖❖❖

3.2

TRANS SHIPMENT MODEL

Trans Shipment Model

Unit Structure :

3.2.1 – INTRODUCTION

In this Unit-III-Chapter3.2, we shall discuss the Trans-Shipment Model. Transshipment or Transshipment is the shipment of goods or containers to an intermediate destination, and then from there to yet another destination. One possible reason is to change the means of transport during the journey (for example from ship transport to road transport), known as trans loading. Another reason is to combine small shipments in to a large shipment (consolidation), dividing the large shipment at the other end (deconsolidation). Transshipment usually takes place in transport hubs. Much international transshipment also takes place in designated customs areas, thus avoiding the need for customs checks or duties, otherwise a major hindrance for efficient transport.

3.2.2 – OBJECTIVES

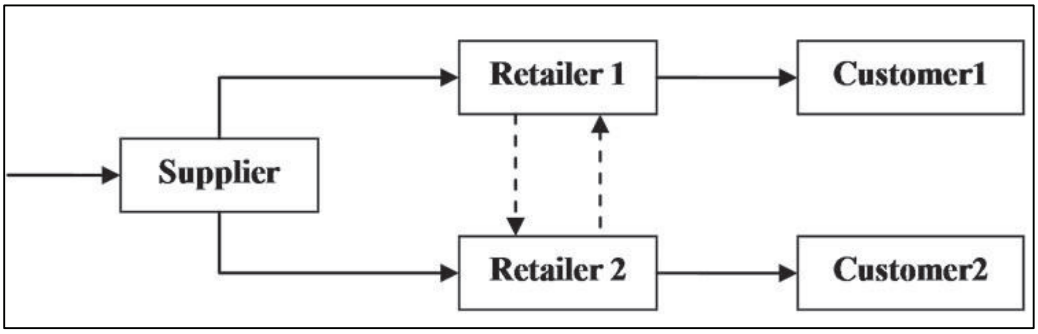
At the end of this unit the learners will be able to

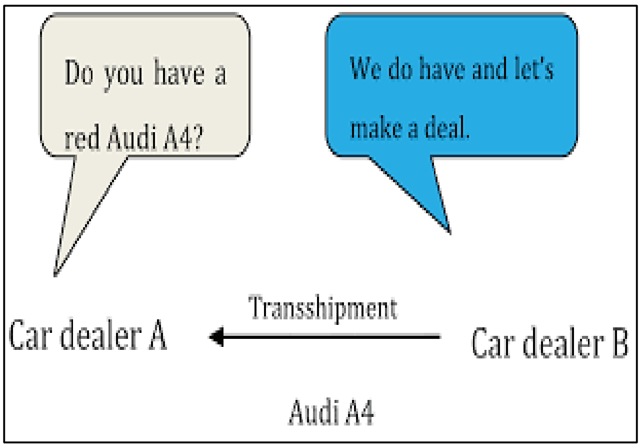
- Understand trans-shipment model and the method of solving tran-shipment problem.

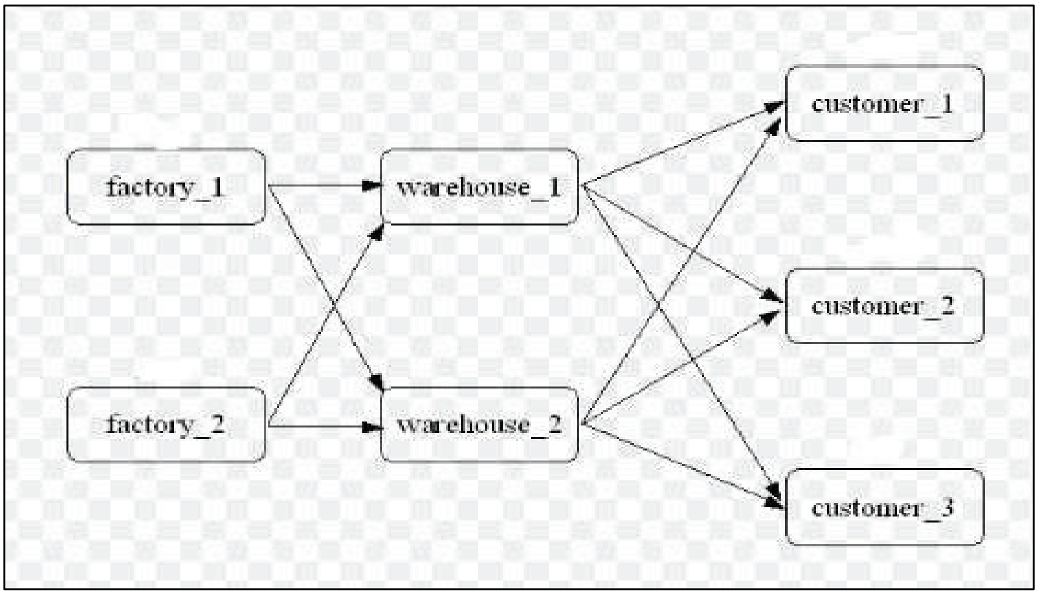
3.2.3 – TRANS-SHIPMENT MODEL

In a transportation problem, shipments are allowed only between source-sink pairs. In many applications, this assumption is too strong. For example, it is often the case that shipments may be allowed between sources and between sinks.

More over, there may also exist points through which units of a product can be transshipped from a source to a sink. Model swith these additional features are called transshipment problems. Interestingly, it turn south at any given transshipment problem can be converted easily into an equivalent transportation problem. The availability of such a conversion procedure significantly broadens the applicability of our algorithm for solving transportation problems.







We will illustrate the conversion procedure with an example. A company manufactures a production two cities, which are Dallas and Houston. The daily production capacities at Dallas and Houston are 160 and 200, respectively. Products are shipped by air to customers in San Francisco and New York. The customers in each city require 140 units of the product per day. Because of the deregulation of air fares, the company believes that it may be cheaper to first fly some products to Chicago or Los Angeles and then fly the products to their final destinations. The costs of flying one unit of the product between these cities are shown in the table below:

	TO					
FROM	Dallas	Houston	Chicago	L. A.	S. F.	N. Y.
Dallas	$0	–	$9	$14	$26	$29
Houston	–	$0	$16	$13	$27	$26
Chicago	–	–	$0	$7	$17	$18
L. A.	–	–	$7	$0	$15	$17
S. F.	–	–	–	–	$0	–
N. Y.	–	–	–	–	–	$0

The company wants to minimize the total cost of daily shipments of the required products to its customers. We shall first define our terminology more carefully. We will define a source as a city that can send products to another city but cannot receive any product from any other city. Similarly, a sink is defined as a city that can receive products from other cities but cannot send products to any other city. Finally, a transshipment point is defined as a city that can both receive products from other cities and send products to other cities.

According to these definitions, Dallas and Houston are sources, with (daily) supplies of 160 and 200 units respectively. Chicago and Los Angeles are transshipment points. San Francisco and New York are sinks, each with a (daily) demand of 140 units.

Observe that the total supply equals 360 and the total demand equals 280. Therefore, we should create a dummy sink, with a demand of 80, to balance the two. With this revision, we have a problem with 2 sources, 3 sinks, and 2 transshipment points.

Since each transshipment point can both receive and send out products, it plays the dual roles of being as in kanda source. This naturally suggests that we could attempt are formulation in which each transshipment point is“ split” into a corresponding sink and a corresponding source. A little bit of reflection, however, leads us to the realization that while the demand and the supply at such a pair of sink and source should be set at the same level (since there is no gain or loss in units), it is not clear what that level should be. This is a consequence of the fact that we do not know a priori how many units will be sent in to and hence shipped out of a transshipment point. Fortunately, upon further reflection, it turns out that this difficulty can actually be circumvented by assigning a “sufficiently-high” value as the demand and the supply for such a sink-source pair and by allowing fictitious shipments from a given transshipment point back to itself at zero cost.

More specifically, suppose the common value of the demand and the supply at the corresponding sink and source of a given transshipment point is set to c; and suppose x units of “real” shipments are sent into and shipped out of that transshipment point. Then, under the assumption that c is no less than x, we can interpret this as saying:(i) a total of c units of the product are being sent into the corresponding sink, of which x units are sent from other points (or cities) and c− x units are sent (fictitiously) from the transshipment point to itself; and (ii) a total of c units of the product are being shipped out of the corresponding source, of which x units are shipped to other points (or cities) and c− x units are shipped (fictitiously) back to the transshipment point itself. Notice that since a shipment from a transshipment point back to itself is assumed to incur no cost, the proposed reformulation preserves the original objective function. The only remaining question now is: What specific value should be assigned to c? The default answer to this question is to let c equal to the total supply in the original problem. Such a choice is clearly sufficient because no shipment can exceed the total available supply. It follows that we have indeed resolved the difficulty alluded to earlier.

In our problem, there are two transshipment points. Therefore, we should replace these by two new sources and two new sinks (all of which will retain their original city names). Furthermore, the new sources and sinks should have a common supply or demand of 360. Our reformulation, therefore, yields an equivalent transportation problem with 4 sources and 5 sinks. This equivalent transportation problem is given in the tableau below:

	S. F.	N. Y.	Chicago	L. A.	Dummy	
Dallas	26	29	9	14	0	160
Houston	27	26	16	13	0	200
Chicago	17	18	0	7	0	360
L. A.	15	17	7	0	0	360
	140	140	360	360	80	

Since the solution method for transportation problems has been explained in detail, we will not attempt to solve this problem. Any method to find the initial basic feasible solution and to test the optimality can be used on the trans-shipment cost matrix given above.

An interesting variation of the above example is to allow shipments between Dallas and Houston, say, at a cost of $ 5per unit either way. This would make Dallas and Houston transshipment points. It follows that we should introduce into the above table au two new sinks to represent Dallas and Houston, respectively.

Both of these two new sinks should have a demand of 360. Correspondingly, it will also be necessary to increase the supply from Dallas by 360, to 520, and the supply from Houston by 360, to 560. These revisions result in the formulation below:

	S. F.	N. Y.	Chicago	L. A.	Dallas	Houston	Dummy	
Dallas	26	29	9	14	0	5	0	520
Houston	27	26	16	13	5	0	0	560
Chicago	17	18	0	7	M	M	0	360
L. A.	15	17	7	0	M	M	0	360
	140	140	360	360	360	360	80	

Notice that we have introduced 4 “big” M’s as the transportation costs in four cells. These are intended to in sure that shipments in to Dallas and Houston from other cities do not appear in the optimal solution. It should be clear that other variations can also be handled similarly.

3.2.4 LET US SUMUP

In this unit we have learnt the concept of the trans-shipment problem and the method of constructing the trans-shipment cost matrix which can be used to optimize using any of the algorithms to find the optimal solution for a balanced transportation problem.

3.2.5. EXERCISES

Question1: Solve the following trans-shipment problem using the optimization techniques of transportation problem.

Wid get co manufactures wid gets at two factories, one in M emphis and one in Denver. The Memphis factory can produce as 150 wid gets, and the Denver factory can produce as many as 200 wid gets per day. Wid gets are shipped by air to customers in LA and Boston. The customers in each city require 130 wid gets per day. Because of the deregulation of air fares, Wid get co believes thatit may be cheaper first fly some wid gets to NY or Chicago and then fly them to their final destinations. The cost of flying a wid get are shown next. Wid get co wants to minimize the total cost of shipping the required widgets to customers.

	NY	Chicago	LA	Boston	Dummy	Supply
Memphis	$8	$13	$25	$28	$0	150
Denver	$15	$12	$26	$25	$0	200
NY	$0	$6	$16	$17	$0	350
Chicago	$6	$0	$14	$16	$0	350
Demand	350	350	130	130	90	

Supply points: Memphis, Denver

Demand Points: LA Boston

Transshipment Points: NY, Chicago

Question 2:

Consider a firm having two factories to ship its products from the factories to three-retail stores. The number of units available at factories X and Y are 200 and 300 respectively, while those demanded at retail stores A, Band C are 100,150 and 250 respectively. Instead of shipping directly from factories to retail stores, it is asked to investigate the possibility of transshipment. The transportation cost (in rupees) per unit is given in the table below:

		Factory		**Retail Stores**		
		X	**Y**	**A**	**B**	**C**
Factory	**X**	0	8	7	8	9
	Y	6	0	5	4	3
Retail Stores	**A**	7	2	0	5	1
	B	1	5	1	0	4
	C	8	9	7	8	0

Find the optimal shipping schedule.

Answer: X ->A, X->B 100 units each, Y->B, Y->C 50, 250 units respectively.
Total cost = 2450

3.2.6 SUGGESTED READINGS

Trans-shipment problem in any of the reference / text books

❖❖❖❖

3.3

TIME MINIMIZATION PROBLEMS

Time Minimization Problems

Unit Structure

3.3.1 INTRODUCTION

In the Time Minimizing Transportation Problem the objective is to minimize the time. This problem is same as the transportation problem of minimizing the cost, expect that the unit transportation cost is replaced by the time t_{ij}

3.3.2 OBJECTIVES

At the end of this unit the learners will be able to

- Understand time minimization problem
- Technique of solving a time minimization problem

3.3.3 TIME MINIMIZATION PROBLEMS

Example

- Determine an initial basic feasible solution using any one of the following:
  1. North West Corner Rule
  2. Matrix Minimum Method
  3. Vogel Approximation Method
- Find Tk for this feasible plan and cross out all the unoccupied cells for which $t_{ij} \geq$ Tk.

- Trace a closed path for the occupied cells corresponding to Tk. If no such closed path can be formed, the solution obtained is optimum otherwise, go to step 2.

The following matrix gives data concerning the transportation timest_{ij}.

Destination							
Origin	D1	D2	D3	D4	D5	D6	Supply
O1	25	30	20	40	45	37	37
O2	30	25	20	30	40	20	22
O3	40	20	40	35	45	22	32
O4	25	24	50	27	30	25	14
Demand	15	20	15	25	20	10	

We compute an initial basic feasible solution by north west corner rule which is shown below:

Destination							
Origin	D1	D2	D3	D4	D5	D6	Supply
O1	25 (15)	30 (20)	20 (2)	40	45	37	37
O2	30	25	20 (13)	30 (9)	40	20	22
O3	40	20	40	35 (16)	45 (16)	22	32
O4	25	24	50	27	30 (4)	25 (10)	14
Demand	15	20	15	25	20	10	

Here, t_{11} = 25, t_{12} = 30, t_{13} = 20, t_{23} = 20, t_{24} = 30, t_{34}= 35, t_{35} = 45, t_{45} =30, t_{46} = 25

Choose maximum from t_{ij}, i.e., T1 = 45. Now, cross out all the unoccupied cells that are ≥ T1. The unoccupied cell (O3D6) enters into the basis as shown below:

Origin	D1	D2	D3	D4	D5	D6	Supply
			Destination				
O1	25 (15)	30 (20)	20 (2)	40	~~45~~	37	37
O2	30	25	20 (13)	30 (9)	40	20	22
O3	40	20	40	35 (16)	- 45 (16)	+ 22	32
O4	25	24	~~50~~	27	+ 30 (4)	- 25 (10)	14
Demand	15	20	15	25	20	10	

Choose the smallest value with a negative position on the closed path, i.e., 10. Clearly only 10 units can be shifted to the entering cell. The next feasible plan is shown in the following table.

Origin	D1	D2	D3	D4	D5	D6	Supply
			Destination				
O1	25 (15)	30 (20)	20 (2)	40	~~45~~	37	37
O2	30	25	20 (13)	30 (9)	40	20	22
O3	40	20	40	35 (16)	45 (6)	22 (10)	32
O4	25	24	~~50~~	27	30 (14)	25	14
Demand	15	20	15	25	20	10	

Here, T2 = Max(25, 30, 20, 20, 20, 35, 45, 22, 30) = 45. Now, cross out all the unoccupied cells that are ≥ T2.

Destination							
Origin	D1	D2	D3	D4	D5	D6	Supply
O1	25 (15)	30 (20)	20 (2)	40	~~45~~	37	37
O2	30	25	20 (13)	- 30 (9)	+ 40	20	22
O3	40	20	40	+ 35 (16)	- 45 (6)	22 (10)	32
O4	25	24	~~50~~	27	30 (14)	25	14
Demand	15	20	15	25	20	10	

By following the same procedure as explained above, we get the following revised matrix.

Destination							
Origin	D1	D2	D3	D4	D5	D6	Supply
O1	25 (15)	30 (20)	20 (2)	~~40~~	~~45~~	37	37
O2	30	25	20 (13)	30 (3)	40 (6)	20	22
O3	~~40~~	20	~~40~~	35 (22)	~~45~~	22 (10)	32
O4	25	24	~~50~~	27	30 (14)	25	14
Demand	15	20	15	25	20	10	

T3 = Max(25, 30, 20, 20, 30, 40, 35, 22, 30) = 40. Now, cross out all the unoccupied cells that are ≥ T3.

Now we cannot form any other closed loop with T3. Hence, the solution obtained at this stage is optimal. Thus, all the shipments can be made within 40 units.

3.3.4 LET US SUM UP

In this unit we have learnt the concept of time minimization problem which is similar to the transportation problem and we have shown the method of solving the time minimization problem.

3.3.5 EXERCISES

Question 1: Consider the following transportation problem with m = 4 sources Ai, i∈I={1,2,3,4}, and n = 5 destinations Bj, j∈J={1,2,3,4,5}. The initial data are presented in the table below. Each row corresponds to a supply point and each column to a demand point. The total supply 65 is equal to the total demand. In each cell (i,j), top left corner represents the time tij required for transporting xij units from source Ai to destination Bj. Optimize this time minimization problem.

		Destinations					Supplies, a_i
	i \ j	B_1	B_2	B_3	B_4	B_5	
Sources	A_1	11 6	3 **3**	10 7	2 **10**	5 **1**	14
	A_2	2 (-) **13**	7 4	3 0	8 6	1 (+) **-1**	13
	A_3	12 7	2 **7**	4 **15**	5 3	7 2	22
	A_4	9 (+) **2**	4 1	6 3	3 1	5 (-) **14**	16
Demands, b_j		15	10	15	10	15	65 \ 65

Question 2:

A concrete company transports concrete from three plants, 1, 2 and 3, to three constructionsites, A, B and C.

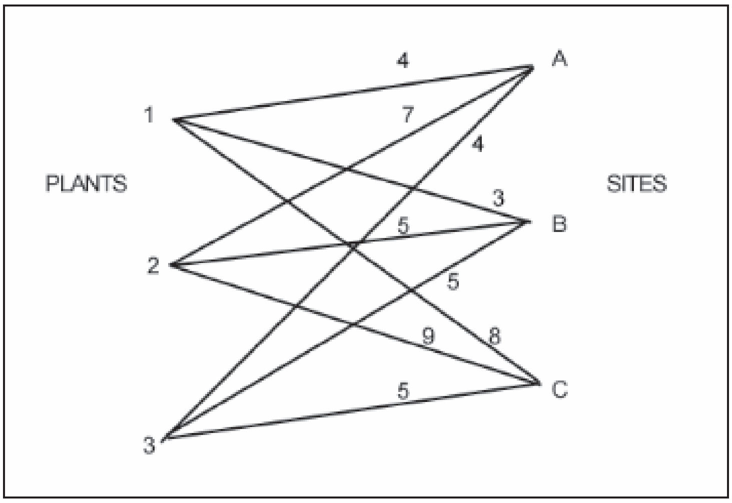
The plants are able to supply the following numbers of tons per week:

Plant	*Supply (capacity)*
1	*300*
2	*300*
3	*100*

The requirements of the sites, in number of tons per week, are:

Construction site	*Demand (requirement)*
A	*200*
B	*200*
C	*300*

The cost of transporting 1 ton of concrete from each plant to each site is shown in the figure below:



For computational purposes it is convenient to put all the above information into a table, as in the simplex method. In this table each row represents a source and each column represents destination.

	Sites				
	From \ *To*	*A*	*B*	*C*	*Supply (Availability)*
Plants	**1**	*4*	*3*	*8*	*300*
	2	*7*	*5*	*9*	*300*
	3	*4*	*5*	*5*	*100*
	Demand (requirement)	*200*	*200*	*300*	

Optimize this transportation problem.

