M.Sc (Mathematics) (Part-II)

Algebra – II

(Paper – I)(OCT-16)

QP Code : 74673

Scheme A (External)] (3 Hours) [Total Marks:100
Scheme B (Internal)] (2 Hours) [Total Marks: 40

Instructions:

- **Mention** on the top of the **answer book** the **scheme** under which you are appearing
- **Scheme A** students should **attempt any five** questions
- **Scheme B** students should **attempt any three** questions
- All **questions** carry **equal marks**

1. (a) Let G be a finite group of order n and p be a prime such that p^k divides n and p^{k+1} does not divide n. Prove that G has a subgroup of order p^k.

   (b) Show that a group of order 42 is not simple.

2. (a) If G is a group such that a normal subgroup H and G/H are both solvable then show that G is also solvable.

   (b) Define a nilpotent group. Show that a group G is nilpotent if and only if there exists a positive integer n such that $G^n = (e)$ where $G^0 = G$ and $G^{i+1} = [G, G^i]$.

3. (a) Let L/F and F/K be field extensions. Prove that $[L : K]$ is finite if and only if $[L : F]$ and $[F : K]$ are finite.

   (b) Show that the characteristic of a field is either zero or a prime integer. Next, show that if a field is a finite field then Char $F \neq 0$. Is the converse true?

4. (a) Define a splitting field of a polynomial $f(x)$ over a field K. If $f(x)$ is a monic polynomial over a field K, prove that there exists a splitting field of $f(x)$ over K.

   (b) Determine the splitting field and its degree over $\mathbb{Q}$ for the polynomial $x^{11} - 1$.

5. (a) State and prove primitive element theorem.

   (b) Prove or Disprove: There exists a field having 80 elements. Justify your answer.

6. (a) Prove that K is normal extension of F if and only if $G(E/K)$ is normal subgroup of $G(E/F)$.Next, show that in that case, $G(E/F)/G(E/K)$ is isomorphic to $G(K/F)$

   (b) Let ω_n be a primitive n-throot of unity in $\mathbb{C}$. Prove that Galois group of $\mathbb{Q}(\omega_n)/\mathbb{Q}$ is isomorphic to the multiplicative group of units $\mathbb{Z}/n\mathbb{Z}$.

7. (a) Show that a submodule of a free module over a PID is free.

   (b) Define free module and torsion module. Give an example of a free module which is not torsion-free.

8. (a) Prove that any Principal ideal domain is Noetherian.

   (b) R is a commutative ring with unity. M is an R-module.Show that the following are equivalent. (i) Ascending chain condition holds in M (ii) Every submodule is finitely generated.

M.Sc (Mathematics) (Part-II)
Analysis - II
(Paper – II) (OCT-16)

QP Code : 74681

[3 hours –Scheme A Idol students] Total Marks : 100

[3 hours –Scheme B] Total Marks : 40

N.B (1)Scheme A (IDOL) students will attempt any Five questions.

Scheme B students will attempt any Three questions

(2) All Questions Carry Equal Marks. Justify the answers with Mathematical justification.

Q.1.

(a) Show that a nonmeasurable Set exists, in the real line

(b) i) State two differences between the outer measure and measure . Suppose A is a set such that for each $\varepsilon > 0$, A $\subset B_\varepsilon$ where B_ε is a set with outer measure $< \varepsilon$. What can be said about the measurability of the set A ?.

ii) Is Lebesgue measure on the real line complete? Justify your answer .

Q. 2

(a) Show that, $\underline{lim}_{n\to\infty} \inf f_n$ is a measurable function if each (f_m) is a measurable function.

(b) Show that product of measurable functions is a measurable function and $\sqrt{f}$ is a measurable function, when f is a nonnegative measurable function ?

Q. 3

(a) State and Prove Fatou's lemma ? Is the analogous statement true for monotone increasing sequence of functions ? Justify your answer.

(b) Show that for a nonnegative Lebesgue integrable function f if the integral of f over a measurable set is zero then f is zero almost everywhere on the set. Show that for a strictly positive Lebesgue integrable function f, $\int_a^b f > 0$, for any closed interval [a, b], $a \neq b$.

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Q. 4

(a) Is the product of Lebesgue integrable functions Lebesgue integrable? Justify ?

What about the product of a Lebesgue integrable function and a measurable function?

(b) Show that a Riemann integrable function over a bounded interval is Lebesgue integrable .

If | f | is Lebesgue integrable, Is f necessarily Lebesgue integrable ? justify

Q. 5

(a) i) Evaluate $\int_0^{\pi/2} \int_0^1 x \cos(xy)\, dx\, dy$. Do both iterated iterated integrals exist ?.Justify

ii) Consider f(x, y) = x –y when x, y are integers or y = 0 and f(x, y) = x/y ,otherwise. Is f integrable over [0, 1] × [0, 1].

b) State Tonneli's theorem . Deduce it from Fubini's theorem.

Q. 6

(a) i)Let g(t) = $\frac{1+(1+t).e^{-t}}{1+t^2}$, t > 0, t $\in \Re$. Show that g is Lebesgue integrable over [0, ∞).

ii) Give an example of a function so that the improper Riemann integral of f exists over some but the Lebesgue integral of f does not exist.

b) Show that a Riemann integrable function is Lebesgue integrable

Q. 7

a) State and Prove Hoder's inequality and Minkowaski's inequality.

b) i) Define Fourier transform. State Plancharel's theorem for $£^2$.

ii) Does the Fourier series of a continuous periodic function convergent pointwise to the function. Justify the answers with Mathematical justification.

Q. 8 (a) State and prove Bessel's inequality and Parsevel's identity for Fourier series.

(b) State and Prove Riesz Fischer's theorem for $£^2$ space .

