M.Sc (Maths) [Part – II]
Algebra - II
(Paper- I)
(May-2017)

Q.P.Code: 011474

[Total Marks: 100]

N.B. 1) Attempt any **five** questions out of **eight**.
2) All questions carry equal marks.

Q. 1. (a) State the class equation for a finite group. Prove it by explaining clearly all the notation used. **(10)**

(b) Let G be a group and let p be a prime dividing the order of G. Prove that any two Sylow p-subgroups of G are conjugate to each other. **(10)**

Q. 2. (a) Let G be a group and let H be a normal subgroup of G. Prove that G is solvable if and only if both H and G/H are solvable. **(10)**

(b) (i) Prove that the center of a group G is a normal subgroup of G. Determine the center of the group of quaternions. **(5)**

(ii) Classify (upto isomorphism) all groups of order 6 with correct justification. **(5)**

Q. 3. (a) State and prove the first isomorphism theorem for modules over a commutative ring R with unity. **(10)**

(b) (i) State (without proof) the Hilbert basis theorem. Define the terms: Noetherian ring, Noetherian module. **(5)**

(ii) Prove that any Artinian ring has finitely many maximal ideals. **(5)**

Q. 4. (a) Construct a finite field of order 9 with correct justification. **(10)**

(b) (i) State (without proof) the structure theorem for modules over a principal ideal domain. Give an example of a ring which is not a principal ideal domain. **(5)**

(ii) Define the terms: free module, torsion module. Give one example of each with correct justification. **(5)**

Q. 5. (a) (i) Determine the degree of the field extension $\mathbb{Q}(\sqrt{2}+\sqrt{3})$ over $\mathbb{Q}$ with correct justification. **(5)**

(ii) State (without proof) primitive element theorem. Give an example of an extension of $\mathbb{Q}$ which is not normal with correct justification. **(5)**

(b) State and prove the fundamental theorem of Galois theory. **(10)**

Q. 6. (a) (i) Determine with correct justification whether the cubic equation $X^3 - 1$ is solvable by radicals over $\mathbb{Q}$. **(5)**

(ii) Is the polynomial $X_1^2 + X_2^2$ symmetric in the variables X_1, X_2? If yes, express it in terms of the elementary symmetric polynomials. **(5)**

(b) Prove that an angle θ is constructible by straightedge and compass if and only if $\cos\theta$ is constructible by straight edge and compass. **(10)**

TURN OVER

Q. 7. (a) (i) Prove that the map $a + b\sqrt{3} \mapsto a - b\sqrt{3}$ is an automorphism of $\mathbb{Q}(\sqrt{3})$. Find the fixed field of this automorphism. **(5)**

(ii) Determine the degree of the splitting field of X^2+X+1 over $\mathbb{Q}$ with correct justification. **(5)**

(b) Let R be a commutative ring with unity and let M be an R-module. Prove that M is a Noetherian R-module if and only if every submodule of M is finitely generated. **(10)**

Q. 8. (a) (i) Define the term: normal extension. Let k be a field and let K be a degree two extension of k. Prove that K is a normal extension of k. **(5)**

(ii) Let S_3 act on itself by left-multiplication. Find the orbit of $(1\ 2), (1\ 2\ 3)$ under this action. **(5)**

(b) Let k be a field. Prove that there exists an algebraically closed field containing k as a subfield. **(10)**

Q. P. Code: 11899

(3 Hours) [Total Marks: 100

N.B.: (1) Attempt any **FIVE** questions.

(2) Figures to the right indicate marks for respective sub-questions.

1. (a) Let E be a Lebesgue measurable subset of $\mathbb{R}$ and $r \in \mathbb{R}$. Show that

    (i) $r + E$ is Lebesgue measurable and $m(r + E) = m(E)$ (5)

    (ii) rE is Lebesgue measurable and $m(rE) = |r|m(E)$ (5)

   (b) Show that there is a non-measurable subset in $\mathbb{R}$. (10)

2. (a) (i) Let $\{f_n\}$ be an increasing sequence of non-negative measurable functions on E. If $f_n \to f$ point-wise a.e. on E, then show that $\lim_{n\to\infty} \int_E f_n = \int_E f$. (5)

    (ii) Let $E_1 \supseteq E_2 \supseteq \cdots$ be measurable subsets of $\mathbb{R}$ with $E = \bigcap_{n=1}^{\infty} E_n$. If $m(E_k) < \infty$ for some k, then show that $m(E) = \lim_{n\to\infty} m(E_n)$. (5)