3.3.6 SUGGESTED READINGS

Time management problem in any of the reference / text books

❖❖❖❖

MODULE - IV

NETWORK ANALYSIS

4.1

CONCEPT OF PROJECT PLANNING

Concept of Project Planning, Scheduling and Controlling, Work Break Down Structure, Basic Tools and Techniques of Project Management, Role of Network Technique in Project Management,

Unit Structure

4.1.1 INTRODUCTION

In this Unit-IV - Chapter 4.1, we shall discuss the concept of project planning, scheduling and controlling, Work Break Down Structure, Basic Tools and Techniques of Project Management, Role of Network Technique in Project Management.

4.1.2 OBJECTIVES

At the end of this unit the learners will be able to

- Project Planning, Scheduling and Controlling
- Work Break Down Structure
- Basic Tools and Techniques of Project Management
- Role of Network Technique in Project Management.

4.1.3 PROJECT PLANNING, SCHEDULING AND CONTROLLING

What is a project?

A project is a one-shot, time-limited, goal-directed, major undertaking, requiring the commitment of varied skills and resources.

A project is a series of inter-related and sequenced activities, managed by a single individual, designed and organized to accomplish a specific goal, within a limited timeframe, frequently with specific budgetary requirements.

Projects are critical to the realization of the performing organization's business strategy because projects are a means by which strategy is implemented.

A project is a temporary endeavor undertaken to create a unique product or service. A project is temporary in that there is a defined start (the decision to proceed) and a defined end (the achievement of the goals and objectives). Ongoing business or maintenance operations are not projects. Energy conservation projects and process improvement efforts that result in better business processes or more efficient operations can be defined as projects. Projects usually include constraints and risks regarding cost, schedule or performance outcome.

Examples of projects

- Developing a new product or service
- Effecting a change in structure, staffing, or style of an organization
- Designing a new transportation vehicle
- Developing or acquiring a new or modified information system
- Constructing or renovating a building or facility
- Building a water system for a community in a developing country
- Running a campaign for political office
- Implementing a new or improved business process or procedure

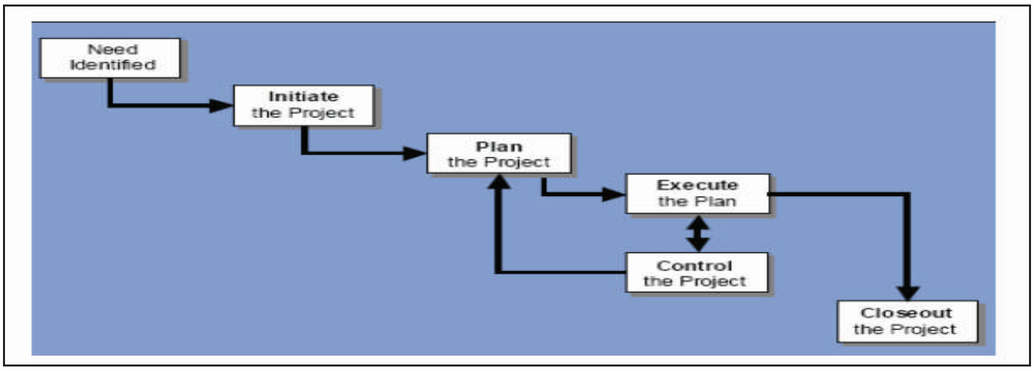
Within a project there can be sub-projects

- Based on project process such as a single phase (e.g. design)
- According to human resource skill requirements (e.g. plumbing)
- By major deliverable (e.g. training)

What is a project management?

Project management is concerned with the overall planning and co-ordination of a project from conception to completion aimed at meeting the stated requirements and ensuring completion on time, within cost and to required quality standards.

Project management is normally reserved for focused, non-repetitive, time-limited activities with some degree of risk and that are beyond the usual scope of operational activities for which the organization is responsible.



The various elements of project management life cycle are

a) Need identification
b) Initiation
c) Planning
d) Executing
e) Controlling.

a) Need Identification

The first step in the project development cycle is to identify components of the project. Projects may be identified both internally and externally:

Internal identification takes place when the energy manager identifies a package of energy saving opportunities during the day-to-day energy management activities, or from facility audits.

External identification of energy savings can occur through systematic energy audits undertaken by a reputable energy auditor or energy service company.

b) Initiation

Initiating is the basic processes that should be performed to get the project started. This starting point is critical because those who will deliver the project, those who will use the project, and those who will have a stake in the project need to reach an agreement on its initiation. Involving all stakeholders in the project phases generally improves the probability of satisfying customer requirements by shared ownership of the project by the stakeholders.

c) Planning

The planning phase is considered the most important phase in project management. Project planning defines project activities that will be performed; the products that will be produced, and describes how these activities will be accomplished and managed. Project planning defines each major task, estimates the time, resources and cost required, and provides a framework for management review and control. Planning involves identifying and documenting scope, tasks, schedules, cost, risk, quality, and staffing needs.

d) Executing

Once a project moves into the execution phase, the project team and all necessary resources to carry out the project should be in place and ready to perform project activities. The project plan is completed and base lined by this time as well. The project team and the project manager’s focus now shifts from planning the project efforts to participating, observing, and analyzing the work being done. In short, it means coordinating and managing the project resources while executing the project plan, performing the planned project activities, and ensuring they are completed efficiently.

e) Controlling

Project Control function that involves comparing actual performance with planned performance and taking corrective action to get the desired outcome when there are significant differences. By monitoring and measuring progress regularly, identifying variances from plan, and taking corrective action if required, project control ensures that project objectives are met.

f) Closing out

Project closeout is performed after all defined project objectives have been met and the customer has formally accepted the project’s deliverables and end product or, in some instances, when a project has been cancelled or terminated early. Although, project closeout is a routine process, it is an important one. By properly completing the project closeout, organizations can benefit from lessons learned and information compiled. The project closeout phase is comprised of contract closeout and administrative closure.

4.1.4 WORK BREAK DOWN STRUCTURE (WBS)

The purpose of work breakdown structure (WBS) is to help plan effectively for a project by breaking key tasks or activities down in to more manageable and smaller units of work.

WBS produces a detailed list of tasks to be performed for a project, helping to deliver better costing, scheduling and resource planning for a project.

Cost breakdown structure (CBS) lists every item classified and its expenditure for the project in order to get a more detailed estimate of cost or expenditure.

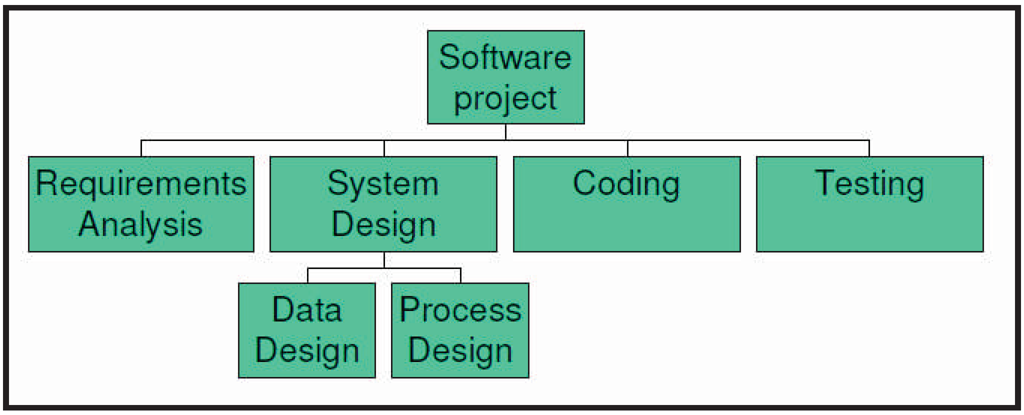
WBS contains a list of activities for a project derived from:
- Previous experience
- Expert brainstorming.

WBS helps in
- identifying the main activities
- break each main activity down into sub-activities which can further be broken down into lower level sub-activities.

WBS problems at times are:
- Too many levels
- Too few levels.

Approaches to creating WBS are:
- Phase based approach
- Product based approach
- Hybrid approach.

WBS Phase-based Approach (PA)



WBS PA Advantage

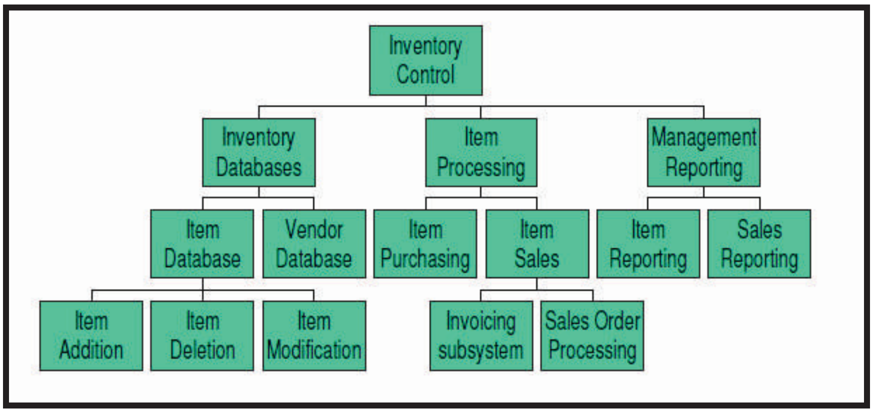
- Activity list likely complete and non-overlapping
- WBS gives a structure that can berefined as the project proceeds
- used for determining dependencies among activities

WBS PA Disadvantage

- May miss some activities related to final product

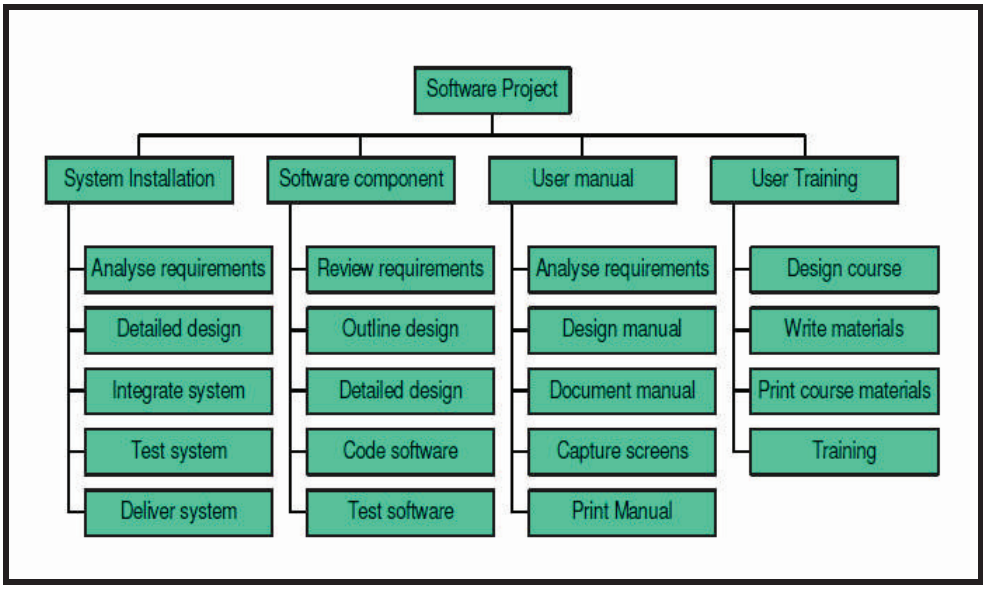
WBS - Product based approach (PBA)

Product Breakdown Structure (PBS) shows how a system can be broken down into different products for development.



WBS – Hybrid Approach (HA)

A mix of the phase-based and product based approaches (most commonly used). The WBS consists of a list of the products of the project and a list of phases for each product.



4.1.5 BASIC TOOLS AND TECHNIQUES OF PROJECT MANAGEMENT

Role of Network Technique in Project Management

Project management is a challenging task with many complex responsibilities. Fortunately, there are many tools available to assist with accomplishing the tasks and executing the responsibilities. Some require a computer with supporting software, while others can be used manually. Project managers should choose a project management tool that best suits their management style. No one tool addresses all project management needs.

Program Evaluation Review Technique (PERT) and Gantt Charts are two of the most commonly used project management tools and are described below. Both of these project management tools can be produced manually or with commercially available project management software.

PERT is a planning and control tool used for defining and controlling the tasks necessary to complete a project. PERT charts and Critical Path Method (CPM) charts are often used interchangeably; the only difference is how task times are computed. Both charts display the total project with all scheduled tasks shown in sequence. The displayed tasks show which ones are in parallel, those tasks that can be performed at the same time. A graphic representation called a "Project Network" or "CPM Diagram" is used to portray graphically the interrelationships of the elements of a project and to show the order in which the activities must be performed.

PERT planning involves the following steps:

a. *Identify the specific activities and milestones.* The activities are the tasks of the project. The milestones are the events that mark the beginning and the end of one or more activities.

b. *Determine the proper sequence of activities.* This step may be combined with #1 above since the activity sequence is evident for some tasks. Other tasks may require some analysis to determine the exact order in which they should be performed.

c. *Construct a network diagram.* Using the activity sequence information, a network diagram can be drawn showing the sequence of the successive and parallel activities. Arrowed lines represent the activities and circles or "bubbles" represent milestones.

d. *Estimate the time required for each activity.* Weeks are a commonly used unit of time for activity completion, but any consistent unit of time can be used. A distinguishing feature of PERT is it's ability to deal with uncertainty in activity completion times. For each activity, the model usually includes three time estimates:

- Optimistic time - the shortest time in which the activity can be completed.
- Most likely time - the completion time having the highest probability.
- Pessimistic time - the longest time that an activity may take.

From this, the expected time for each activity can be calculated using the following weighted average:

Expected Time = (Optimistic + 4 x Most Likely + Pessimistic) / 6

This helps to bias time estimates away from the unrealistically short timescales normally assumed.

e. *Determine the critical path.* The critical path is determined by adding the times for the activities in each sequence and determining the longest path in the project. The critical path determines the total calendar time required for the project. The amount of time that a non-critical path activity can be delayed without delaying the project is referred to as slack time.

If the critical path is not immediately obvious, it may be helpful to determine the following four times for each activity:

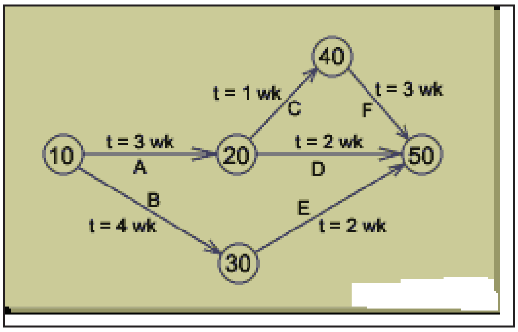
- ES - Earliest Start time
- EF - Earliest Finish time
- LS - Latest Start time
- LF - Latest Finish time

These times are calculated using the expected time for the relevant activities. The earliest start and finish times of each activity are determined by working forward through the network and determining the earliest time at which an activity can start and finish considering its predecessor activities. The latest start and finish times are the latest times that an activity can start and finish without delaying the project. LS and LF are found by working backward through the network. The difference in the latest and earliest finish of each activity is that activity's slack. The critical path then is the path through the network in which none of the activities have slack.

The variance in the project completion time can be calculated by summing the variances in the completion times of the activities in the critical path. Given this variance, one can calculate the probability that the project will be completed by a certain date assuming a normal probability distribution for the critical path. The normal distribution assumption holds if the number of activities in the path is large enough for the central limit theorem to be applied.

f. *Update the PERT chart as the project progresses.* As the project unfolds, the estimated times can be replaced with actual times. In cases where there are delays, additional resources may be needed to stay on schedule and the PERT chart may be modified to reflect the new situation. An example of a PERT chart is provided below:

g.



Benefits to using a PERT chart or the Critical Path Method include:

- Improved planning and scheduling of activities.
- Improved forecasting of resource requirements.
- Identification of repetitive planning patterns which can be followed in other projects, thus simplifying the planning process.
- Ability to see and thus reschedule activities to reflect interproject dependencies and resource limitations following know priority rules.
- It also provides the following: expected project completion time, probability of completion before a specified date, the critical path activities that impact completion time, the activities that have slack time and that can lend resources to critical path activities, and activity start and end dates.

Gantt chart

Gantt charts are used to show calendar time task assignments in days, weeks or months. The tool uses graphic representations to show start, elapsed, and completion times of each task within a project. Gantt charts are ideal for tracking progress. The number of days actually required to complete a task that reaches a milestone can be compared with the planned or estimated number. The actual workdays, from actual start to actual finish, are plotted below the scheduled days. This information helps target potential timeline slippage or failure points. These charts serve as a valuable budgeting tool and can show dollars allocated versus dollars spent.

To draw up a Gantt chart, follow these steps:

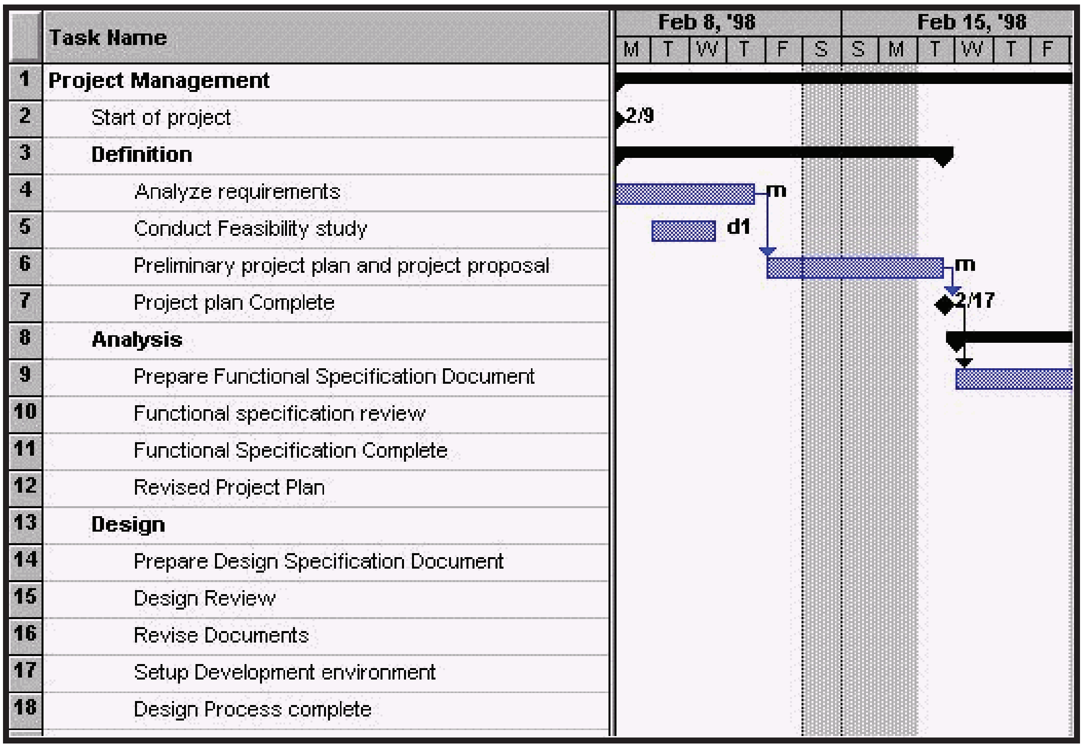
a. *List all activities in the plan.* For each task, show the earliest start date, estimated length of time it will take, and whether it is parallel or sequential. If tasks are sequential, show which stages they depend on.

b. *Head up graph paper with the days or weeks through completion.*

c. *Plot tasks onto graph paper.* Show each task starting on the earliest possible date. Draw it as a bar, with the length of the bar being the length of the task. Above the task bars, mark the time taken to complete them.

d. *Schedule activities.* Schedule them in such a way that sequential actions are carried out in the required sequence. Ensure that dependent activities do not start until the activities they depend on have been completed. Where possible, schedule parallel tasks so that they do not interfere with sequential actions on the critical path. While scheduling, ensure that you make best use of the resources you have available, and do not over-commit resources. Also, allow some slack time in the schedule for holdups, overruns, failures, etc.

e. *Presenting the analysis.* In the final version of your Gantt chart, combine your draft analysis (#3 above) with your scheduling and analysis of resources (#4 above). This chart will show when you anticipate that jobs should start and finish. An example of a Gantt chart is provided below:



Benefits of using a Gantt chart include:

- Gives an easy to understand visual display of the scheduled time of a task or activity.
- Makes it easy to develop "what if" scenarios.
- Enables better project control by promoting clearer communication.
- Becomes a tool for negotiations.
- Shows the actual progress against the planned schedule.
- Can report results at appropriate levels.
- Allows comparison of multiple projects to determine risk or resource allocation.
- Rewards the project manager with more visibility and control over the project.

At the end of this unit the learners will be able to

- Project Planning, Scheduling and Controlling
- Work Break Down Structure
- Basic Tools and Techniques of Project Management
- Role of Network Technique in Project Management.

4.1.6 LET US SUM UP

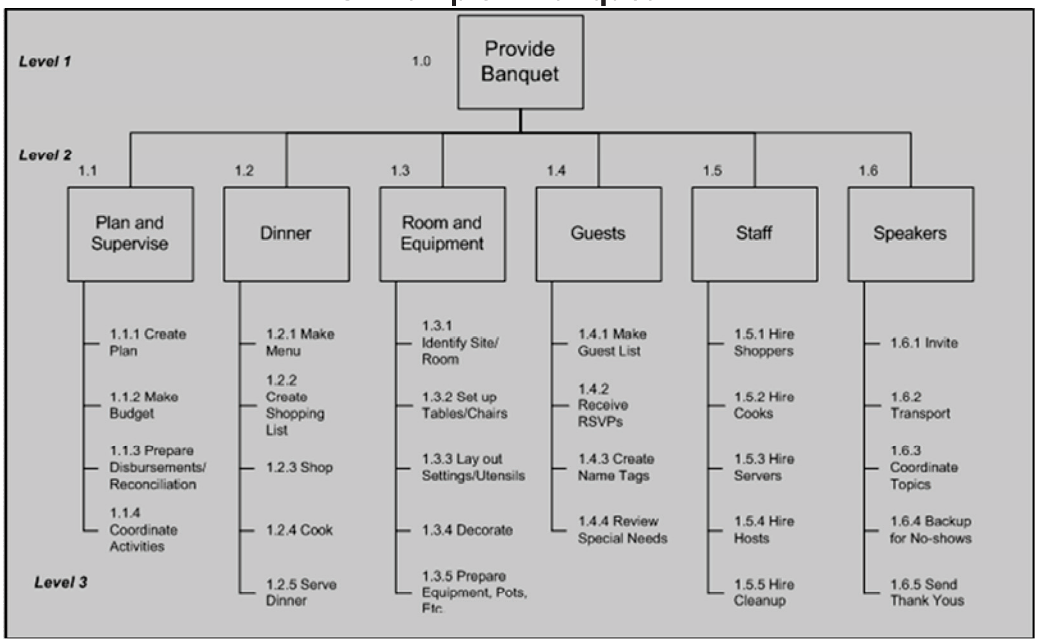
In this unit you have learnt Project Planning, Scheduling and Controlling, Work Break down Structure, Basic Tools and Techniques of Project Management and Role of Network Technique in Project Management.

4.1.7 EXERCISES

Question 1. Discuss with your own examples project planning and scheduling techniques.

Question 2. An example of the work break-down structure (WBS) diagram is shown.

WBS Example – Banquet



In a WBS, every level item has a unique assigned number so that work can be identified and tracked over time. A WBS may have varying numbers of decomposition levels, but there is a general scheme for how to number each level so that tasks are uniquely numbered and correctly summarized. Below is the general convention for how tasks are decomposed:

- **Level 1** – Designated by 1.0. This level is the top level of the WBS and is usually the project name. All other levels are subordinate to this level.
- **Level 2** – Designated by 1.X (e.g., 1.1, 1.2). This level is the summary level.

- **Level 3** – Designated by 1.X.X (e.g., 1.1.1, 1.1.2). This third level comprises the subcomponents to each level 2 summary element. This effort continues down until progressively subordinate levels are assigned for all work required for the entire project.

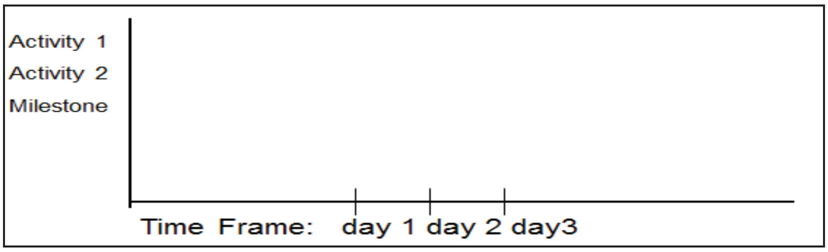
Please name the different hierarchical levels in the above WBS diagram.

Question 3. Draw the Gantt Chart for the following activities:

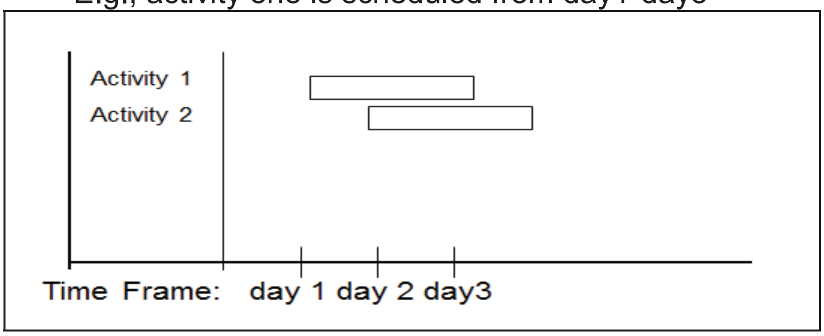
Number	Activity	Predecessor	Duration
1	Design house and obtain financing	--	3 months
2	Lay foundation	1	2 months
3	Order and receive materials	1	1 month
4	Build house	2,3	3 months
5	Select paint	2, 3	1 month
6	Select carper	5	1 month
7	Finish work	4, 6	1 month

Suggested Steps

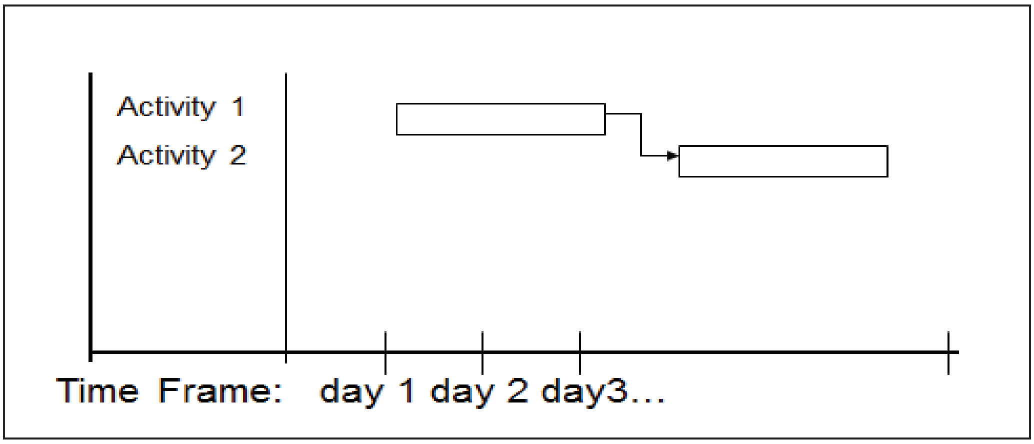
- List all tasks and milestones from the project along the vertical axis
- List time frame along the horizontal axis



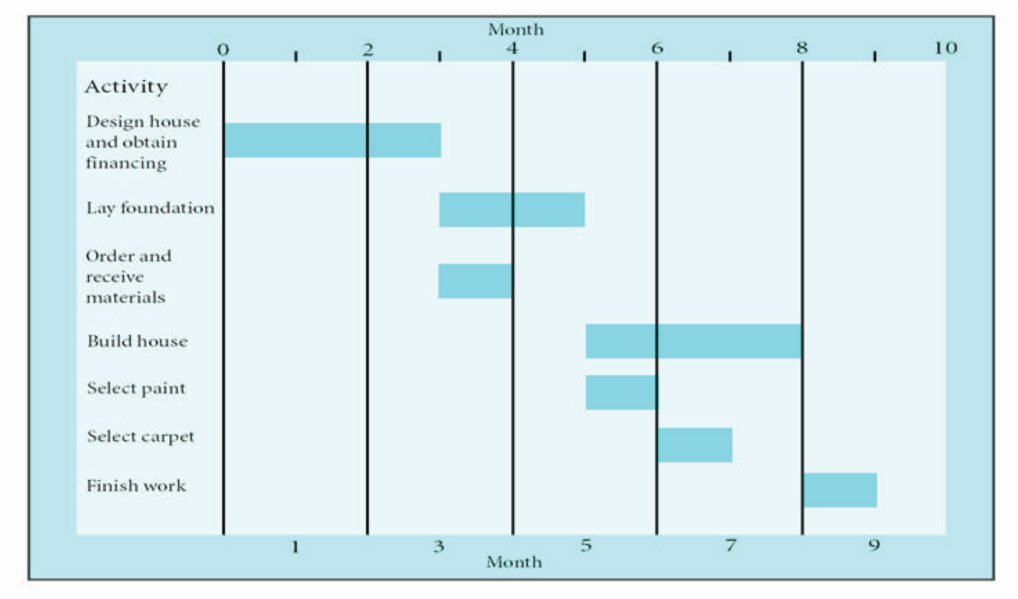
- Activities: Create box the length of each activity time duration – E.g., activity one is scheduled from day1-day3



- Dependencies: Show dependencies between activities with arrows
  - E.g., activity 2 cannot start until activity 1 is complete



Answer to the question:



4.1.8 SUGGESTED READINGS

Network Analysis section in any of the reference / text books

❖❖❖❖

4.2

CONCEPT OF NETWORK

Concept of Network or Arrow Diagram, Activity on Node Diagram, Critical Path Method

Unit Structure

4.2.1 INTRODUCTION

In this Unit-IV - Chapter 4.2, we shall discuss the concept of Project Network or Arrow Diagram, Activity on Node Diagram, Critical Path Method.

The project network arrow diagram shows the required order of tasks in a project or process, the best schedule for the entire project, and potential scheduling and resource problems and their solutions. The arrow diagram lets you calculate the “critical path” of the project. This is the flow of critical steps where delays will affect the timing of the entire project and where addition of resources can speed up the project.

4.2.2 OBJECTIVES

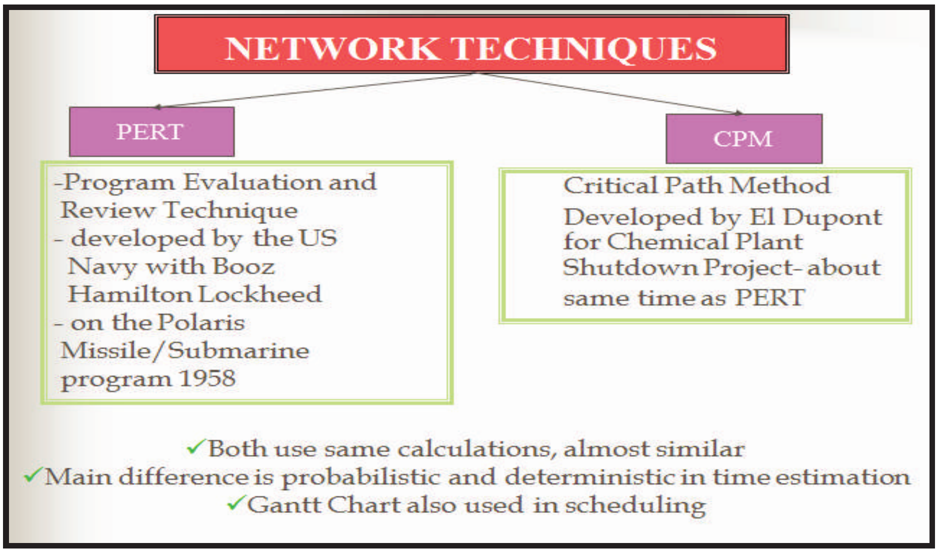
At the end of this unit the learners will be able to

- Draw a project network
- Calculate different times estimates for different events on the network activities
- Find the critical path

4.2.3 CONCEPT OF NETWORK OR ARROW DIAGRAM, ACTIVITY ON NODE DIAGRAM, CRITICAL PATH METHOD

Project schedule converts action plan into operating time table and forms the basis for monitoring and controlling project.

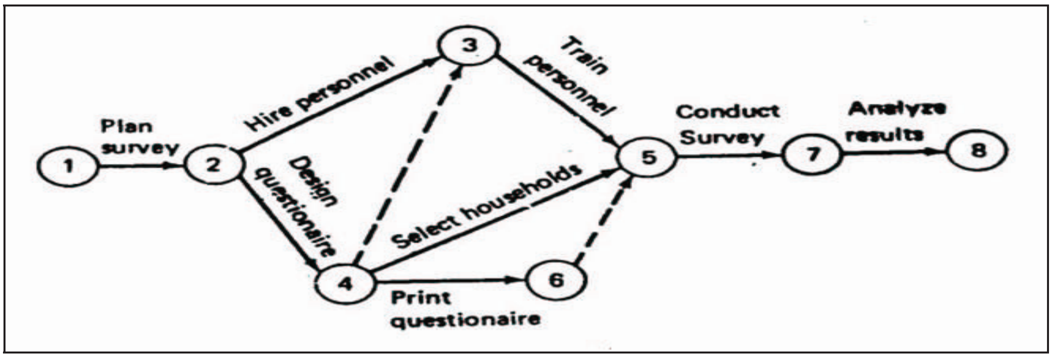
Scheduling is more important in projects than in production, because of the unique nature of the projects.



A network is:

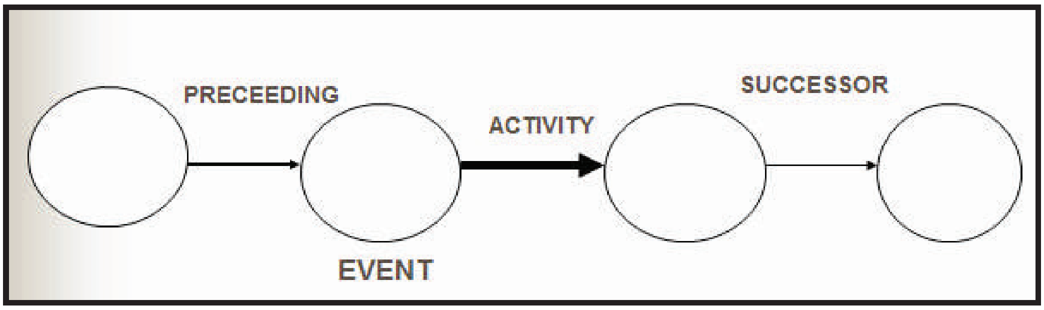
- Graphical portrayal of activities and event
- Shows dependency relationships between tasks/activities in a project
- Clearly shows tasks that must precede (precedence) or follow (succeeding) other tasks in a logical manner
- Clear representation of plan – a powerful tool for planning and controlling project

Example of a Simple Network – Survey



Definition of Terms in a Network

- Activity: any portions of project (tasks) which required by project, uses up resource and consumes time – may involve labor, paper work, contractual negotiations, machinery operations Activity on Arrow (AOA) showed as arrow, AON – Activity on Node
- Event: beginning or ending points of one or more activities, instantaneous point in time, also called 'nodes'
- Network: Combination of all project activities and the events



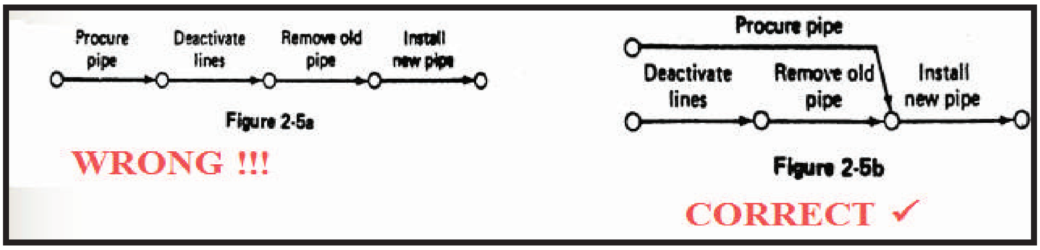
Emphasis on Logic in Network Construction

Construction of network should be based on logical or technical dependencies among activities

Example

Before activity 'Approve Drawing' can be started, the activity 'Prepare Drawing' must be completed

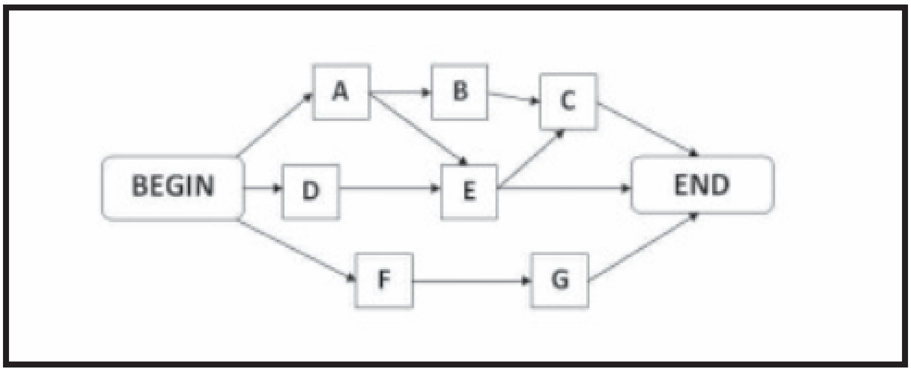
Build network on the basis of time logic (a feeling for proper sequence) see example below:



Activity on Node Diagram

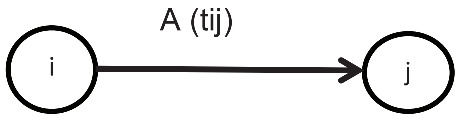
Activity-on-node is a project management term that refers to a precedence diagramming method which uses boxes to denote schedule activities. These various boxes or "nodes" are connected from beginning to end with arrows to depict a logical progression of the dependencies between the schedule activities. Each node is coded with a letter or number that correlates to an activity on the project schedule.

Typically, an activity-on-node diagram will be designed to show which activities must be completed in order for other activities to commence. This is referred to as "finish-to-start" precedence – meaning one activity must be finished before the next one can start. In the diagram below, activities A and D must be done so that activity E can begin. It is also possible to create other variations of this type of diagram. For example, a "start-to-start" diagram is one in which a predecessor activity must simply be started rather than fully completed in order for the successor activity to be initiated.



An activity-on-node diagram can be used to provide a visual representation of the network logic of an entire project schedule. Or, it can be used for any smaller section of the schedule that lends itself to being represented as having a defined beginning and end. To keep the logic in the diagram simple, it may be most effective to include only critical path schedule activities. The planned start date of each node may also be listed in the diagram legend in accordance with the project management timeline.

But in the network analysis an activity is indicated by an arrow with circles at the start and at the end of the arrow as follows.



The activity is named by an alphabet that is placed on the top of the arrow. The beginning of an activity which is the tail of the arrow is the start event of that activity; the start event is denoted by a circle with a number inside it. The ending of an activity which is the head of the arrow is the end event of that activity; the end event is denoted by a circle with a number inside it. The progress of an activity is from the tail to the head of the activity arrow.

In the above figure activity i-j is named as A, i is the start event and j is the end event of that activity, (tij) is the time duration of the activity A.

Example 1- A simple network

Consider the list of four activities for making a simple product.

Activity	Description	Immediate predecessors
A	Buy Plastic Body	-
B	Design Component	-
C	Make Component	B
D	Assemble product	A,C

Immediate predecessors for a particular activity are the activities that, when completed, enable the start of the activity in question.

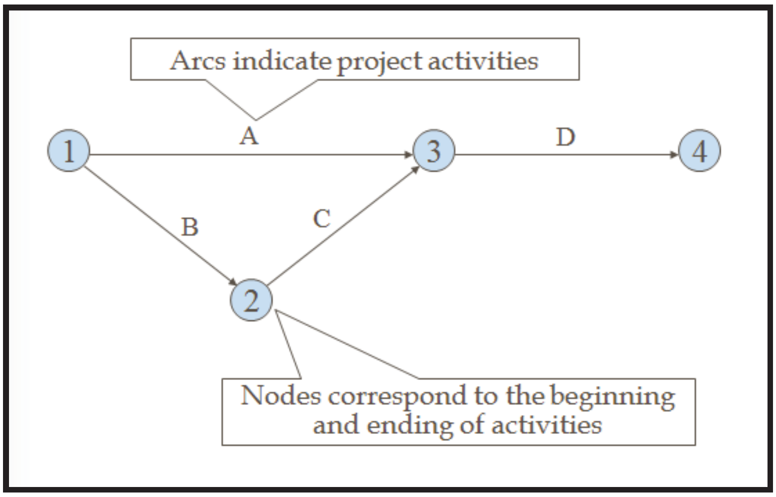
Can start work on activities A and B anytime, since neither of these activities depends upon the completion of prior activities.