   (b) State and prove Fatou's lemma. Show by an example that the inequality in Fatou's lemma may be a strict inequality. (10)

3. (a) (i) Let f be a bounded function defined on the closed and bounded interval $[a, b]$. If f is Riemann integrable over $[a, b]$, then show that it is Lebesgue integrable over $[a, b]$ and the two integrals are equal. (5)

    (ii) Show by an example that a Lebesgue integrable function may not be Riemann integrable. (5)

   (b) If f is a measurable function, then show that for $\lambda \in \mathbb{R}$,

    (i) λf is measurable. (5)

    (ii) $\lambda + f$ is measurable. (5)

4. (a) (i) Let A be a subset of $\mathbb{R}$. Show that the characteristic function χ_A is measurable if and only if the set A is measurable. (5)

    (ii) Let f and g be non-negative integrable functions on a measurable subset E of $\mathbb{R}$. Show that if $f \leq g$ a.e. then $\int_E f \leq \int_E g$. (5)

   (b) State Fubini's theorem. Use Fubini's theorem to evaluate $\int_A (ye^x - x\sin y)dxdy$, where $A = [-1, 1] \times [0, \pi/2]$. (10)

[TURN OVER

5. (a) Let (f_n) be a sequence of measurable functions. Show that $\inf_n\{f_n\}$ and $\lim_{n\to\infty} f_n$ are measurable functions. (10)

(b) Let f and g be two non-negative measurable functions on a measaurable set E and λ be a non-negative real number. Show that (10)

(i) $\int_E (f+g) = \int_E f + \int_E g$

(ii) $\int_E (\lambda f) = \lambda \int_E f$

6. (a) Let S be a closed subspace of a Hilbert space H. Show that $S^\perp$ is also a closed subspace of H and $H = S \oplus S^\perp$. (10)

(b) State and prove Minkowski's inequality. (5)

(c) State and prove Bessel's inequality. (5)

7. (a) Let $\{e_n\}_{n\in\mathbb{N}}$ be an arbitrary orthonormal set in $L^2[-\pi,\pi]$ and let $c_1, c_2, \ldots$ be complex numbers such that the series $\sum_{k=1}^{\infty} c_k$ converges. Show that there exist a function $f \in L^2[-\pi,\pi]$ such that $c_k = \langle f, e_k\rangle$ and $\sum_{k=1}^{\infty} c_k^2 = \|f\|^2$. (10)

(b) Show that $\ell^2(\mathbb{N})$ is a complete metric space. (10)

8. (a) Let f be an integrable function on the circle which is differentiable at a point x_0. Show that $S_N(f)(x_0) \to f(x_0)$ as $N \to \infty$, where $S_N f(x)$ is the N-th partial sum of the Fourier series of f. (10)

(b) (i) Find the solution of the Dirichlet's problem $\Delta u = 0$ on the unit disc, with the boundary condition $u(1,\theta) = \sin^2\theta$. (5)

(ii) Find the Fourier series of the function $f(x) = x$ in $-\pi \le x \le \pi$. (5)

M.Sc (Maths) [Part – II]
Differential Geometry
(Paper- III)
(May-2017)

Q.P. Code :09904

[Time: 3 Hours] **[Marks:100]**

Please check whether you have got the right question paper.

N.B:
1. **Attempt any five questions**
2. **All questions carry equal marks.**
3. **Parts (a), (b) in each question carry ten marks each.**

Q.1 (a) (i) Define the metric topology of the Euclidean space $\mathbb{R}^n$ and prove that the topology is complete and Separable.
(ii) Define the orthogonal group *O(n)* and prove that det(A) = ± 1 for every $A \in O(n)$.
(b) (i) State and prove the Gram- Schmidt ortho-normalization process of a linearly independent set of vectors in $\mathbb{R}^n$.
(ii) Obtain an orthonormal basis of $\mathbb{R}^3$ from the set $\{\vec{a}, \vec{b}, \vec{c}\}$ where $\vec{a}$ = (1, 2, 3), $\vec{b}$ = (1, 0, 3) and $\vec{c}$ = (2, 1, 1).

Q.2 (a) Define an isometry of $\mathbb{R}^n$ and state and prove the basic result regarding the structure of an isometry of $\mathbb{R}^n$.
(b) Describe, with justification, the set of all isometries of $\mathbb{R}^3$.

Q.3 (a) Let $c : I \rightarrow \mathbb{R}^3$ be a smooth curve. Define the following terms.
(i) *c* being a regular curve
(ii) A reparametrization of *c*,
(iii) *c* being a unit speed curve.
Prove that a unit speed curve can be reparametrized so as to get a unit speed curve.

(b) Reparametrize the curve $c : (-1,1) \rightarrow \mathbb{R}^3$ given by *c(t) = (4 Sint, 4 Cost, 5t)* and get a unit speed reparametrization of it.

Q.4 (a) Define:
(i) the curvature $k(p)$and (ii) the torsion $\tau(p)$ of a regular curve $c : I \rightarrow \mathbb{R}^3$ at a point *p* of it.
Calculate *k(0)*, $\tau(o)$ for the curve $c: (-1,1) \rightarrow \mathbb{R}^3$ given by *c(t)= (2(t- Sin t), 2(1-Cos t), 3t)*
(b) Let $c : I \rightarrow \mathbb{R}^3$ be a smooth curve with $\dot{c}(t) \neq 0$ and $\dot{c}(t)$ not parallel to $\ddot{c}(t)$ for all $t \in I$.
Derive the following fomulae:
$$k(t) = \frac{||\dot{c}(t) * \ddot{c}(t)||}{||\dot{c}(t)||^3} \text{ and } \tau(t) = \frac{\det[\dot{c}(t), \ddot{c}(t), \dddot{c}(t)]}{||\dot{c}(t) * \ddot{c}(t)||^2}$$

Q.5 (a) Define the following terms:
(i) A regular surface *M* in $\mathbb{R}^3$
(ii) Smoothness of a curve $c : I \rightarrow M$
(iii) A vector v being tangential to *M* at a point $p \in M$
(iv) A smooth vector field X tangential to *M*

[P.T.O]

(b) Prove that the set $M = \{(x, y, z) \in \mathbb{R}^3 : z = +\sqrt{x^2 + y^4}\}$ is a regular surface.
Describe a vector basis of $T_p\ (M)$ for the above M at its point $p=(0, 0, 0)$.

Q.6 (a) (i) Prove that the inner product $< ; >$ of $\mathbb{R}^3$ gives rise to an inner product $< ; >_p$ on each tangent space $T_p\ (M)$ of a regular surface M.
(ii) If X, Y are smooth vector fields tangential to M prove that the function : $M \to \mathbb{R}$ given by $p \to < X(p), Y(p) >$ is smooth on M.
(b) When is a surface said to be oriented? Give with justification, an example of a regular surface which is not orientable.

Q.7 Let p be a point of a regular oriented surface M.
(a) (i) Define the Weingarten map $W_p: T_p(M) \to T_p(M)$ and prove that it is self-adjoint with respect to the inner product induced on $T_p(M)$ by the inner product of $\mathbb{R}^3$.
(ii) Define the following terms :
(1) Principal directions of M at P
(2) The Gaussian and mean curvatures of M at p.
(b) Let M be the graph of the function $f: \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = 2x^2 + 3y^2$. Calculate the Gauss and mean curvatures of M at the point $p = (0,0,0)$.

Q.8 Sate and prove Gauss Theorema Egragium.

M.Sc (Maths) [Part – II]

Graph Theory

(Paper- IV) (Revised)

(May-2017)

Q.P. Code: 13189

Duration: 3 hrs **Marks: 80**

N.B. 1) Both the sections are compulsory.

2) Attempt **ANY TWO** questions from each section.

Section I

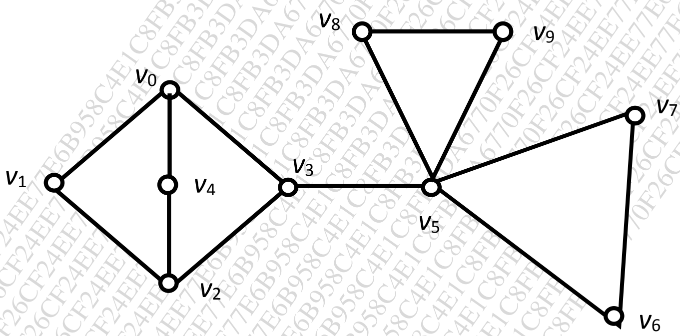
1 (a) Prove the following. 6

i. Every simple graph on finite vertices has at least two vertices of same degree.

ii. Complete graph on n vertices has $\frac{n(n-1)}{2}$ edges.