Activity C cannot be started until activity B has been completed.

Activity D cannot be started until both activities A and C have been completed.

The graphical representation (next slide) is referred to as the PERT/CPM network.

Example 1 - Network of Four Activities



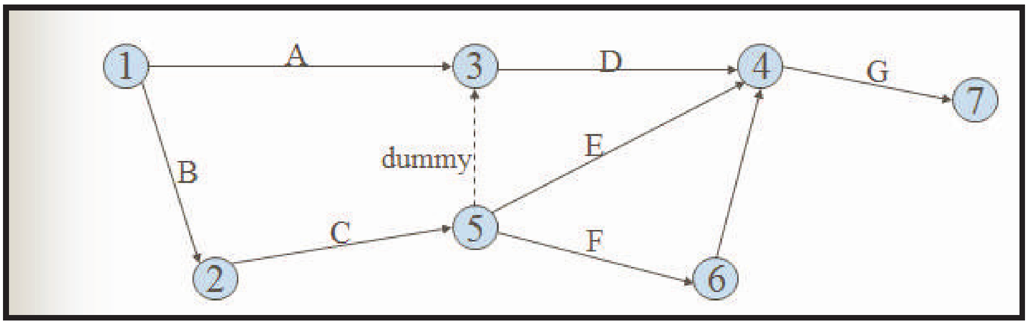
Example 2

Develop the network for a project with following activities and immediate predecessors:

Activity	Immediate predecessors
A	-
B	-
C	B
D	A, C
E	C
F	C
G	D,E,F

Try to do for the first five (A,B,C,D,E) activities

Example 2 – Network of Seven Activities

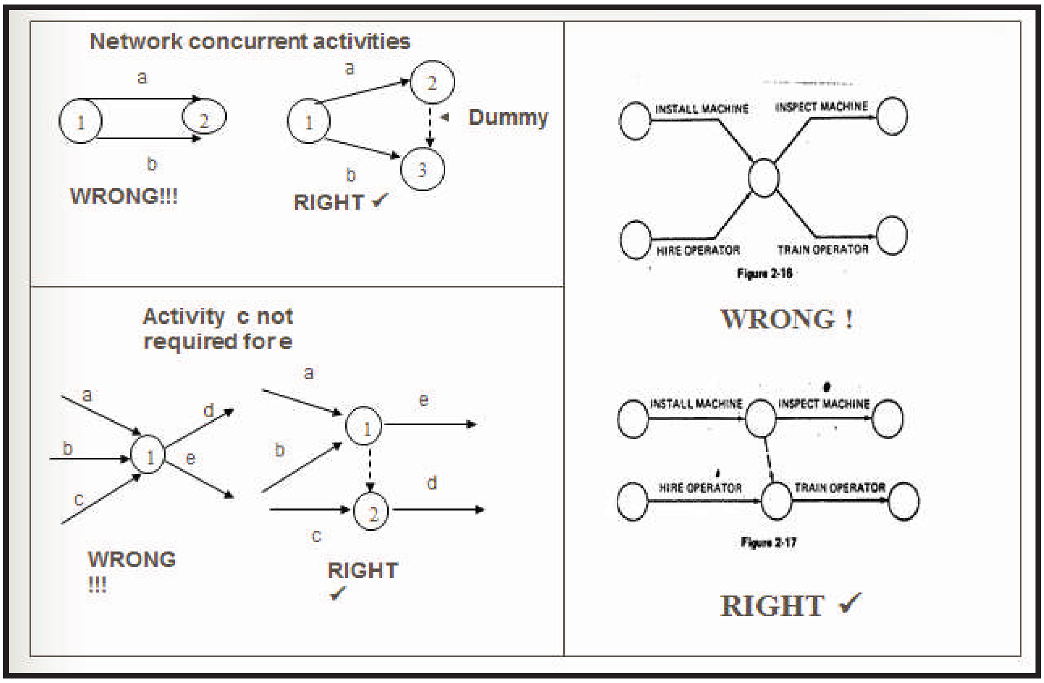


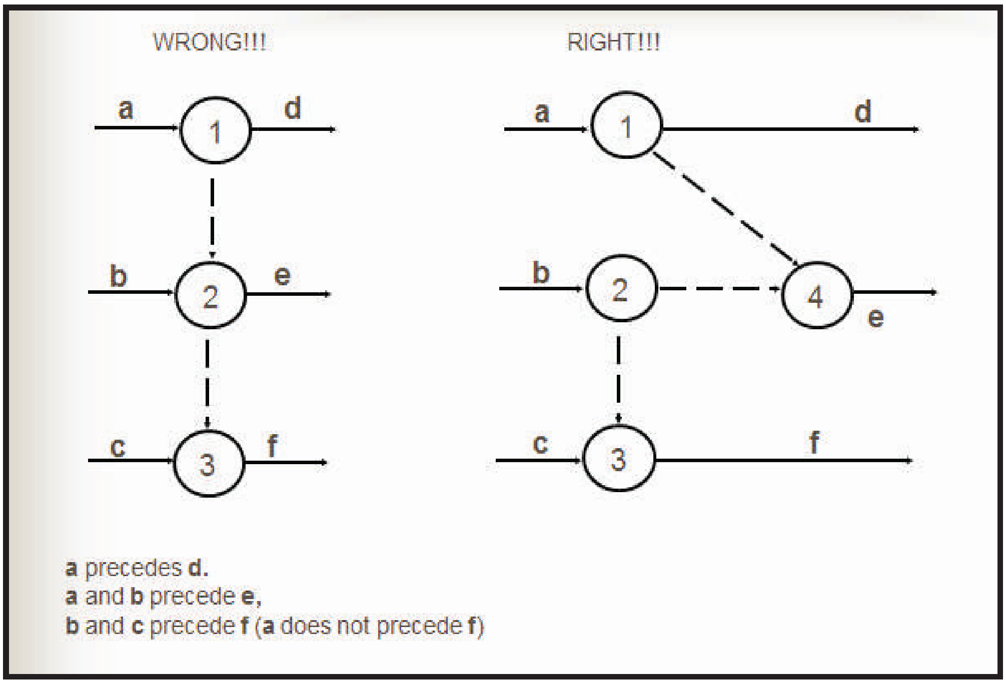
Note how the network correctly identifies D, E, and F as the immediate predecessors for activity G.

Dummy activities are used to identify precedence relationships correctly and to eliminate possible confusion of two or more activities having the same starting and ending nodes.

Dummy activities have no resources (time, labor, machinery, etc), their purpose is to preserve the logic of the network

Examples of the Use of Dummy Activity





Scheduling with activity time

Activity	Immediate predecessors	Completion Time (week)
A	-	5
B	-	6
C	A	4
D	A	3
E	A	1
F	E	4
G	D,F	14
H	B,C	12
I	G,H	2
	Total	51

This information indicates that the total time required to complete activities is 51 weeks. However, we can see from the network that several of the activities can be conducted simultaneously (A and B, for example).

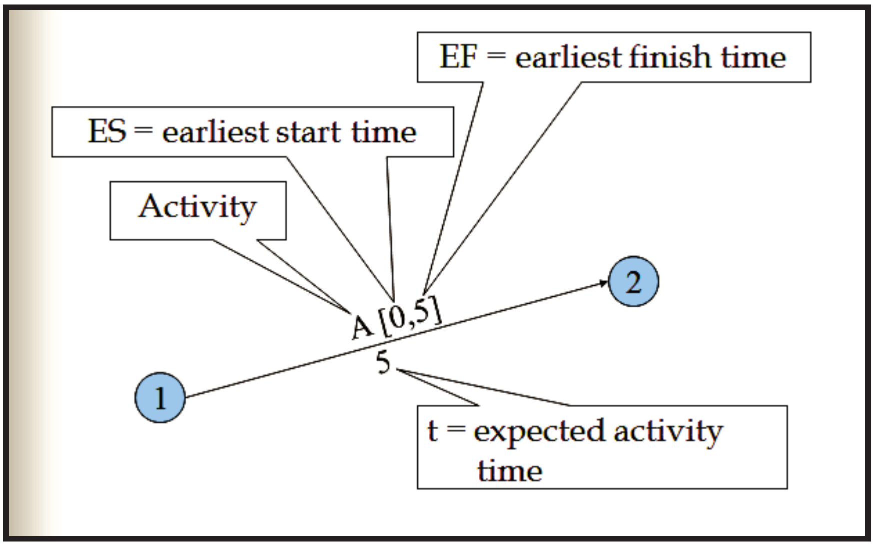
Earliest Start & Earliest Finish Time

We are interested in the longest path through the network, i.e., the critical path. Starting at the network's origin (node 1) and using a starting time of 0, we compute an earliest start (ES) and earliest finish (EF) time for each activity in the network.

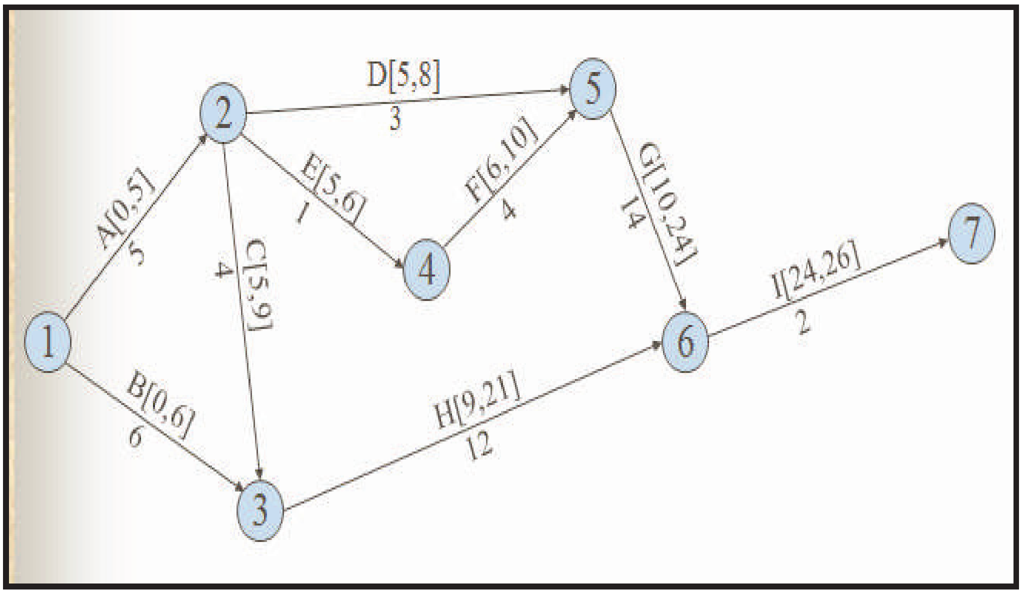
The expression EF = ES + t can be used to find the earliest finish time for a given activity.

For example, for activity A, ES = 0 and t = 5; thus the earliest finish time for activity A is EF = 0 + 5 = 5

Arc with ES & EF time



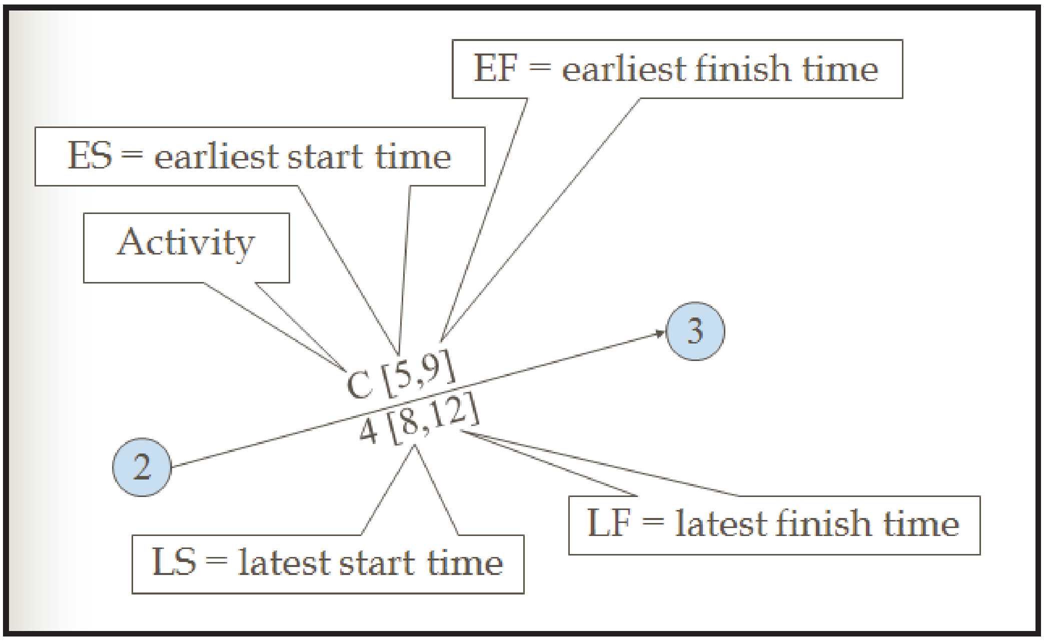
Network with ES & EF time



Earliest start time rule

The earliest start time for an activity leaving a particular node is equal to the largest of the earliest finish times for all activities entering the node.

Activity, duration, ES, EF, LS, LF



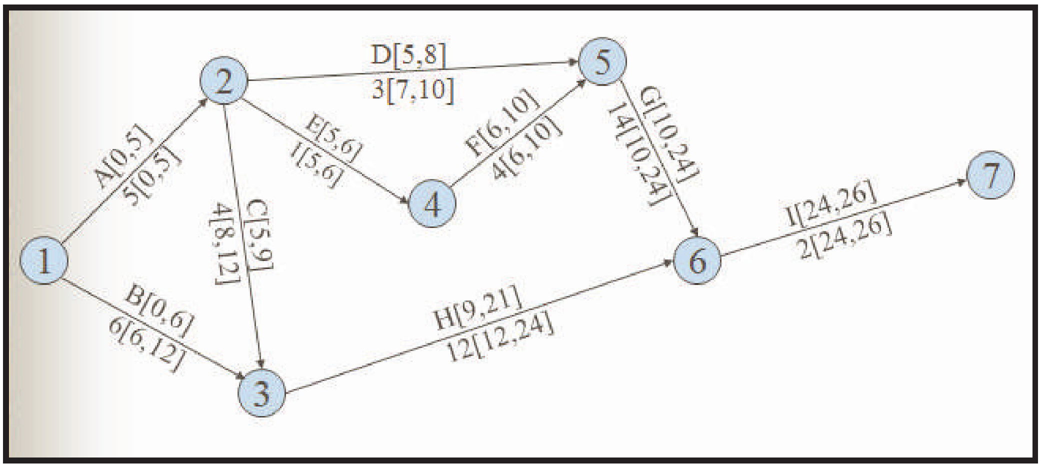
Latest start & latest finish time

To find the critical path we need a backward pass calculation.

Starting at the completion point (node 7) and using a *latest finish time* (LF) of 26 for activity I, we trace back through the network computing a *latest start* (LS) and latest finish time for each activity.

The expression LS = LF – t can be used to calculate latest start time for each activity. For example, for activity I, LF = 26 and t= 2, thus the latest start time for activity I is LS = 26 – 2 = 24.

Network with LS & LF time



Latest finish time rule

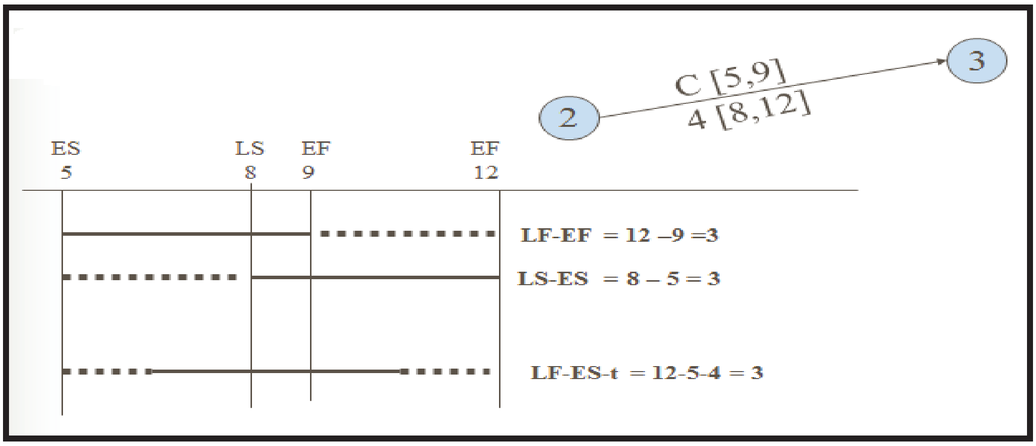
The latest finish time for an activity entering a particular node is equal to the smallest of the latest start times for all activities leaving the node.

Slack or Free Time or Float

Slack is the length of time an activity can be delayed without affecting the completion date for the entire project.

For example, slack for C = 3 weeks, i.e. Activity C can be delayed up to 3 weeks

(Start, anywhere between weeks 5 and 8).



Activity schedule for the example

Activity	Earliest start (ES)	Latest start (LS)	Earliest finish (EF)	Latest finish (LF)	Slack (LS-ES)	Critical path
A	0	0	5	5	0	Yes
B	0	6	6	12	6	
C	5	8	9	12	3	
D	5	7	8	10	2	
E	5	5	6	6	0	Yes
F	6	6	10	10	0	Yes
G	10	10	24	24	0	Yes
H	9	12	21	24	3	
I	24	24	26	26	0	Yes

Important Questions

1. What is the total time to complete the project?

   26 weeks if the individual activities are completed on schedule.

2. What are the scheduled start and completion times for each activity?

   ES, EF, LS, LF are given for each activity.

3. What activities are *critical* and must be completed as scheduled in order to keep the project on time?

   Critical path activities: A, E, F, G, and I.

4. How long can *non-critical* activities be delayed before they cause a delay in the project's completion time

   Slack time that is available for all activities are given.

Critical Path Method

Slack or Float shows how much allowance each activity has, i.e how long it can be delayed without affecting completion date of project

Critical path is a sequence of activities from start to finish with zero slack. Critical activities are activities on the critical path.

Critical path identifies the minimum time to complete project If any activity on the critical path is shortened or extended, project time will be shortened or extended accordingly.

So, a lot of effort should be put in trying to control activities along this path, so that project can meet due date. If any activity is lengthened, be aware that project will not meet deadline and some action needs to be taken.

If can spend resources to speed up some activity, do so only for critical activities.

Don't waste resources on non-critical activity; it will not shorten the project time.

If resources can be saved by lengthening some activities, do so for non-critical activities, up to limit of float.

Total Float belongs to the path.

4.2.4 LET US SUM UP

In this chapter you have learnt how to draw a network and find the total project duration and the critical path.

4.2.5 EXERCISES

Question 1: Table below shows the activities within a small project.

Activity	Start node	End node	Completion time (weeks)
1	1	2	2
2	1	3	3
3	1	4	2
4	2	5	3
5	3	6	7
6	4	6	5
7	5	7	4
8	6	7	9
9	7	8	3

- Draw the network diagram.
- Calculate the minimum overall project completion time.
- Calculate the float time for each activity and hence identify the activities which are critical.

Ans: The critical activities (those with a float of zero) are 2,5,8,9 and 10 and these form the critical path from the start node (node 1) to the finish node (node 8) in the network. The overall project completion time is 22 weeks. Drawing a network is left as an exercise.

Question 2: Table below shows the activities within a small project.

Activity	Start node	End node	Completion time (weeks)
1	1	2	2
2	1	3	3
3	3	4	7
4	3	5	6
5	2	7	10
6	4	5	1
7	5	6	3
8	6	7	4
9	4	7	7
10	7	8	3

- Draw the network diagram.
- Calculate the minimum overall project completion time.
- Calculate the float time for each activity and hence identify the activities which are critical.

Ans: The critical activities (those with a float of zero) are 2,3,6,7,8,10 and 13 forming the critical path and the total project duration is 21 weeks. Drawing a network is left as an exercise.