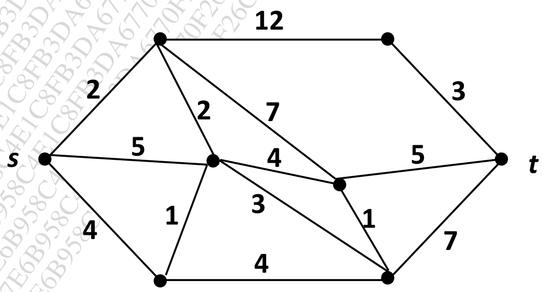
iii. For u, v, and w vertices in a simple connected graph, prove triangle inequality $d(u, v) + d(v, w) \geq d(u, w)$.

(b) For the graph below, find its cut vertices, bridges and blocks. 6



(c) i. For any graph G, prove that $k \leq k' \leq \delta$. 4
k: Vertex connectivity, k': Edge connectivity and
δ: Minimum degree of Graph.

ii. Use Dijkstra's algorithm to find the shortest path from s to t in the following graph. 4

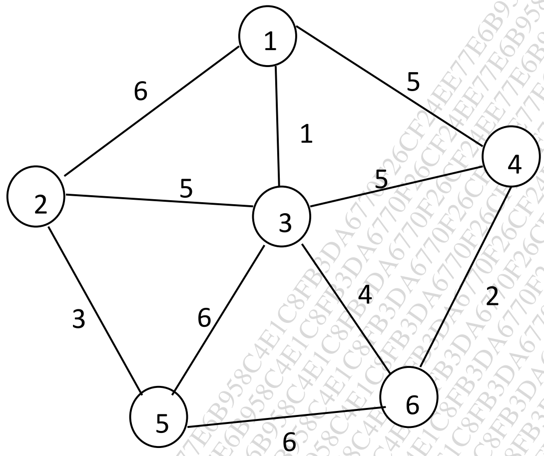


2 (a) . Prove the following with reference to a tree. 6

i. Show that there is unique path between any two vertices of a tree.

ii. A connected graph with n vertices and n edges has exactly one cycle if and only if it has n edges.

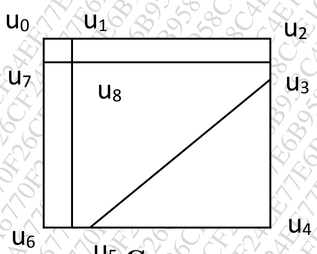
(b) Find minimum cost spanning tree for the following graph using Prim's algorithm. 6



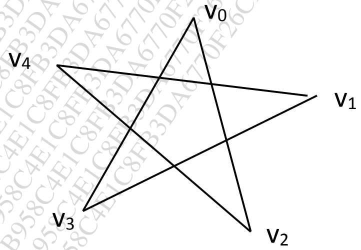
(c) i. Use Huffman coding to encode these symbols with given frequencies: A: 0.10, B: 0.25, C: 0.05, D: 0.15, E: 0.30, F: 0.07, G: 0.08. What is the average number of bits required to encode a symbol? 4

ii. Outline Depth First Search (DFS) algorithm. 4

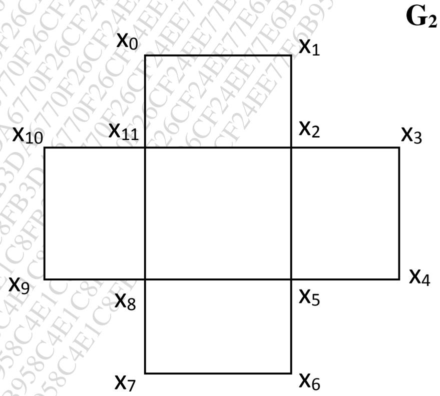
3 (a) Which of the following graphs are Eulerian? Justify you answer. For an Eulerian graph, give one Eulerian circuit in it. 6



G_1



G_2



G_3

(b) State and prove Dirac's theorem that gives sufficient condition for a graph to be Hamiltonian. 6

(c) i. Explain closure of a graph with a suitable example. 8

ii. Write a short note on Chinese postman's problem.

4. (a) State and prove Berge theorem of matching theory. 10

(b) Prove that i) R(2, k) = k for all $k \geq 2$, ii) R(3, 3) = 6 , where R(s, t) is Ramsey number. 10

Section II

5. (a) If G is a planar graph then prove that $\chi(G) \leq 5$. (where $\chi(G)$ is chromatic number of Graph G.) 10

(b) Define Line graph of a graph G. And prove that, if two graphs are isomorphic then their respective line graph are also isomorphic. 10

6. (a) State and prove Euler's formula for planar graph G. 10

(b) Prove that for a planar graph G, G is bipartite if and only if every face of G has even length. 10

7. (a) Prove that a diagraph D is strongly Connected if and only if D contains a directed closed walk containing all its vertices. 10

(b) Define tournament and prove that every tournament D contains a directed Hamiltonian Path. 10

8. (a) Define spectrum of graph and find the spectra of path P_3. 10

(b) Prove that the following are equivalent statements about a graph G 10

(i) G is bipartite.

(ii) The non-zero eigenvalues of G occurs in pairs λ_i , λ_j such that $\lambda_i + \lambda_j = 0$ (with the same multiplicity).

(iii) p(G, x) is a polynomial in x^2 after factoring out the largest common power of x.

(iv) $\sum \lambda_i^{2t+1} = 0 \;\forall t \in N$

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M.Sc (Maths) [Part – II]
Graph Theory
(Paper- IV) (Old)
(May-2017)

Q.P.Code:13373

(3 hours) **Total Marks : 100**

N.B. : 1) Answer any FIVE questions.

2)All questions carry EQUAL marks.

1. (a) Show that the graph G is bipartite if and only if it contains no odd cycle.
   (b) Show that for any edge e of graph G, $\omega(G) \leq \omega(G-e) \leq \omega(G)+1$ where $\omega(G)$ denotes number of components of G. Show further that, in general, $G-e$ cannot be replaced by $G-v$ where v is any vertrx of G.
2. (a) State Kruskasl's algorithm. Prove that any spanning tree constructed by Kruskal's algorithm is optimal.
   (b) (i)Show that every tree with exactly two vertices of degree one is a path.
   (ii)Show that a connected simple graph that has exactly two non cut vertices is a path.
3. (a) If G is simple graph on $p \geq 3$ vertices and $\delta \geq p/2$ then show that G is Hamiltonian.
   (b) A graph G with $p \geq 3$ is 2-connected if and only if any two vertices of it are connected by at least two internally disjoint paths.
4. (a) Show that matching M in a graph G is maximum matching if and only if G contains no Maugmenting path.
   (b) Show that every k regular bipartite graph ($k > 0$) has a perfect matching.
5. (a) Let , $\pi_k(G)$ denote number proper k colorings of G on p vertices. Show that $\pi_k(G)$ is a monic polynomial in k of degree p with integer coefficients, constant term zero. Also prove that the terms alternate in sign. What is coefficient of k^{p-1}.
   (b) Determine vertex and edge chromatic number of K_n and Peterson graph.
6. (a) Define line graph of a graph. Show that line graph of connected graph is isomorphic to the graph if and only if it is cycle.
   (b) State and prove Havel Hakimi theorem about graphic degree sequence. Hence determine if the sequence 7,7,6,6,5,5,4,4,2,2,2 is graphic.
7. (a) Show that every planar graph is five vertex colorable.
   (b) Define dual of a planar graph. If G^*denote dual of a planar graph G, then show that $G^{**} \cong G$ if and only if G is connected.
8. (a) Define Ramsey number $r(p,q);\ p,q \geq 2$. Show that $r(p,q) \leq \binom{p+q-2}{p-1}$.
   (b) If T is a tree on m vertices then show that $r(T, K_n) = (m-1)(n-1)+1$.

M.Sc (Maths) [Part – II]
Numerical Analysis
(May-2017) (Revised)

Q. P. Code : 11371

Revised **(3 Hours)** **Total Marks : 80**

Instructions :

- **Attempt any two questions from each section**
- **All questions carry equal marks. Scientific calculator can be used.**
- **Answers to section I and section II should be written in the same answer book**

Section I(Attempt any two questions)

Q1 a) Define: Absolute error, Relative error and Percentage error. Find the Relative and Absolute error in calculation of $Z = 3x - 3$ by taking approximate value of x as 3.45, and true value of x as 3.457.

b) i) Convert decimal number $(0.859375)_{10}$ to corresponding binary number.
ii) Convert binary number $(10110101.110011100)_2$ to Octal number.

Q2 a) Prove that Newton Raphson method has quadratic rate of convergence. Hence find correct root using Newton-Raphson Method for $f(x) = x^4 - x - 10$ with initial approximation $x_0 = 1$ upto two decimal places.

b) Derive the Muller's formula to find a root of the algebraic or transcendental equation $f(x) = 0$. Perform one iteration with muller method for

$$f(x) = x^2 + x - 1 \;\&\; x_0 = 0, x_1 = 0.5, x_2 = 1$$

Q3 a) Solve the system by using cholesky method

$12x + 4y - z = 15$
$4x + 7y + z = 12$
$-x + y + 6z = 6$

b) Determine the largest eigenvalues and the corresponding eigenvector of the matrix $\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$ correct to three decimal places using power method. Take the initial approximate vector as $v^{(0)} = [1 \quad 1]^t$.