4.2.6 SUGGESTED READINGS

Network Analysis section in any of the reference / text books

❖❖❖❖

4.3

CONCEPT OF PERT/CRASHING

Concept of PERT, Concept of CPM, Cost Analysis and Crashing the Network

Unit Structure

4.3.1 INTRODUCTION

The program (or project) evaluation and review technique, commonly abbreviated PERT, is a statistical tool, used in project management, which was designed to analyze and represent the tasks involved in completing a given project. First developed by the United States Navy in the 1950s, it is commonly used in conjunction with the critical path method (CPM).

4.3.2 OBJECTIVES

In this unit you will learn the following:

- Concept of PERT
- Concept of CPM and
- Cost analysis and crashing the network.

4.3.3 CONCEPT OF PERT

The Navy's Special Projects Office, charged with developing the Polaris-Submarine weapon system and the Fleet Ballistic Missile capability, has developed a statistical technique for measuring and forecasting progress in research and development programs. This program evaluation and review technique (code-named PERT) is applied as a decision-making tool designed to save time in achieving end-objectives, and is of particular interest to those engaged in research and development programs for which time is a critical factor.

The new technique takes recognition of three factors that influence successful achievement of research and development program objectives: time, resources, and technical performance specifications. PERT employs time as the variable that reflects planned resource-applications and performance specifications. With units of time as a common denominator, PERT quantifies knowledge about the uncertainties involved in developmental programs requiring effort at the edge of, or beyond, current knowledge of the subject — effort for which little or no previous experience exists.

Through an electronic computer, the PERT technique processes data representing the major, finite accomplishments (events) essential to achieve end-objectives; the inter-dependence of those events; and estimates of time and range of time necessary to complete each activity between two successive events. Such time expectations include estimates of "most likely time", "optimistic time", and "pessimistic time" for each activity. The technique is a management control tool that sizes up the outlook for meeting objectives on time; highlights danger signals requiring management decisions; reveals and defines both methodical ness and slack in the flow plan or the network of sequential activities that must be performed to meet objectives; compares current expectations with scheduled completion dates and computes the probability for meeting scheduled dates; and simulates the effects of options for decision — before decision.

The concept of PERT was developed by an operations research team staffed with representatives from the Operations Research Department of Booz, Allen and Hamilton; the Evaluation Office of the Lockheed Missile Systems Division; and the Program Evaluation Branch, Special Projects Office, of the Department of the Navy.

PERT is a method of analyzing the tasks involved in completing a given project, especially the time needed to complete each task, and to identify the minimum time needed to complete the total project.

PERT was developed primarily to simplify the planning and scheduling of large and complex projects. It was developed for the U.S. Navy Special Projects Office in 1957 to support the U.S. Navy's Polaris nuclear submarine project. It was able to incorporate uncertainty by making it possible to schedule a project while not knowing precisely the details and durations of all the activities. It is more of an event-oriented technique rather than start- and completion-oriented, and is used more in projects where time is the major factor rather than cost. It is applied to very large-scale, one-time, complex, non-routine infrastructure and Research and Development projects. An example of this was for the 1968 Winter Olympics in Grenoble which applied PERT from 1965 until the opening of the 1968 Games.

This project model was the first of its kind, a revival for scientific management, founded by Frederick Taylor (Taylorism) and later refined by Henry Ford (Fordism). DuPont's critical path method was invented at roughly the same time as PERT.

4.3.4 CONCEPT OF CPM

The critical path method (CPM) is a step-by-step methodology, technique or algorithm for planning projects with numerous activities that involve complex, interdependent interactions. CPM is an important tool for project management because it identifies critical and non-critical tasks to prevent conflicts and bottlenecks. CPM is often applied to the analysis of a project network logic diagram to produce maximum practical efficiency.

CPM is commonly employed in many diverse types of projects. These include product development, engineering, construction, aerospace and defense, software development and research projects. Several CPM software solutions are available.

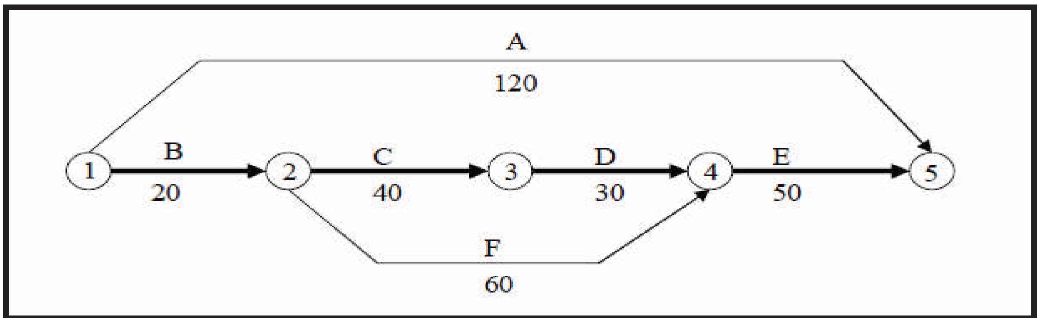
The basic steps employed in CPM are:

1. Determine required tasks
2. List required tasks in sequence
3. Create a flowchart including each required task
4. Identify all critical and non-critical relationships (paths) among required tasks
5. Assign an expected completion/execution time for each required task
6. Study all critical relationships to determine all possible alternatives or backups for as many as possible.

Often a major objective in CPM is to complete the project in the shortest time possible. One way to do this is called fast tracking, which involves performing activities in parallel (simultaneously) and adding resources to shorten critical path durations (called crashing the critical path). This may result in expansion, which leads to increasing project complexity, duration or both.

4.3.5 COST ANALYSIS AND CRASHING THE NETWORK

Crashing Example: The network and durations given below shows the normal schedule for a project. You can decrease (crash) the durations at an additional expense. The Table given below summarizes the time-cost information for the activities. The owner wants you to you to finish the project in 110 days. Find the minimum possible cost for the project if you want to finish it on 110 days. (Assume that for each activity there is a single linear, continuous function between the crash duration and normal duration points).



Activity	Normal duration (days)	Crash Duration (days)	Normal Cost	Crash Cost
A	120	100	12000	14000
B	20	15	1800	2800
C	40	30	16000	22000
D	30	20	1400	2000
E	50	40	3600	4800
F	60	45	13500	18000

Assume that the duration-cost relationship for each activity is a single linear, continuous function between the crash duration and normal duration points. Using the normal duration(ND), crash duration (CD), normal cost (NC), and crash cost (CC), the crash cost slope for each activity can be determined as follows:

$$S_A = \frac{CC - NC}{ND - CD}$$

$$S_A = \frac{14000 - 12000}{120 - 100} = \$100/day$$

$S_B = \$200/day$

$S_C = \$600/day$

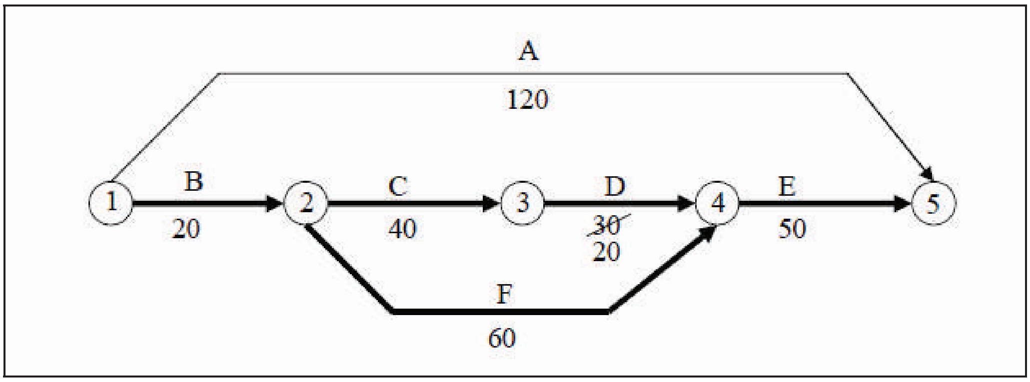
$S_D = \$60/day$

$S_E = \$120/day$

$S_F = \$300/day$

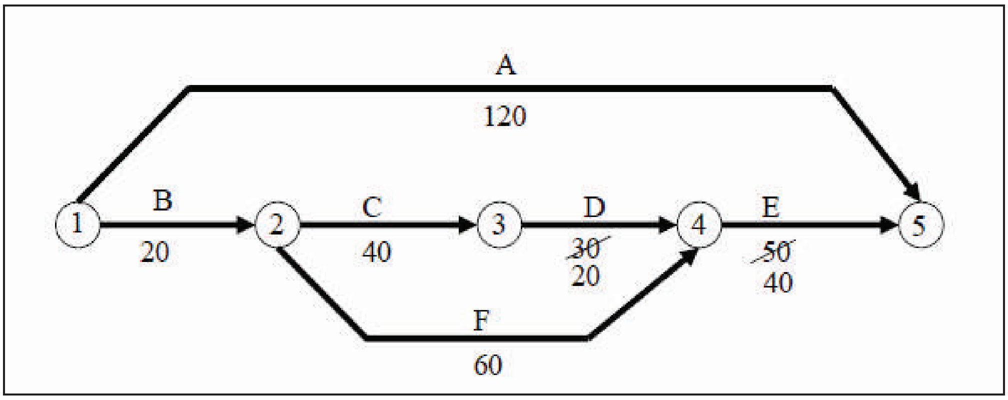
The normal cost for the project is the sum of a normal cost for each activity. The normal cost for the project is $48300 and the normal duration is 140 days. The activity which should be crashed is the one on the critical path which will add the least amount to the overall project cost. This will be the activity with the flattest or least-cost slope. The duration can be reduced as long as the critical path is not changed or a new critical path is created. In addition, the activity duration cannot be less than the crash duration.

SD = $60/day (least-cost slope) Maximum of 10 days can be cut from this schedule by reducing the duration of activity D to the crash duration of 20 days.



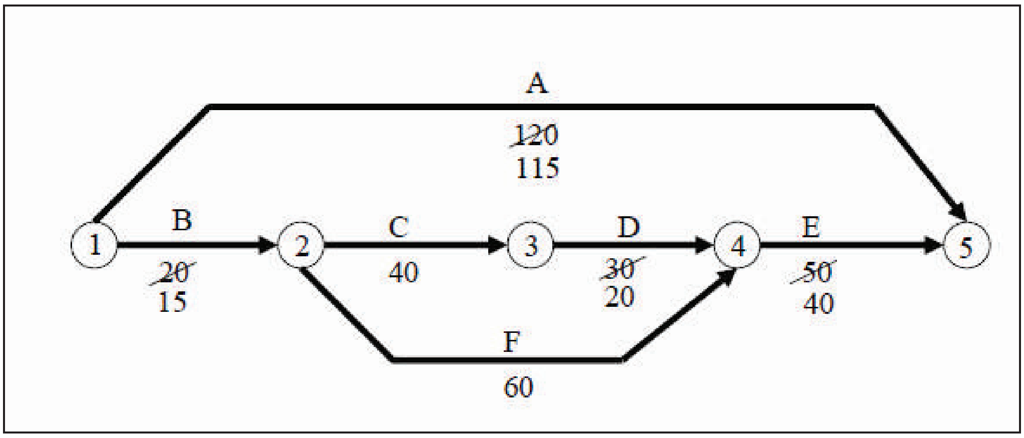
Overall duration is 130 days and there are multiple critical paths (B-F-E and B-C-D-E). Total project cost at this duration is the normal cost of $48300 plus the cost of crashing the activity by 10 days (60 * 10 = $600) for a total of $48900.

The next activity to be crashed would be the activity E, since it has the least-cost slope ($120per day) of any of the activities on the critical path. Activity E can be crashed by a total of 10days. Crashing the activity E by 10 days will cost an additional $120 per day or $1200.

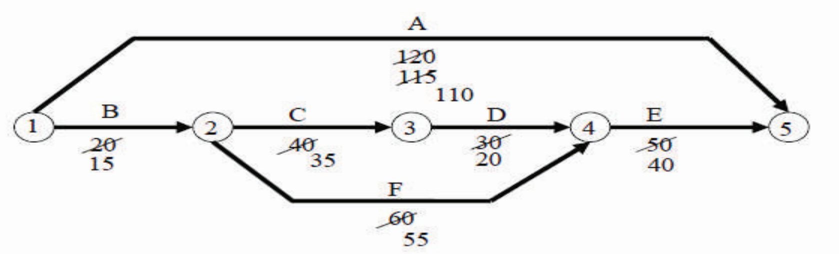


The project duration is now 120 days and the total project cost is $50100. There are now three critical paths (A, B-C-D-E, and B-F-E). The next stage of crashing requires a more through analysis since it is impossible to crash one activity alone and achieve a reduction in the overall project duration. Activity A is paired with each of the other activities to determine which has the least overall cost slope for those activities which have remaining days to be crashed.
Activity A ($100) + activity B ($200)
Activity A ($100) + activity C ($600) + activity F ($300)

The least-cost slope will be activity A + activity B for a cost increase of $300 per day. Reducing the project duration by 5 days will add 5*300 = $1500 dollar crashing cost and the total project cost would be $51600. Activity B cannot be crashed any more.



Final step in crashing the project to 110 days would be accomplished by reducing the duration of activity A by 5 days to 110 days, reducing activity C by 5 days to 35 days, and reducing activity F by 5 days to 55 days. The combined cost slope for the simultaneous reduction of activity A, activity C, and activity F would be $1000 per day. For 5 days of reduction this would be an additional $5000 in total project cost. The total project cost for the crashed schedule to 110 days of duration would be $56600.



4.3.6 LET US SUM UP

In this unit you have understood the concept of PERT and CPM and the method of cost analysis and crashing the network has been explained to you.

4.3.7 EXERCISES

Question 1: You are given the following data about the project tasks, network, and crash times/costs. Calculate the cost of the project at all-time durations until you can no longer crash the project any further.

ID	Direct costs				Slope	Maximum Crash Time
	Normal		Crash			
	Time	Cost	Time	Cost		
A	5	$500	4	$600	**$100**	**1**
B	10	$1200	6	$2000	**$200**	**4**
C	13	$3600	11	$4800	**$600**	**2**
D	13	$300	11	$600	**$150**	**2**
E	5	$1000	4	$1400	**$400**	**1**
F	10	$2400	8	$5400	**$1500**	**2**
G	5	$700	5	$700	**$0**	**0**
		$9700				

Answer is not given as the question is left as an exercise.

Question 2:

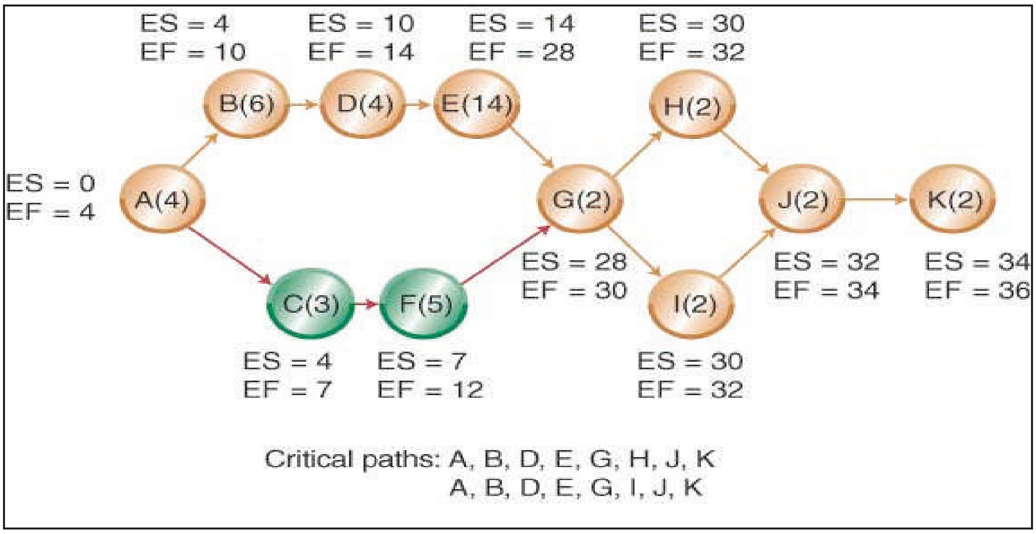
Crash the project activities given in the following table:

Activity	Normal Time (wk)	Normal Cost ($)	Crash Time	Crash Cost ($)	Max. weeks of reduction	Reduce cost per week
A	4	8,000	3	11,000	1	3,000
B	6	30,000	5	35,000	1	5,000
C	3	6,000	3	6,000	0	0
D	6	24,000	4	28,000	2	2,000
E	14	60,000	12	72,000	2	6,000
F	5	5,000	4	6,500	1	1500
G	2	6,000	2	6,000	0	0
H	2	4,000	2	4,000	0	0
I	3	4,000	2	5,000	1	1,000
J	4	4,000	2	6,400	2	1,200
K	2	5,000	2	5,000	0	0

Answer:

Suppose the project manager wants to reduce the new product project from 41 to 36 weeks:

- Crashing Costs are considered to be linear
- Look to crash activities on the critical path
- Crash the least expensive activities on the critical path first (based on cost per week)
  a. Crash activity I from 3 weeks to 2 weeks $1000
  b. Crash activity J from 4 weeks to 2 weeks $2400
  c. Crash activity D from 6 weeks to 4 weeks <u>$4000</u>
  d. <u>Recommend Crash Cost</u> $7400
- Will crashing <u>5 weeks</u> return more than it costs?



4.3.8 SUGGESTED READINGS

Network Analysis section in any of the reference / text books

❖❖❖❖

MODULE – V
GAME THEORY

5.1

INTRODUCTION TO THEORY OF GAMES

Introduction to Theory of Games, Characteristics of Games, Game Models

Unit Structure

5.1.1 INTRODUCTION

Game theory is "the study of mathematical models of conflict and cooperation between intelligent rational decision-makers." Game theory is mainly used in economics, political science and psychology as well as in logic, computer science and biology. Originally, it addressed zero-sum games, in which one person's gains result in losses for the other participants. Today, game theory applies to a wide range of behavioral relations, and is now an umbrella term for the science of logical decision making in humans, animals, and computers.

This theory was developed extensively in the 1950s by many scholars. Game theory was later explicitly applied to biology in the 1970s, although similar developments go back at least as far as the 1930s. Game theory has been widely recognized as an important tool in many fields.

Game theory is a tool used to analyze strategic behavior by taking into account how participants expect others to behave.

Game theory is used to find the optimal outcome from a set of choices by analyzing the costs and benefits to each independent party as they compete with each other.

5.1.2 OBJECTIVES

After studying this Unit – V Chapter 5.1, you will be able to understand the following:

- Introduction to Theory of Games
- Characteristics of Games
- Characteristics of Game Models.

5.1.3 INTRODUCTION TO THEORY OF GAMES

Game theory is the formal study of conflict and cooperation. Game theoretic concepts apply whenever the actions of several agents are interdependent. These agents may be individuals, groups, firms, or any combination of these. The concepts of game theory provide a language to formulate structure, analyze, and understand strategic scenarios.

The earliest example of a formal game-theoretic analysis is the study of a duopoly by Antoine Cournot in 1838. The mathematician Emile Borel suggested a formal theory of games in 1921, which was furthered by the mathematician John von Neumann in 1928 in a “theory of parlor games.” Game theory was established as a field in its own right after the 1944 publication of the monumental volume Theory of Games and Economic Behavior by von Neumann and the economist Oskar Morgenstern. This book provided much of the basic terminology and problem setup that is still in use today.

In 1950, John Nash demonstrated that finite games have always had an equilibrium point, at which all players choose actions which are best for them given their opponents’ choices. This central concept of non-cooperative game theory has been a focal point of analysis since then. In the 1950s and 1960s, game theory was broadened theoretically and applied to problems of war and politics. Since the 1970s, it has driven a revolution in economic theory. Additionally, it has found applications in sociology and psychology, and established links with evolution and biology. Game theory received special attention in 1994 with the awarding of the Nobel Prize in economics to Nash, John Harsanyi, and ReinhardSelten.

At the end of the 1990s, a high-profile application of game theory has been the design of auctions. Prominent game theorists have been involved in the design of auctions for allocating rights to the use of bands of the electromagnetic spectrum to the mobile telecommunications industry. Most of these auctions were designed with the goal of allocating these resources more efficiently than traditional governmental practices, and additionally raised billions of dollars in the United States and Europe.

How does it work (example)?