Q4 a) Obtain the Newton's forward interpolating polynomial, for the following tabular data and interpolate the value of the function at $x = 0.0045$.

x	0	0.001	0.002	0.003	0.004	0.005
y	1.121	1.123	1.1255	1.127	1.128	1.1285

b) From the following table, find x for which y is minimum and find this value of y.

x	3	4	5	6	7
y	2.7	6.4	12.5	21.6	34.3

Section II (Attempt any two questions)

Q5 a) Derive two point Guassian quadrature formula to evaluate the integral $\int_{-1}^{1} f(x)\, dx$.

b) Evaluate $\int_0^{\pi} \frac{\sin^2 x}{5 + 4\cos x} dx$ by taking 5 ordinates by Simpson's $\left(\frac{1}{3}\right)^{rd}$ rule.

Q6 a) Obtain the least squares approximation of second degree for $f(x) = sinx$ on $[0, \frac{\pi}{2}]$ with respect to the weight function $w(x) = 1$.

b) Explain the term Discrete Fourier Transform(D.F.T) and compute the (4-point) D.F.T of the sequence x = (1,2,3,4)

TURN OVER

Q7 a) Derive the Milne's Method to solve the differential equation $\frac{dy}{dx} = f(x, y)$with $y(x_0) = y_0$.

b) Solve

$$\frac{dx}{dt} = y - t, \qquad \frac{dy}{dt} = x + t$$

With $x(0) = 1, y(0) = 1$ for $x(0.1)$and $y(0.1)$ by Runge –Kutta Method.

Q8 a) Derive the Bender-Schmidt method to obtain the numerical solution of one dimensional heat equation with initial and boundary conditions.

b) Solve $u_t = u_{xx}$ subject to the initial condition $u(x, 0) = sin\pi x \ \forall\, x \in [0,1]$ and $u(0, t) = 0, u(0, t) = 1 \ \forall\, t > 0$ by the Gauss-Seidel Method.

M.Sc (Maths) [Part – II]
Numerical Analysis
(May-2017) (Old)

QP Code : 74664

External (Scheme A) (3 Hours) [Total Marks:100
Internal (Scheme B) (2 Hours) [Total Marks:40

Note:

(1) External (Scheme A) students answer any five questions.

(2) Internal (Scheme B) students answer any three questions.

(3) All questions carry equal marks. Scientific calculator can be used.

(4) Write on top of your answer book the scheme under which you are appearing.

Que. 1 (a) Define: Absolute error and Percentage error.
Evaluate the sum $S = \sqrt{5} + \sqrt{7} + \sqrt{11}$ upto 4 significant digits and find its absolute and relative errors.

(b) Convert the decimal fraction $(391.6875)_{10}$ to the binary form and then convert to the octal form.

Que. 2 (a) Define the term rate of convergence of iterative method and also find the rate of convergence of the Iteration method.

(b) Perform two iterations of the Birge-Vieta method to find a root (correct upto four decimal places) of the equation $x^3 + 2x^2 + 10x - 20 = 0$. Use initial approximation $p_0 = 1$.

Que. 3 (a) Describe Crout's method to solve the following system of linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3. \end{aligned}$$

(b) Find the Singular Value Decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Que. 4 (a) Estimate the error in Newton's backward difference interpolation formula.

(b) From the following data obtain the first and second derivatives of $y = \log_e x$ at $x = 550$.

x:	510	520	530	540	550
y:	6.2344	6.2538	6.2729	6.2916	6.3099.

Que. 5 (a) Derive Newton-Cotes quadrature formula and use it to derive Simpson's three eighth rule for numerical integration.

(b) Use Romberg's method to evaluate $\int_0^1 \frac{1}{1+x^2}\, dx$. Take $h = 0.5, 0.25, 0.125$.

[TURN OVER

Que. 6 (a) Using the least-squares method, obtain the normal equations to find the values of a, b and c when the curve $y = c + bx + ax^2$ is to be fitted for the data points $(x_i, y_i), i = 1, 2, 3 \ldots\ldots, n$.

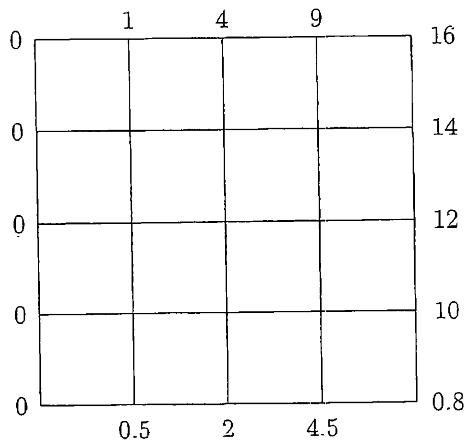
(b) Using Chebyshev polynomials, obtain the least squares approximation of second degree for $f(x) = x^4 + x^3 - x - 9$ on $[-1, 1]$ with respect to the weight function $w(x) = \dfrac{1}{\sqrt{1-x^2}}$.

Que. 7 (a) Derive the Adams-Bashforth corrector formula to solve the differential equation $\dfrac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.

(b) Use Milne's method to compute $y(0.8)$ correct upto four decimal places, given that $\dfrac{dy}{dx} = 1 + y^2$ with $y(0) = 0,\ y(0.2) = 0.2027,\ y(0.4) = 0.4228,\ y(0.6) = 0.6841$.

Que. 8 (a) Derive a Crank-Nicolson's numerical method to obtain the numerical solution of one dimensional heat equation with initial and boundary conditions.

(b) Use Liebmann's method to solve the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ at the interior mesh points of the square region with boundary values given in the following figure.



[Take 2 iterations and obtain result correct upto three decimal places.]

Q.P.Code:16332

Duration: 3 Hours] **[Max. Marks: 100**

N.B. 1) Solve any **Five** questions from question number 1 to 8.
2) All questions carry equal marks.
3) K denote either $I\!R$, the set of real numbers or $\mathbb{C}$, the set of complex numbers.

1. (a) (i) Define the normed linear space. For $1 \leq p < \infty$, consider the set l^p, the set of scalar sequences and $x = (x(1), x(2), \ldots) \in l^p$, define (5)

$$||x||_p = (|x(1)|^p + |x(2)|^p + \ldots)^{1/p}.$$

Then show that l^p is a linear space and $||.||_p$ is a norm on it. Hence, prove that for $1 \leq p < r < \infty$, $l^p \subset l^r$.

(ii) Let Y be a closed subspace of normed space X. For $x + Y$ in the quotient space X/Y, let (5)

$$||(|x + Y|)|| = \inf\{||x + y|| : y \in Y\}$$

be the norm defined on X/Y. Then show that a sequence $(x_n + Y)$ converges to $(x + Y)$ in X/Y if and only if there is a sequence (y_n) in Y such that $(x_n + y_n)$ converges to x in X.

(b) Let X be a normed space and Y be a subspace of X.

(i) If $x \in X$, $y \in Y$ and $k \in K$ then show that $||kx + y|| \geq \text{dist}(x, Y)$. (4)

(ii) If the subset $\{x \in X : ||x|| \leq 1\}$ is compact then X is finite dimensional. (6)

2. (a) (i) Define a strictly convex normed linear space. Hence, prove that for $n \geq 2$ K^n with either of the norms $||.||_1$ and $||.||_\infty$ is not strictly convex. (5)

(ii) let X and Y be normed spaces. If X is finite dimensional, then show that every linear map from X to Y is continuous. (5)

(b) (i) Define the Banach space. Let $C[a, b]$ be the set of real valued continuous functions defined on interval $[a, b]$ and for $x \in C[a, b]$, if we define the norm as $||x|| = \max\limits_{t \in [a,b]} |x(t)|$ then show that $C[a, b]$ is a Banach space. (5)

(ii) If $||.||$ and $||.||_0$ are equivalent norms on X, show that the Cauchy sequences in $(X, ||.||)$ and $(X, ||.||_0)$ are the same. (5)

3. (a) (i) Define the bounded linear operator from the normed space X to normed space Y. Hence define the norm of operator. (4)

(ii) Let X be a normed space and X' be a dual space of X. If X' is separable, then show that X is separable. (6)

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(b) (i) If a normed space X is finite dimensional then show that every linear operator on X is bounded. (5)

(ii) Let X and Y be normed spaces, $T : X \to Y$ be a linear operator. Prove that the operator T is continuous if and only if T is bounded. (5)

4. (a) (i) Prove that a Banach space cannot have denumerable basis. (4)

(ii) Give an example of linear map on normed space X such that the map is continuous with respect to some norm on X, but discontinuous with respect to another map on X. (6)

(b) Let $B(X,Y)$ be the set of bounded linear operators from normed sapce X into normed space Y. $B(x,Y)$ is a normed linear space with norm (10)

$$||T|| = \sup_{x\in X, ||x||=1} ||Tx||.$$

If Y is Banach space then show that $B(X,Y)$ is Banach space.