Game theory explores the possible outcomes of a situation in which two or more competing parties look for the course of action that best benefits them. No variables are left to chance, so each possible outcome is derived from the combinations of simultaneous actions by each party.

Game theory is best exemplified by a classic hypothetical situation called the Prisoners' Dilemma. In this scenario, two people are arrested for stealing a car. They will each serve 2 years in prison for their crime.

The case is air-tight, but the police have reason to suspect that the two prisoners are also responsible for a recent string of high-profile bank robberies. Each prisoner is placed in a separate cell. Each is told he is suspected of being a bank robber and questioned separately regarding the robberies. The prisoners cannot communicate with each other.

The prisoners are told that a) if they both confess to the robberies, they'll each serve 3 years for the robberies and the car theft, and b) if only one confesses to the robbery and the other does not, the one who confesses will be rewarded with a 1 year sentence while the other will be punished with a 10 year sentence.

In the game, the prisoners have only two possible actions: confess to the bank robbery, or deny having participated in the bank robbery.

Since there are two players, each with two different strategies, there are four outcomes that are possible:

		Prisoner 2	
		Confess	Deny
Prisoner 1	Confess	Both prisoners serve 3 years in prison	Prisoner 2 serves 10 years, Prisoner 1 serves 1 year
	Deny	Prisoner 1 serves 10 years, Prisoner 2 serves 1 year	Both prisoners serve 2 years in prison

The best option for both prisoners is to deny committing the robberies and face 2 years in prison for the car theft. But because neither can be guaranteed that the other won't confess, the most likely outcome is that both prisoners will hedge their bets and confess to the robberies -- effectively taking the 10 year sentence off the table and replacing it with the 3 year sentence.

5.1.4 CHARACTERISTICS OF GAMES AND GAME MODELS

A Game is defined as an activity among two or more persons as per a set of rules at the end of which each person gets some benefit or bears loss. The set of rules and procedures defines the **game**. Going with the set of rules and procedures once by the participants defines the **play**.

Characteristics of Games

- There are finite number of competitors known as 'players'
- All the strategies and their impacts are specified to the players but player does not know which strategy is to be selected.
- Each player has a limited number of possible courses of action known as 'strategies'
- A game is played when every player selects one of his strategies. The strategies are supposed to be prepared

simultaneously with an outcome such that no player recognizes his opponent's strategy until he chooses his own strategy.

- The figures present as the outcomes of strategies in a matrix form are known as 'pay-off matrix'.
- The game is a blend of the strategies and in certain units which finds out the gain or loss.
- The player playing the game always attempts to select the best course of action which results in optimal pay off known as 'optimal strategy'.
- The expected pay off when all the players of the game go after their optimal strategies is called as 'value of the game'. The main aim of a problem of a game is to determine the value of the game.
- The game is said to be 'fair' if the value of the game is zero or else it is known as 'unfair'.

1. Competitive game

A competitive situation is known as **competitive game** if it has the four properties

a. There are limited number of competitors such that $n \geq 2$. In the case of $n = 2$, it is known as a **two-person game** and in case of $n > 2$, it is known as **n-person game**.

b. Each player has a record of finite number of possible actions.

c. A play is said to takes place when each player selects one of his activities. The choices are supposed to be made simultaneously i.e. no player knows the selection of the other until he has chosen on his own.

d. Every combination of activities finds out an outcome which results in a gain of payments to every player, provided each player is playing openly to get as much as possible. Negative gain means the loss of same amount.

2. Strategy

The strategy of a player is the determined rule by which player chooses his strategy from his own list during the game. The two types of strategy are:

- Pure strategy
- Mixed strategy.

Pure Strategy

If a player knows precisely what another player is going to do, a deterministic condition is achieved and objective function is to maximize the profit. Thus, the pure strategy is a decision rule always to choose a particular startegy.

Mixed Strategy

If a player is guessing as to which action is to be chosen by the other on any particular instance, a probabilistic condition is achieved and objective function is to maximize the expected profit. Hence the mixed strategy is a choice among pure strategies with fixed probabilities.

Repeated Game Strategies

- In repeated games, the chronological nature of the relationship permits for the acceptance of strategies that are dependent on the actions chosen in previous plays of the game.
- Most contingent strategies are of the kind called as "trigger" strategies.
- For Example trigger strategies

- In prisoners' dilemma: At start, play doesn't confess. If your opponent plays Confess, then you need to play Confess in the next round. If your opponent plays don't confess, then go for doesn't confess in the subsequent round. This is called as the "tit for tat" strategy.

- In the investment game, if you are sender: At start play Send. Play Send providing the receiver plays Return. If the receiver plays keep, then never go for Send again. This is called as the "grim trigger" strategy.

3. Number of persons

When the number of persons playing is 'n' then the game is known as 'n' person game. The person here means an individual or a group aims at a particular objective.

Two-person, zero-sum game

A game with just two players (player A and player B) is known as 'two-person, zero-sum game', if the losses of one player are equal to the gains of the other one so that the sum total of their net gains or profits is zero.

Two-person, zero-sum games are also known as rectangular games as these are generally presented through a payoff matrix in a rectangular form.

4. Number of activities

The activities can be finite or infinite.

5. Payoff

Payoff is referred to as the quantitative measure of satisfaction a person obtains at the end of each play.

6. Payoff matrix

Assume the player A has 'm' activities and the player B has 'n' activities. Then a payoff matrix can be made by accepting the following rules

- Row designations for every matrix are the activities or actions available to player A
- Column designations for every matrix are the activities or actions available to player B
- Cell entry V_{ij} is the payment to player A in A's payoff matrix when A selects the activity i and B selects the activity j.
- In a zero-sum, two-person game, the cell entry in the player B's payoff matrix will be negative of the related cell entry V_{ij} in the player A's payoff matrix in order that total sum of payoff matrices for player A and player B is finally zero.

7. Value of the game

Value of the game is the maximum guaranteed game to player A (maximizing player) when both the players utilizes their best strategies. It is usually signifies with 'V' and it is unique.

Game Models

Simultaneous v. Sequential Move Games

- Games where players select activities simultaneously are simultaneous move games.
  a. Examples: Sealed-Bid Auctions, Prisoners' Dilemma.
  b. Must forecast what your opponent will do at this point, finding that your opponent is also doing the same.

- Games where players select activities in a particular series or sequence are sequential move games.
  a. Examples: Bargaining/Negotiations, Chess.
  b. Must look forward so as to know what action to select now.
  c. Many sequential move games have deadlines on moves.

- Many strategic situations include both sequential and simultaneous moves

One-Shot versus Repeated Games

- One-shot: play of the game takes place once.
  a. Players likely not know much about each another.

b. Example - tipping on vacation

- Repeated: play of the game is recurring with the same players.
  a. Finitely versus Indefinitely repeated games
  b. Reputational concerns do matter; opportunities for cooperative behavior may emerge.
- Advise: If you plan to follow an *aggressive* strategy, ask yourself whether you are in a one-shot game or in repeated game. If a repeated game then *think again*.

Usually games are divided into

- Pure strategy games
- Mixed strategy games

The technique for solving for these two types does change. By solving a game, we require to determine best strategies for both the players and also to get the value of the game.

Saddle point method can be used to solve pure strategy games.

The diverse methods for solving a mixed strategy game are

- Dominance rule
- Analytical method
- Graphical method
- Simplex method

In the next chapter you will learn solution of 2x2, MxN games using the dominance rule and analytical method. The Simplex method for solving a mixed strategy game is out of this course.

5.1.5 LET US SUM UP

In this unit you have learnt Project Planning, Scheduling and Controlling, Work Break down Structure, Basic Tools and Techniques of Project Management and Role of Network Technique in Project Management.

Limitations of Game Theory

The main limitations are

- The hypothesis that the players have the information about their own payoffs and others is rather impractical

- As the number of players adds in the game, the analysis of the gaming strategies turns out to be increasingly intricate and complicated.
- The assumptions of maximin and minimax presents that the players are risk-averse and have whole information of the strategies. It doesn't look practical.
- Rather than each player in an oligopoly condition working under uncertain situations, the players will permit each other to share the secrets of business so as to work out collusion. Then the mixed strategies are not very helpful.

5.1.6 EXERCISES

Question 1: Discuss the properties of a game.

Question 2: What do you understand by Game Theory?

5.1.7 SUGGESTED READINGS

Game Theory section in any of the reference / text books

❖❖❖❖

5.2

RULES FOR GAME THEORY

Rules for Game Theory, Concept of Pure Game, Mixed Strategies – 2x2 Games

Unit Structure

5.2.1 INTRODUCTION

Game Theory is the process of modeling the strategic interaction between two or more players in a situation containing set rules and outcomes. While used in a number of disciplines, game theory is most notably used as a tool within the study of economics. The economic application of game theory can be a valuable tool to aide in the fundamental analysis of industries, sectors and any strategic interaction between two or more firms. Here, we'll take an introductory look at game theory and the terms involved, and introduce you to a simple method of solving games.

Definitions

Any time we have a situation with two or more players that involves known payouts or quantifiable consequences, we can use game theory to help determine the most likely outcomes. Let's start out by revisiting a few terms commonly used in the study of game theory:

- *Game:* Any set of circumstances that has a result dependent on the actions of two of more decision makers ("players")
- *Players:* A strategic decision maker within the context of the game

- *Strategy:* A complete plan of action a player will take given the set of circumstances that might arise within the game
- *Payoff:* The payout a player receives from arriving at a particular outcome. The payout can be in any quantifiable form, from dollars to utility.
- *Information Set:* The information available at a given point in the game. The term information set is most usually applied when the game has a sequential component.

Equilibrium: The point in a game where both players have made their decisions and an outcome is reached.

Assumptions

As with any concept in economics, there is the assumption of rationality. There is also an assumption of maximization. It is assumed that players within the game are rational and will strive to maximize their payoffs in the game. (The question of rationality has been applied to investor behavior as well).

When examining games that are already set up, it is assumed on your behalf that the payouts listed include the sum of all payoffs that are associated with that outcome. This will exclude any "what if" questions that may arise.

The number of players in a game can theoretically be infinite, but most games will be put into the context of two players. One of the simplest games is a sequential game involving two players.

5.2.2 OBJECTIVES

After studying this unit V – Chapter 5.2, you will be able to understand the following:

- Rules of Game Theory
- Concept of Pure Game
- Mixed Strategies – 2X2 Games.

5.2.3 RULES OF GAME THEORY

The game theory provides an appropriate solution of a problem if its conditions are properly satisfied. These conditions are often termed as the assumptions of the game theory which can be considered as the rules of game theory.

Some of these assumptions are as follows:

i. Assumes that a player can adopt multiple strategies for solving a problem

ii. Assumes that there is an availability of pre-defined outcomes

iii. Assumes that the overall outcome for all players would be zero at the end of the game

iv. Assumes that all players in the game are aware of the game rules as well as outcomes of other players

v. Assumes that players take a rational decision to increase their profit.

Among the aforementioned assumptions, the last two assumptions make the application of the game theory confined in real world.

5.2.4 CONCEPT OF PURE GAME

In a game theory, the pay-off for the players is given in the pay-off matrix. In a two person game denoted below, player A is maximizing player and player B is minimizing player. The player A tries to maximize all his gains while player B tries to minimize all his losses when opposite player plays his strategies.

		Player B	
		B1	**B2**
Player A	**A1**	a11	a12
	A2	a21	a22

Player A has two strategies A1, A2 whereas player B has two strategies B1, B2. The pay-off values a11, a12, a21, a22 are given in the above pay-off matrix across the strategies when player A and B adopt their strategies.

The simplest type of game is one where the best strategies for both players are **pure strategies**. This is the case if and only if, the pay-off matrix contains a saddle point.

A **saddle point** is a payoff that is simultaneously a row minimum and a column maximum. To locate saddle points, circle the row minima and box the column maxima. The saddle points are those entries that are both circled and boxed. A game is strictly determined if it has at least one saddle point.

A pure strategy defines a specific move or action that a player will follow in every possible attainable situation in a game. Such moves may not be random, or drawn from a distribution, as in the case of mixed strategies. So a pure strategy can be considered as a single strategy.

Example: What is the optimal strategy for both the players? Use the pay-off matrix given below:

		Player B				
		I	II	III	IV	V
	I	-2	0	0	5	3
Player A	II	4	2	1	3	2
	III	-4	-3	0	-2	6
	IV	5	3	-4	2	-6

We use the maximin (minimax) principle to analyze the game.

		Player B					
		I	II	III	IV	V	Minimum
	I	-2	0	0	5	3	-2
Player A	II	4	2	1	3	2	1
	III	-4	-3	0	-2	6	-4
	IV	5	3	-4	2	-6	-6
Maximum		5	3	1	5	6	

Select minimum from the maximum of columns.
Minimax = 1
Player A will choose II strategy, which yields the maximum payoff of 1.

Select maximum from the minimum of rows.
Maximin = 1
Similarly, player B will choose III strategy.

Since the value of maximin coincides with the value of the minimax, therefore, saddle point (equilibrium point) = 1.

The amount of payoff at an equilibrium point is also known as value of the game.

The optimal strategies for both players are: Player A must select II strategy and player B must select III strategy. The value of game is 1, which indicates that player A will gain 1 unit and player B will sacrifice 1 unit.

5.2.5 MIXED STRATEGIES – 2X2 GAMES

Mixed strategy means a situation where a saddle point does not exist, the maximin (minimax) principle for solving a game problem breaks down. The concept is illustrated with the help of following example.

Example: Two companies A and B are competing for the same product. Their different strategies are given in the following pay-off matrix:

		Company B		
		I	II	III
Company A	I	-2	14	-2
	II	-5	-6	-4
	III	-6	20	-8

Determine the optimal strategies for both the companies.

First, we apply the maximin (minimax) principle to analyze the game.

		Company B			
		I	II	III	Minimum
Company A	I	-2	14	-2	-2
	II	-5	-6	-4	-6
	III	-6	20	-8	-8
Maximum		-2	20	-2	

Minimax = -2
Maximin = -2

There are two elements whose value is –2. Hence, the solution to such a game is not unique.

In the above problem, there is no saddle point. In such cases, the maximin and minimax principle of solving a game problem can't be applied. Under this situation, both the companies may resort to what is known as mixed strategy.

In a mixed strategy, each player moves in a random fashion.

A mixed strategy game can be solved by algebraic method.

We will now talk about the algebraic method used to solve mixed strategy games. Here we have provided formulas and examples of algebraic method.

Consider the zero sum two person game given below:

		Player B	
		I	II
Player A	I	a	b
	II	c	d

The solution of the game is:
A play's (p, 1 - p)

where:

$$p = \frac{d - c}{(a + d) - (b + c)}$$

B play's (q, 1 - q)
where:

$$q = \frac{d - b}{(a + d) - (b + c)}$$

$$\text{Value of the game, } V = \frac{ad - bc}{(a + d) - (b + c)}$$

Example: Consider the game of matching coins. Two players, A & B, put down a coin. If coins match (i.e., both are heads or both are tails) A gets rewarded, otherwise B. However, matching on heads gives a double premium. Obtain the best strategies for both players and the value of the game.

		Player B	
		I	II
Player A	I	2	-1
	II	-1	1

This game has no **saddle point**

$$. \ p = \frac{1 - (-1)}{(2 + 1) - (-1 - 1)} = \frac{2}{5}$$

$1 – p = 3/5$

$$q = \frac{1 - (-1)}{(2 + 1) - (-1 - 1)} = \frac{2}{5}$$

$1 – q = 3/5$

$$V = \frac{2 \times 1 - (-1) \times (-1)}{(2 + 1) - (-1 - 1)} = \frac{1}{5}$$

5.2.6 LET US SUM UP

In this Unit –IV, Chapter – 5.2, you have learnt the Rules of Game Theory, Concept of Pure Game, Solution of 2X2 game with saddle point and mixed strategies for 2X2 game.

5.2.7 EXERCISES

Question 1:
Solve the game whose pay-off matrix is given below:

		Player B	
		I	II
Player A	I	1	7
	II	6	2

Question 2:
Solve the game whose pay-off matrix is given below:

Strategy		Player 2: 1	Player 2: 2
Player 1	1	2	-2
	2	-2	2

Question 3:
Discuss and define important terms in Game Theory.

5.2.8 SUGGESTED READINGS

Game Theory section in any of the reference / text books

❖❖❖❖

5.3

MIXED STRATEGIES – MXN GAMES

Mixed Strategies – 2xN or Mx2, Mixed Strategies – MxN Games

Unit Structure

5.3.1 INTRODUCTION

In the theory of games a player is said to use a mixed strategy whenever he or she chooses to randomize over the setof available actions. Formally, a mixed strategy is a probability distribution that assigns to each available action a likelihood of being selected. If only one action has a positive probability of being selected, the player is said to use a pure strategy.

A mixed strategy profile is a list of strategies, one for each player in the game. A mixed strategy profile induces a probability distribution or lottery over the possible out-comes of the game. A Nash equilibrium (mixed strategy)is a strategy profile with the property that no single player can, by deviating unilaterally to another strategy, induce a lottery that he or she finds strictly preferable. In 1950 the mathematician John Nash proved that every game with afinite set of players and actions has at least one equilibrium.

To illustrate, one can consider the children's game Matching Pennies, in which each of two players can choose either heads (H) or tails (T); player 1 wins a dollar from player 2 if their choices match and loses a dollar to player 2 if they do not. This game can be represented as follows:

	H	T
H	(1, −1)	(−1, 1)
T	(−1, 1)	(1, −1)

Here player 1's choice determines a row, player 2'schoice determines a column, and the corresponding cell indicates the payoffs to players 1 and 2 in that order. This game has a unique Nash equilibrium that requires each player to choose each action with probability one-half.

5.3.2 OBJECTIVES

In this Unit – V – Chapter 5.3, you will learn about the solution of game using mixed strategies for the following game models:

- 2xN
- Mx2
- MxN Games.

5.3.3 SOLUTION OF 2XN GAME

Graphical Method can only be used in games with no saddle point, and having a pay-off matrix of type n X 2 or 2 X n.
Consider the following pay-off matrix:

		Player B	
		B_1	B_2
Player A	A_1	-2	4
	A_1	8	3
	A_1	9	0

The game does not have a saddle point as shown in the following table.

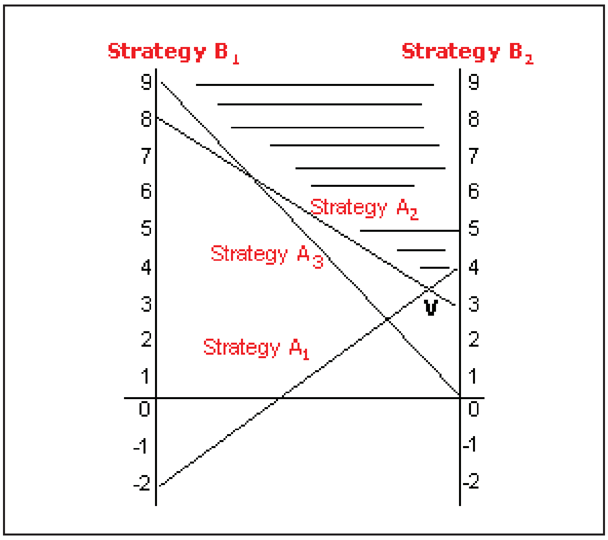
		Player B		Minimum	Probability
		B_1	B_2		
Player A	A_1	-2	4	-2	q_1
	A_2	8	3	3	q_2
	A_3	9	0	0	q_3
Maximum		9	4		
Probability		p_1	p_1		

Maximin = 4, Minimax = 3

First, we draw two parallel lines 1 unit distance apart and mark a scale on each. The two parallel lines represent strategies of player B.

If player A selects strategy A1, player B can win –2 (i.e., lose 2 units) or 4 units depending on B's selection of strategies. The value -2 is plotted along the vertical axis under strategy B_1 and the value 4 is plotted along the vertical axis under strategy B_2. A straight line joining the two points is then drawn.

Similarly, we can plot strategies A2 and A3 also. The problem is graphed in the following figure.



The lowest point V in the shaded region indicates the value of game. From the above figure, the value of the game is 3.4 units. Likewise, we can draw a graph for player B.