5. (a) let $a \le p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For fixed $y \in l^p$, let (10)

$$f_y(x) = \sum_{j=1}^{\infty} x(j)y(j), \quad x \in l^p.$$

Then show that f_y belongs to dual space $(l^p)'$ and $||f_y|| = ||y||_q$. Also, show that the map $F : l^q \to (l^p)'$ defined by $F(y) = f_y$ for $y \in l^q$ is a linear isometry from l^q into $(l^p)'$.

(b) Let X, Y and Z be normed spaces.

(i) Let F_1 and F_2 be in $BL(X,Y)$, and $k \in K$. Then prove that (4)

$$(F_1 + F_2)' = F_1' + F_2', \quad (kF_1)' = kF_1'$$

(ii) Let $F \in BL(X,Y)$. Then prove that (6)

$$||F'|| = ||F|| = ||F''|| \quad \text{and} \quad F''J_X = J_Y F,$$

where J_X and J_Y are the canonical embedding of X and Y into X' and Y', respectively.

6. (a) (i) Assume that a bounded linear operator T from a Banach space X onto a Banach space Y has the property that the image $T(B_0)$ of open unit ball $B_0 = B(0,1) \subset X$ contains an open ball about $0 \in Y$. Hence, prove that $T : X \to Y$ is an open mapping. (4)

(ii) State and prove the closed graph theorem. (6)

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(b) (i) Let $X = C[0,1]$ and $T : \mathcal{D}(T) \to X$, $x \to x'$ where the prime denotes differentiation and $\mathcal{D}(T)$ is the subspace of functions $x \in X$ which have a continuous derivative. Then prove that T is closed but not bounded. (4)

(ii) Let $T : \mathcal{D}(T) \to Y$ be a linear operator, where $\mathcal{D}(T) \subset X$ and X, Y are normed spaces. Does the closedness of linear operator T imply boundedness and vice-versa? Justify your answer. (6)

7. (a) Let $\langle , \rangle$ be an inner product on a linear space X, $x, y \in X$. Then prove that:

(i) $4\langle x, y\rangle = \langle x+y, x+y\rangle - \langle x-y, x-y\rangle + i\langle x+iy, x+iy\rangle - i\langle x-iy, x-iy\rangle$ (2)

(ii) $\langle x, y\rangle = 0$ for all y if and only if $x = 0$. (2)

(iii) $|\langle x, y\rangle|^2 \leq \langle x, y\rangle\langle x, y\rangle$, where equality holds if and only if the set $\{x, y\}$ is linearly independent. (6)

(b) (i) Show that the space $C[a,b]$ is not a Hilbert space. (2)

(ii) Give an example of Banach space which is not a Hilbert space. (4)

(iii) Let S and T be linear operators which are defined on all of Hilbert space H and satisfy $\langle Tx, y\rangle = \langle y, Sx\rangle$ for all $x, y \in H$, then show that T is bounded and S is its Hilbert adjoint operator. (4)

8. (a) (i) Let $x(s) - \mu \int_a^b k(s,t)x(t)dt = \tilde{y}(s)$ be the Fredholm integral equation of second kind. Setting $\mu = \frac{1}{\lambda}$ and $\tilde{y}(s) = -y(s)/\lambda$, where $\lambda \neq 0$, we have $Tx - \lambda x = y$ with T defined by $(Tx)(s) = \int_a^b k(s,t)x(t)dt$. If k and $T : X \to X$ as defined above and T is compact linear operator on a normed space X, then show that the Fredholm alternative holds for T_λ. (5)

(ii) Let $J = [a,b]$ be any compact interval and suppose that k is continuous on $J \times J$. Then show that the operator $T : X \to X$ defined by (5)

$$(Tx)(s) = \int_a^b k(s,t)x(t)dt,$$

where $X = C[a,b]$, is a compact linear operator.

(b) Consider the linear integral equation (10)

$$x(s) - \mu \int_0^1 k(s,t)x(t)dt = 1,$$

where $k(s,t) = s(1+t)$. Determine the eigenvalues and eigenfunctions.

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