The point of optimal solution (i.e., maximin point) occurs at the intersection of two lines:

$E1 = -2p_1 + 4p_2$ and

$E2 = 8p_1 + 3p_2$

Comparing the above two equations, we have

$-2p_1 + 4p_2 = 8p_1 + 3p_2$

Substituting $p_2 = 1 - p_1$

$-2p_1 + 4(1 - p_1) = 8p_1 + 3(1 - p_1)$

$p_1 = 1/11$

$p_2 = 10/11$

5.3.4 SOLUTION OF MX2 GAME

Draw two vertical axes 1 unit apart. The two lines are x1=0, x1= 1.

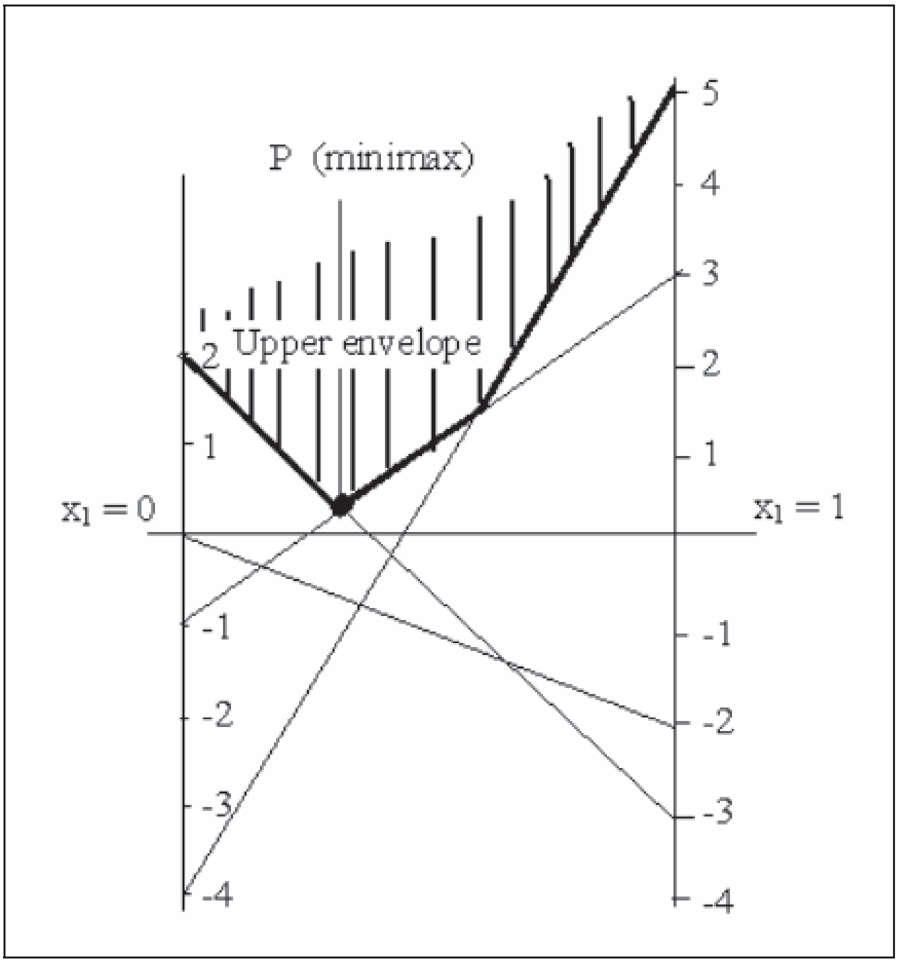
Take the points of the first row in the payoff matrix on the vertical line x1= 1 and the points of the second row in the payoff matrix on the vertical line x1= 0.

The point a1jon axis x1= 1 is then joined to the point a2jon the axis x1= 0 to give a straight line. Draw'n' straight lines for j=1, 2... n and determine the lowest point of the upper envelope obtained. This will be the **minimax point**.

The two or more lines passing through the minimax point determines the required 2 x 2 payoff matrix. This in turn gives the optimum solution by making use of analytical method.

Example 1: Solve by Graphical Method**.**

$$\begin{array}{c c} & \begin{array}{cc} B1 & B2 \end{array} \\ \begin{array}{c} A1 \\ A2 \\ A3 \\ A4 \end{array} & \begin{bmatrix} -2 & 0 \\ 3 & -1 \\ -3 & 2 \\ 5 & -4 \end{bmatrix} \end{array}$$



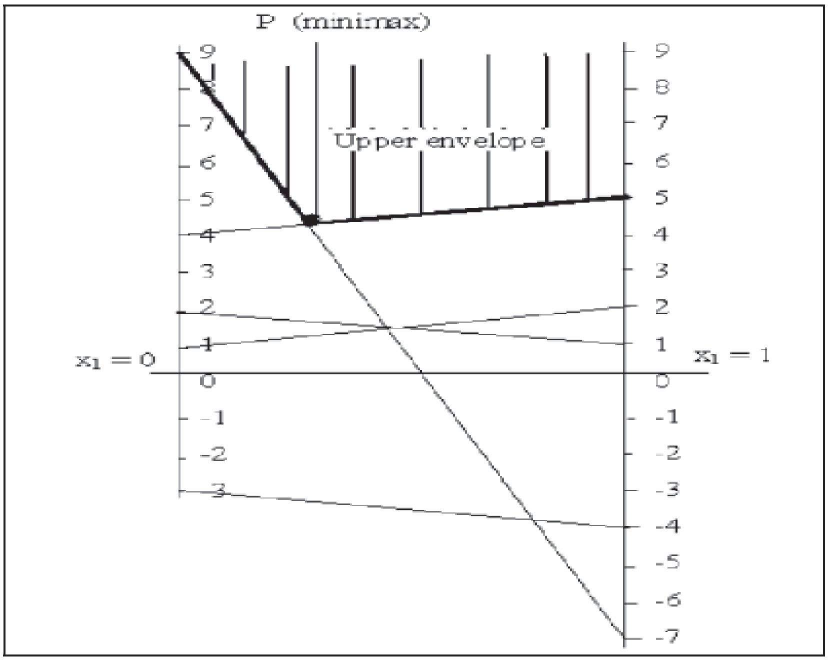
	B1	B2	
A2	3	-1	5
A3	-3	2	4
	3	6	

$$V = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{6-3}{5+4}$$

V = 3/9 = 1/3
SA = (0, 5 /9, 4/9, 0)
SB = (3/9, 6 /9)

Example 2: Solve by Graphical Method.

$$\begin{array}{c} \\ A1 \\ A2 \\ A3 \\ A4 \\ A5 \end{array} \begin{array}{c} \begin{array}{cc} B1 & B2 \end{array} \\ \begin{bmatrix} 1 & 2 \\ 5 & 4 \\ -7 & 9 \\ -4 & -3 \\ 2 & 1 \end{bmatrix} \end{array}$$



$$\begin{array}{c} \\ A2 \\ A3 \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} B1 & B2 \end{array} \\ \begin{bmatrix} 5 & 4 \\ -7 & 9 \end{bmatrix} \\ \begin{array}{cc} 5 & 12 \end{array} \end{array} \begin{array}{c} \\ 16 \\ 1 \\ \\ \end{array}$$

$$V = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{45 + 28}{14 + 3}$$

V = 73/17
SA= (0, 16/17, 1/17, 0, 0)
SB= (5/17, 12 /17)

5.3.5 MIXED STRATEGIES – MXN GAME

The MxN game is reduced to 2xn or Mx2 game using the principle of dominance and then the Graphical Method is used on the revised matrix.

In a game, sometimes a strategy available to a player might be found to be preferable to some other strategy / strategies. Such a strategy is said to dominate the other one(s). The rules of dominance are used to reduce the size of the payoff matrix. These rules help in deleting certain rows and/or columns of the payoff matrix, which are of lower priority to at least one of the remaining rows, and/or columns in terms of payoffs to both the players. Rows / columns once deleted will never be used for determining the optimal strategy for both the players.

This concept of domination is very usefully employed in simplifying the two – person zero sum games without saddle point. In general the following rules are used to reduce the size of payoff matrix.

The Rules (Principles of Dominance) you will have to follow are:

Rule 1: If all the elements in a row (say ithrow) of a payoff matrix are less than or equal to the corresponding elements of the other row (say jthrow) then the player A will never choose the ithstrategy then we say ithstrategy is dominated by jthstrategy and will delete the ith row.

Rule 2: If all the elements in a column (say rthcolumn) of a payoff matrix are greater than or equal to the corresponding elements of the other column (say sthcolumn) then the player B will never choose the rthstrategy or in the other words
the rthstrategy is dominated by the sthstrategy and we delete rth column.

Rule 3: A pure strategy may be dominated if it is inferior to average of two or more other pure strategies.

Example:

Given the payoff matrix for player A, obtain the optimum strategies for both the players and determine the value of the game.

		Player B	
Player A	6	-3	7
	-3	0	4

Solution

		Player B		
		B1	B2	B3
Player A	A1	6	-3	7
	A2	-3	0	4

When A chooses strategy A1 or A2, B will never go to strategy B3. Hence strategyB3 is redundant.

		Player B		
		B1	B2	Row minima
Player A	A1	6	-3	-3
	A2	-3	0	-3
Column maxima		6	0	

Minimax (=0), maximin (= -3).Hence this isnot a pure strategy with a saddle point.

Let the probability of mixed strategy of A for choosing AI and A2 strategies are p1and 1-p1respectively. We get

$$6\, p_1 - 3\,(1 - p_1) = -3\, p_1 + 0\,(1 - p_1) \quad \text{or} \quad p_1 = 1/4$$

Again, q1and 1 - q1being probabilities of strategy B, we get

$$6\, q_1 - 3\,(1 - q_1) = -3\, q_1 + 0\,(1 - q_1) \quad \text{or} \quad q_1 = 1/4$$

Hence optimum strategies for players A and B will be as follows:

$$S_A = \begin{bmatrix} A1 & A2 \\ 1/4 & 3/4 \end{bmatrix}$$

and

$$S_B = \begin{bmatrix} B1 & B2 & B3 \\ 1/4 & 3/4 & 0 \end{bmatrix}$$

Expected value of the game = $q_1 (6 p_1 - 3(1 - p_1)) + (1 - q_1)(3 q_1 + 0(1 - q_1)) = ¾$

5.3.6 LET US SUM UP

In this Unit V, Chapter 5.3, you have learnt Game Theory Mixed Strategies – 2xN or Mx2 Games, Mixed Strategies – MxN Games and theory of dominance.

5.3.7 EXERCISES

Exercise 1: Given the payoff table for player 1 (politician 1), which strategy should each player select?

		Player 2		
Strategy		1	2	3
Player 1	1	1	2	4
	2	1	0	5
	3	0	1	−1

Exercise 2: Solve the following game:

		Player 2				
Strategy		1	2	3	4	5
Player 1	1	1	−3	2	−2	1
	2	2	3	0	3	−2
	3	0	4	−1	−3	2
	4	−4	0	−2	2	−1

Exercise 3: Solve the following games:

	B1	B2	B3
A1	1	3	12
A2	8	6	2

	B1	B2
A1	-2	0
A2	3	-1
A3	-3	2
A4	5	-4

5.3.8 SUGGESTED READINGS

Game Theory section in any of the reference / text books

❖❖❖❖

MODULE - VI

MARKOV CHAINS

6.1

INTRODUCTION TO MARKOV CHAINS

Introduction to Markov Chains, Brand Switching Examples

Unit Structure

6.1.1 INTRODUCTION

Most of our study of probability has dealt with independent trials processes. These processes are the basis of classical probability theory and much of statistics.

We have seen that when a sequence of chance experiments forms an independent trials process, the possible outcomes for each experiment are the same and occur with the same probability. Further, knowledge of the outcomes of the previous experiments does not influence our predictions for the outcomes of the next experiment. The distribution for the outcomes of a single experiment is sufficient to construct a tree and a tree measure for a sequence of n experiments, and we can answer any probability question about these experiments by using this tree measure.

Modern probability theory studies chance processes for which the knowledge of previous outcomes influences predictions for future experiments. In principle, when we observe a sequence of chance experiments, all of the past outcomes could influence our predictions for the next experiment. For example, this should be the case in predicting a student's grades on a sequence of exams in a course. But to allow this much generality would make it very difficult to prove general results.

In 1907, A. A. Markov began the study of an important new type of chance process. In this process, the outcome of a given experiment can affect the outcome of the next experiment. This type of process is called a Markov chain.

Markov chains, named after Andrey Markov, are mathematical systems that hop from one "state" (a situation or set of values) to another. For example, if you made a Markov chain model of a baby's behavior, you might include "playing," "eating", "sleeping," and "crying" as states, which together with other behaviors could form a 'state space': a list of all possible states. In addition, on top of the state space, a Markov chain tells you the probability of hopping, or "transitioning," from one state to any other state---e.g., the chance that a baby currently playing will fall asleep in the next five minutes without crying first.

6.1.2 OBJECTIVES

After studying this Unit VI – Chapter 6.1, you will be able to understand:

- Markov Chain
- Brand Switching Examples.

6.1.3 WHAT IS MARKOV CHAINS?

Markov Property: The state of the system at time t+1 depends only on the state of the system at time t

Specifying a Markov Chain

We describe a Markov chain as follows: We have a set of *states,* $S = \{s1; s2, \ldots, sr\}$.The process starts in one of these states and moves successively from one state to another. Each move is called a *step.* If the chain is currently in state si, then it moves to state sjat the next step with a probability denoted by pij, and this probability does not depend upon which states the chain was in before the current state.

The probabilities $pij$are called *transition probabilities.* The process can remainin the state it is in, and this occurs with probability pii. An initial probability distribution, defined on S, specifies the starting state. Usually this is done by specifying a particular state as the starting state.

A picturesque description of a Markov chain is a frog jumping on a set of lily pads. The frog starts on one of the pads and then jumps from lily pad to lily pad with the appropriate transition probabilities.

Example 1: Transition Matrix

Newzeland is blessed by many things, but not by good weather. They never have two nice days in a row. If they have a nice day, they are just as likely to have snow as rain the next day. If they have snow or rain, they have an even chance of having the same the next day. If there is change from snow or rain, only half of the time is this a change to a nice day. With this information we form a Markov chain as follows:

We take as states the kinds of weather R, N, and S. From the above information we determine the transition probabilities. These are most conveniently represented in a square array as

$$\mathbf{P} = \begin{array}{c} \\ \text{R} \\ \text{N} \\ \text{S} \end{array} \begin{array}{c} \begin{array}{ccc} \text{R} & \text{N} & \text{S} \end{array} \\ \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \end{array} .$$

The entries in the first row of the matrix P in the above example represent the probabilities for the various kinds of weather following a rainy day. Similarly, the entries in the second and third rows represent the probabilities for the various kinds of weather following nice and snowy days, respectively. Such a square array is called the *matrix of transition probabilities*, or the *transition matrix*.

We consider the question of determining the probability that, given the chain is in state *i* today, it will be in state *j* two days from now. We denote this probability by *pij*(2).

We see that if it is rainy today then the event that itis snowy two days from now is the disjoint union of the following three events:

- it is rainy tomorrow and snowy two days from now,
- it is nice tomorrow and snowy two days from now, and
- it is snowy tomorrow and snowy two days from now.

The probability of the first of these events is the product of the conditional probability that it is rainy tomorrow, given that it is rainy today, and the conditional probability that it is snowy two days from now, given that it is rainy tomorrow.

Using the transition matrix P, we can write this product as p11p13.The other two events also have probabilities that can be written as products of entries of P. Thus, we have

$$p_{13}^{(2)} = p_{11}p_{13} + p_{12}p_{23} + p_{13}p_{33} \ .$$

In general, if a Markov chain has *r* states, then

$$p_{ij}^{(2)} = \sum_{k=1}^{r} p_{ik} p_{kj} \, .$$

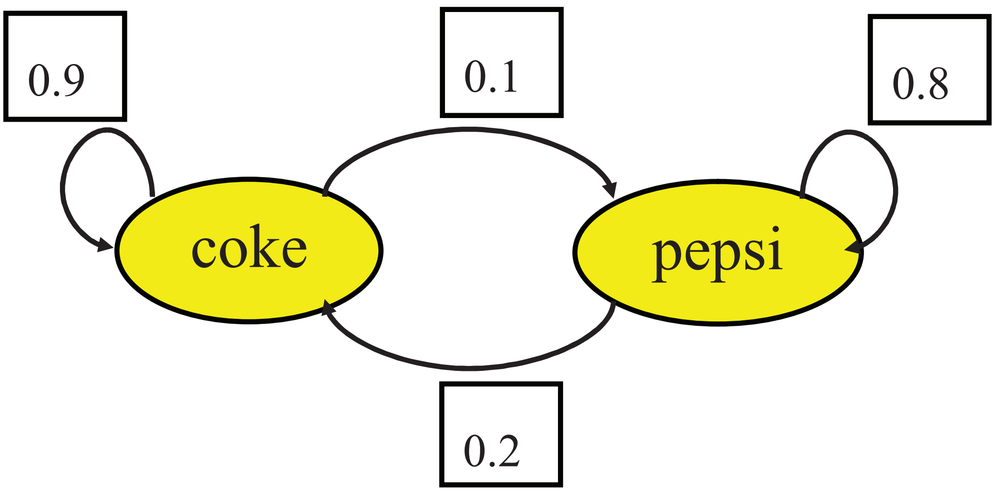
6.1.4 BRAND SWITCHING EXAMPLE

Coke vs. Pepsi Example

- Given that a person's last cola purchase was Coke, there is a 90% chance that his next cola purchase will also be Coke.
- If a person's last cola purchase was Pepsi, there is an 80% chance that his next cola purchase will also be Pepsi.

Transition Matrix

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$



Given that a person is currently a Pepsi purchaser, what is the probability that he will purchase Coke two purchases from now?

Pr[Pepsi→?→Coke] =
Pr[Pepsi→Coke→Coke] + Pr[Pepsi→Pepsi→Coke] =
0.2 * 0.9 + 0.8 * 0.2 = 0.34

6.1.5 LET US SUM UP

In this UNIT VI – Chapter 6.1, you have been taught Markov Chain, brand switching example and calculation of probabilities.

6.1.6 EXERCISES

Question 1: Given that a person is currently a Coke purchaser, what is the probability that he will purchase Pepsi three purchases from now?

Answer: 0.219

$$P^3 = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix} = \begin{bmatrix} 0.781 & 0.219 \\ 0.438 & 0.562 \end{bmatrix}$$

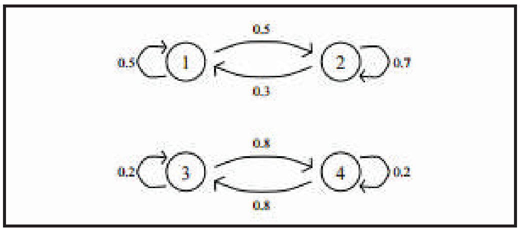
Question 2:

Consider a Markov Chain with the following transition matrix:

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.8 & 0.2 \end{bmatrix}.$$

Draw the transition graph.

Answer:



6.1.7 SUGGESTED READINGS

Markov Chain section in any of the reference / text books

❖❖❖❖

6.2

MARKOV PROCESS

Markov Process

Unit Structure

6.2.1 INTRODUCTION

As we have discussed earlier, a *Markov process* is a random process in which the future is independent of the past, given the present. Markov processes, named for Andrei Markov are among the most important of all random processes. In a sense, they are the stochastic analogs of differential equations and recurrence relations, which are of course, among the most important deterministic processes.

6.2.2 OBJECTIVES

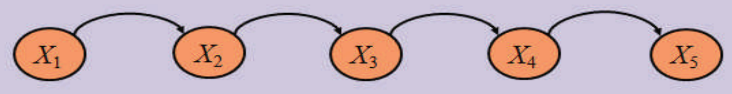
In this Unit VI, Chapter 6.2 you will learn about the Markov Process and the State Transition Matrix.

6.2.3 WHAT IS A MARKOV PROCESS

Suppose that we perform, one after the other, a sequence of experiments that have the same set of outcomes. If the probabilities of the various outcomes of the current experiment depend (at most) on the outcome of the preceding experiment, then we call the sequence a Markov process.

Every Markov Process has a Markov Property.The state of the system at time t+1 depends only on the state of the system at time t.

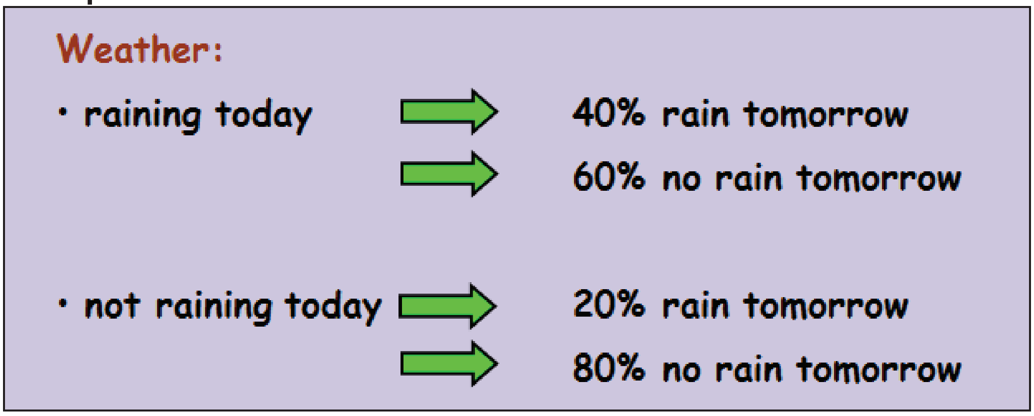
$$\Pr[X_{t+1} = x_{t+1} \mid X_1 \cdots X_t = x_1 \cdots x_t] = \Pr[X_{t+1} = x_{t+1} \mid X_t = x_t]$$



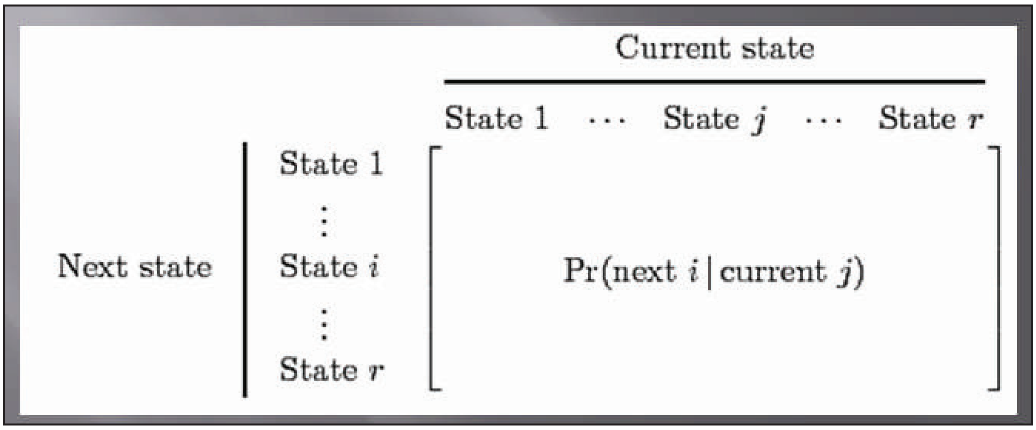
The experiments of a Markov process are performed at regular time intervals and have the same set of outcomes. These outcomes are called states, and the outcome of the current experiment is referred to as the current state of the process. The states are represented as column matrices.

Markov Process – Transition Matrix Examples

Example 1:



The transition matrix records all data about transitions from one state to the other. The form of a general transition matrix is



A stochastic matrix is any square matrix that satisfies the following two properties:

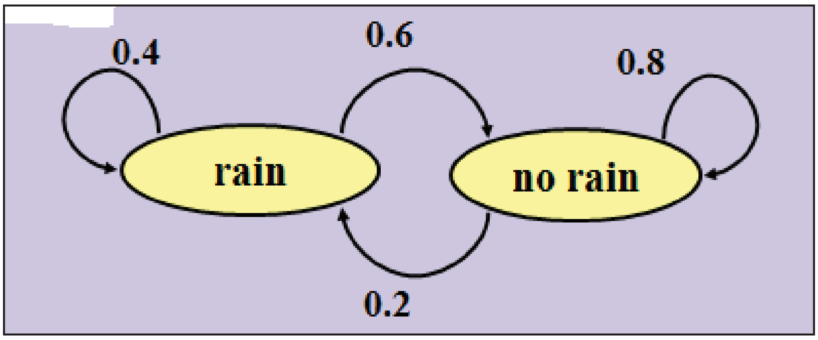
- All entries are greater than or equal to 0;
- The sum of the entries in each column is 1.

In a double stochastic matrix rows and columns sum up to one.

All transition matrices are stochastic matrices. The transition matrix for given example is as under:

$$P = \begin{pmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{pmatrix}$$

Transition Matrix – State Diagram



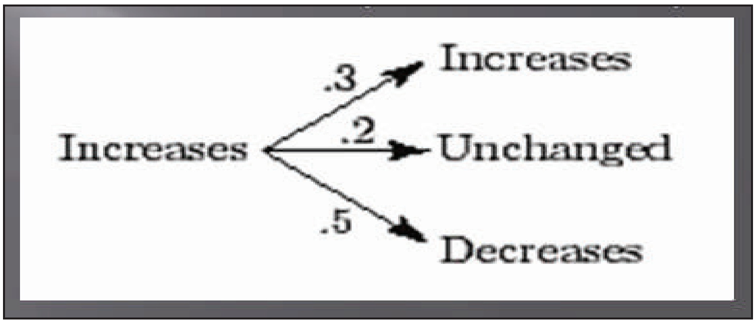
Example 2:

A particular utility stock is very stable and, in the short run, the probability that it increases or decreases in price depends only on the result of the preceding day's trading. The price of the stock is observed at 4 P.M. each day and is recorded as "increased," "decreased," or "unchanged." The sequence of observations forms a Markov process.

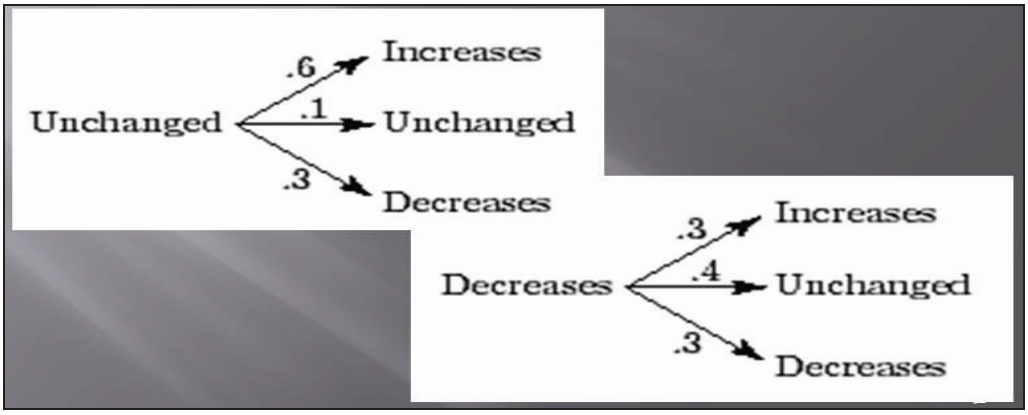
For the utility stock, if the stock increases one day, the probability that on the next day it increases are .3, remains unchanged .2 and decreases .5. If the stock is unchanged one day, the probability that on the next day it increases is .6, remains unchanged .1, and decreases .3. If the stock decreases one day, the probability that it increases the next day is .3, is unchanged .4, decreases .3. Find the transition matrix.

The Markov process has three states: "increases," "unchanged," and "decreases."

The transitions from the first state ("increases") to the other states are



The transitions from the other two states are:



Putting this information into a single matrix so that each column of the matrix records the information about transitions from one particular state is the transition matrix.

		Current state		
		Increases	Unchanged	Decreases
Next state	Increases	.3	.6	.3
	Unchanged	.2	.1	.4
	Decreases	.5	.3	.3

Distribution Matrix

The matrix that represents a particular state is called a distribution matrix. Whenever a Markov process applies to a group with members in r possible states, a distribution matrix for n is a column matrix whose entries give the percentages of members in each of the r states after n time periods.

Distribution Matrix for *n*

Let A be the transition matrix for a Markov process with initial distribution matrix then the distribution matrix after n time periods is given by

$$A^n \begin{bmatrix} \; \end{bmatrix}_0 = \begin{bmatrix} \; \end{bmatrix}_n$$

Example 3:

Census studies from the 1960s reveal that in the US 80% of the daughters of working women also work and that 30% of daughters of nonworking women work. Assume that this trend remains unchanged from one generation to the next. If 40% of women worked in 1960, determine the percentage of working women in each of the next two generations.

There are two states, "work" and "don't work."

The first column of the transition matrix corresponds to transitions from "work".

The probability that a daughter from this state "works" is 0.8 and "doesn't work" is 1 - 0.8 = 0.2.

Similarly, the daughter from the "don't work" state "works" with probability 0.3 and "doesn't work" with probability 0.7.

Transition Matrix is as under:

		Current generation	
		Work	Don't work
Next generation	Work	.8	.3
	Don't work	.2	.7

The Initial Distribution Matrix is as under:

$$\begin{bmatrix} .4 \\ .6 \end{bmatrix}_0$$

Let us show the Distribution Matrix for n(4).
In one generation,

$$\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} .4 \\ .6 \end{bmatrix}_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}_1$$

So 50% women work and 50% don't work.

For the second generation,

$$\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}^2\begin{bmatrix} .4 \\ .6 \end{bmatrix}_0 = \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix}\begin{bmatrix} .4 \\ .6 \end{bmatrix}_0 = \begin{bmatrix} .55 \\ .45 \end{bmatrix}_2$$

So 55% women work and 45% don't work.

Interpretation of the Entries of A^n

The entry in the i^{th} row and j^{th} column of the matrix A^n is the probability of the transition from state *j* to state *i* after *n* periods.

Example Interpretation of the Entries

Interpret from the last example.

$$\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}^2 = \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix}$$

If a woman works, the probability that her granddaughter will work is .7 and not work is .3.

If a woman does not work, the probability that her granddaughter will work is .45 and not work is .55.

6.2.4 LET US SUM UP

In this UNIT VI – Chapter 6.2, you have learnt that:

- A Markov process is a sequence of experiments performed at regular time intervals involving *states*. As a result of each experiment, transitions between states occur with probabilities given by a matrix called the *transition matrix*. The ij^{th} entry in the transition matrix is the conditional probability Pr(moving to state i|in state j).
- A *stochastic matrix* is a square matrix for which every entry is greater than or equal to 0 and the sum of the entries in each column is 1. Every transition matrix is a stochastic matrix.
- The n^{th} distribution matrix gives the percentage of members in each state after n time periods.
- A^n is obtained by multiplying together n copies of A. Its ij^{th} entry is the conditional probability Pr(moving to state i after n time periods | in state j). Also, A^n times the initial distribution matrix gives the n^{th} distribution matrix.

6.2.5 EXERCISES

Question 1: Explain Markov Process, Markov Chain and State Transition Diagram.

Question 2:

Each time a certain horse runs in a three-horse race, he has probability 1/2 of winning, 1/4 of coming in second, and 1/4 of coming in third, independent of the outcome of any previous race. We have an independent trials process, but it can also be considered from the point of view of Markov chain theory. Draw the transition probability matrix.

Answer:

$$\mathbf{P} = \begin{array}{c} \\ \text{W} \\ \text{P} \\ \text{S} \end{array} \begin{array}{c} \begin{array}{ccc} \text{W} & \text{P} & \text{S} \end{array} \\ \begin{pmatrix} .5 & .25 & .25 \\ .5 & .25 & .25 \\ .5 & .25 & .25 \end{pmatrix} \end{array} .$$

Question 3:

We have two urns that, between them, contain four balls. At each step, one of the four balls is chosen at random and moved from the urn that it is in into the other urn. We choose, as states, the number of balls in the first urn. Draw the transition matrix.

Answer:

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array} .$$

6.2.6 SUGGESTED READINGS

Markov Chain section in any of the reference / text book

6.3

MARKOV PROCESS

Markov Analysis – Input and Output

Unit Structure

6.3.1 INTRODUCTION

Markov analysis, like decision analysis, is a probabilistic technique. However, Marko analysis is different in that it does not provide a recommended decision. Instead, Markov analysis provides probabilistic information about a decision situation that can aid the decision maker in making a decision. In other words, Markov analysis is not an optimization technique; it is a descriptive technique that results in probabilistic information.

Markov analysis is specifically applicable to systems that exhibit probabilistic movement from one state (or condition) to another, over time. For example, Markov analysis can be used to determine the probability that a machine will be running one day and broken down the next or that a customer will change brands of cereal from one month to the next. This latter type of example—referred to as the “brand-switching” problem—will be used to demonstrate the principles of Markov analysis in the following discussion.

6.3.2 OBJECTIVES

In this Unit VI, Chapter 6.3 you will learn about the Markov Analysis – Input and Output

6.3.3 MARKOV ANALYSIS – INPUT AND OUTPUT

Markov analysis can be used to analyze a number of different decision situations; however, one of its most popular applications has been the analysis of customer brand switching. This is basically a marketing application that focuses on the loyalty of customers to a particular product brand, store, or supplier. Markov analysis provides information on the probability of customers' switching from one brand to one or more other brands. An example of the brand-switching problem will be used to demonstrate Markov analysis.

A small community has two gasoline service stations, Petroco and National. The residents of the community purchase gasoline at the two stations on a monthly basis. The marketing department of Petroco surveyed a number of residents and found that the customers were not totally loyal to either brand of gasoline. Customers were willing to change service stations as a result of advertising, service, and other factors. The marketing department found that if a customer bought gasoline from Petroco in any given month, there was only a 0.60 probability that the customer would buy from Petroco the next month and a 0.40probability that the customer would buy gas from National the next month. Likewise, if a customer traded with National in a given month, there was an0.80 probability that the customer would purchase gasoline from National in the next month and a 0.20 probability that the customer would purchase gasoline from Petroco. These probabilities are summarized in the table below:

	Next Month	
This Month	PETROCO	NATIONAL
Petroco	.60	.40
National	.20	.80

Markov assumptions in this example are:

(1) the probabilities of moving from a state to all others sum to one,
(2) the probabilities apply to all system participants, and
(3) the probabilities are constant over time.

This example contains several important assumptions. First, notice that in the above table the probabilities in each row sum to one because they are mutually exclusive and collectively exhaustive. This means that if a customer trades with Petroco one month, the customer must trade with either Petroco or National the next month (i.e., the customer will not give up buying gasoline, nor will the customer trade with both in one month). Second, the probabilities in the table apply to every customer who purchases gasoline. Third, the probabilities in the table will not change over time. In other words, regardless of when the customer buys gasoline, the probabilities of trading with one of the service stations the next month will be the values in the table. The probabilities in the table will not change in the future if conditions remain the same.

It is these properties that make this example a Markov process. In Markov terminology, the service station a customer trades at in a given month is referred to as a state of the system. Thus, this example contains two states of the system—a customer will purchase gasolineat either Petroco or National in any given month. The probabilities of the various states in the table are known as transition probabilities. In other words, they are the probabilities of a customer's making the transition from one state to another during one time period. The table contains four transition probabilities.

A transition probability is the probability of moving from one state to another during one time period.

The state of the system is where the system is at a point in time.

The properties for the service station example just described define a Markov process. They are summarized in Markov terminology as follows:

- *Property 1:* The transition probabilities for a given beginning state of the system sum to one.

- *Property 2:* The probabilities apply to all participants in the system.
- *Property 3:* The transition probabilities are constant over time.
- *Property 4:* The states are independent over time.

Now that we have defined a Markov process and determined that our example exhibits the Markov properties, the next question is, What information will Markov analysis provide?

The most obvious information available from Markov analysis is the probability of being in a state at some future time period, which is also the sort of information we can gain from a decision tree.

For example, suppose the service stations wanted to know the probability that a customer would trade with it in month 3, given that the customer traded with it this month.

(I). This analysis can be performed for each service station by using decision trees, as in the figures below.

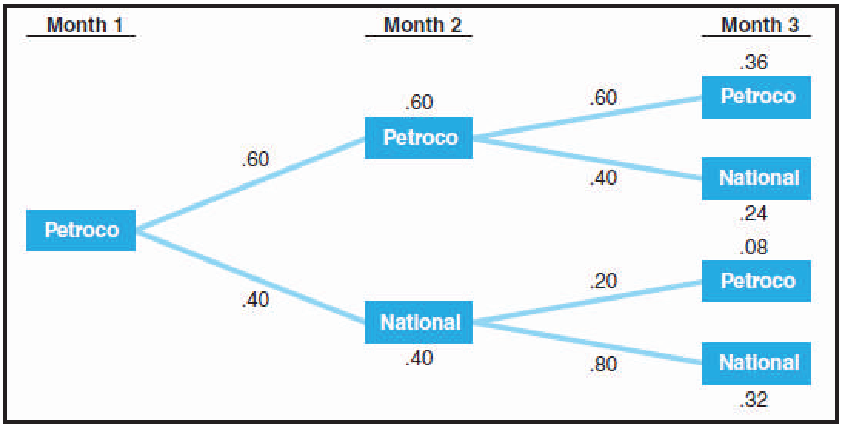
To determine the probability of a customer's trading with Petroco in month 3, given that the customer initially traded with Petroco in month 1, we must add the two branch probabilities in figure associated with Petroco:

0.36 + 0.08 = 0.44, the probability of a customer's trading with Petroco in month 3

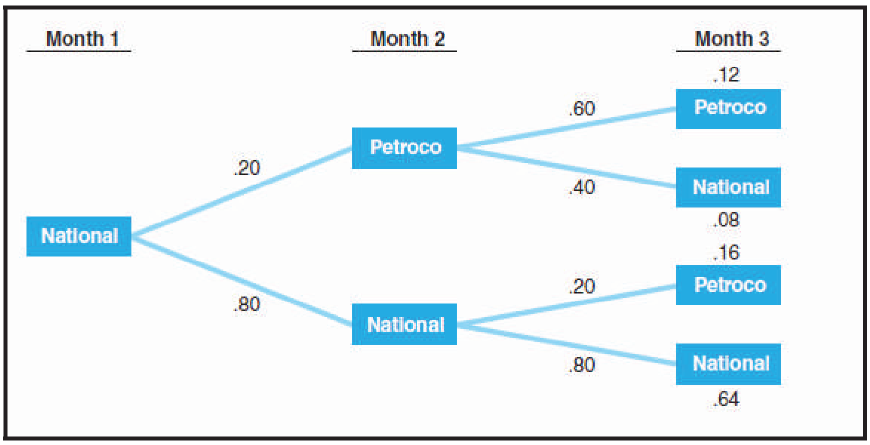
Likewise, to determine the probability of a customer's purchasing gasoline from National in month 3, we add the two branch probabilities in figure associated with National:

0.24 + 0.32 = 0.56, the probability of a customer's trading with National in month 3

Probabilities of future states, given that a customer trades with Petroco this month



Probabilities of future states, given that a customer trades with National this month



This same type of analysis can be performed under the condition that a customer initially purchased gasoline from National, as shown in the figure. Given that

National is the startingstate in month 1, the probability of a customer's purchasing gasoline from National in month 3 is

$$0.08 + 0.64 = .72$$

and the probability of a customer's trading with Petroco in month 3 is

$$0.12 + 0.16 = 0.28$$

Notice that for each starting state, Petroco and National, the probabilities of ending upin either state in month 3 sum to one:

	Probability of Trade in Month 3		
Starting State	PETROCO	NATIONAL	SUM
Petroco	.44	.56	1.00
National	.28	.72	1.00

Although the use of decision trees is perfectly logical for this type of analysis, it is time consuming and cumbersome. For example, if Petroco wanted to know the probability that a customer who traded with it in month 1 will trade with it in month 10, a rather large decision tree would have to be constructed. Alternatively, the same analysis performed previously using decision trees can be done by using *matrix algebra* techniques.

6.3.4 LET US SUM UP

In this UNIT VI – Chapter 6.3, you have learn Markov Analysis with examples.

6.3.5 EXERCISES

Question 1:
Discuss the properties that must exist for the transition matrix in to be considered a Markov process.

Question 2:
The only grocery store in a community stocks milk from two dairies: Cream wood and Cheese dale.

The following transition matrix shows the probabilities of a customer's purchasing each brand of milk next week, given that he or she purchased a particular brand this week:

This Week	*Next Week*	
	Creamwood	Cheesedale
Creamwood	.7	.3
Cheesedale	.4	.6

Given that a customer purchases Cream wood milk this week, use a decision tree to determine the probability that he or she will purchase Cheese dale milk in week 4.

6.3.6 SUGGESTED READINGS

Markov Chain section in any of the reference / text book

❖❖❖❖