M.Sc (Mathematics) (Part-II)

Differential Geometry

(Paper – III)

(OCT-16)

QP Code : 74721

Duration:[3 Hours] [Marks: 100]

N.B. 1) All questions carry equal marks.
2) Attempt any **five** questions.

1. (a) (i) Let V is an inner product space then show that V has an orthonormal basis. (5)
(ii) For any $x, y \in V$, where V is an inner product space, show that $||x-y||^2 = ||x||^2 + ||y||^2$ if and only if x is orthogonal to y. (5)
(a) (i) Find an equation of the plane that passes through the two points $(1, 0, -1)$ and $(-1, 2, 1)$ and is parallel to the line of intersection of the planes $3x + y - 2Z = 6$ and $4x - y + 3z = 0$. (5)
(ii) Let $m : I\!R^n \longrightarrow I\!R^n$. Show that m is an isometry which fixes the origin if and only if $\langle m(x), m(y) \rangle = \langle x, y \rangle$ for all $x, y \in I\!R^n$. (5)

2. (a) Explain Picard's scheme of approximation for the solution of initial value problem $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$ and hence find approximate solution of $\frac{dy}{dx} = x + y$ with $y(0) = 1$. (10)
(b) Find approximate solution upto t^4 of the initial value problem $\frac{dx}{dt} = 2x + ty$, $\frac{dx}{dt} = xy$ with $x(0) = 1$ and $y(0) = 1$. (10)

3. (a) If $f : U \longrightarrow I\!R$ is a differentiable function in an open set U of $I\!R^2$ then show that the subset of $I\!R^3$ given by $(x, y, f(x,y))$ for $(x, y) \in U$ is a regular surface and hence or otherwise prove that every plane in $I\!R^3$ is a regular surface. (10)
(b) (i) Define orientable surface. The surface S be defined by a smooth function $f(x, y, z) = 0$ such that f_x, f_y and f_z do not vanish simultanously at any point of S. Show that the vector $\nabla f = (f_x, f_y, f_z)$ is perpendicular to the tangent plane at every point of S. Is S is orientable? Justify. (5)
(ii) Find the values of c for which the set $f(x, y, z) = c$ is a regular surface, where $f(x, y, z) = (x + y + z - 1)^2$. (5)

4. (a) Let $\gamma(t)$ be a regular curve in $I\!R^3$ with nowhere vanishing curvature. Then show that its torsion is given by $\tau = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{||\dot{\gamma} \times \ddot{\gamma}||^2}$, where $\cdot$ represent differentiation w.r.t. t and hence compute the torsion of the circular helix $\gamma(t) = (acost, asint, bt)$. (10)
(b) (i) Write parametric equation of circle and show that the curvature of a circle is inversly proportional to its radius. (5)
(ii) Show that the curve $\gamma(t) = (\frac{1+t^2}{t}, t+1, \frac{1-t}{t})$ is planar. (5)

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5. (a) State and prove the generalized Stoke's theorem for the integration of exterior forms. (10)

(b) (i) Prove that the local maxima and local minima of function f are critical points of f. (5)

(ii) If $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function then show that (5)

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_n}dx_n.$$

6. (a) (i) Define a self adjoint linear map and show that the differential $dN_p : T_p(S) \to T_p(S)$ of the Gauss map is a self adjoint linear map. (5)

(ii) Define normal curvature and compute normal curvature along a direction of $T_p(S)$. (5)

(b) Calculate Gaussian curvature and mean curvature of the points of torus (10)

$\sigma(u,v) = ((a + rcosu)cosv, (a + rcosu)sinv, rsinu)$, $0 < u < 2\pi$ and $0 < v < 2\pi$.

7. (a) Define and derive the expression for first fundamental forms of regular surface in $\mathbb{R}^3$ and hence show that $\|\sigma_u \times \sigma_v\| = (EG - F^2)^{\frac{1}{2}}$ where E,F and G are notations as in first fundamental form. (10)

(b) (i) Let S_1 be the infinite strip in the xy plane given by $0 < x < 2\pi$ and S_2 be the circular surface $x^2 + y^2 = 1$ with the rulling given by $x = 1, y = 0$ removed. Prove or disprove the map $f : S_1 \to S_2$ is an isometry. (5)

(ii) Find a unit speed reparametrization of the curve $\gamma(t) = (e^t cost, e^t sint)$. (5)

8. (a) Compute curvature k, torsion τ, tangent t, normal n and binormal b for parametrized curve $\gamma(t) = (\frac{4}{5}cost, 1 - sint, \frac{-3}{5}cost)$. (5)

(b) Find the equation of tangent plane to the surface patch $\sigma(u,v) = (u, v, u^2 - v^2)$ at $(1,1,0)$. (5)

(c) Define an isometry of $\mathbb{R}^n$. Prove or disprove composition of an isometry is an isometry. (5)

(d) Find the length of the part of the curve $\sigma(u,v) = (ucosv, usinv, u)$ with $0 \le t \le \pi$ where $u = e^{\lambda t}$,$v = t$ and λ is constant. (5)

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M.Sc (Mathematics) (Part-II)

Graph Theory

(OCT-16)

QP Code : 74795

External (Scheme A)	**(3 Hours)**	**Total marks : 100**
Internal/External (Scheme B)	**(2Hours)**	**Total marks : 40**

N.B. 1) **Scheme A** students answer any **five** questions.
2) **Scheme B** students answer any **three** questions.
3) **All** questions carry **equal** marks.
4) Write on **top** of your answer book the **scheme** under which you are **appearing**.

1. (a) Show that a simple (p,q) graph G with $q > p^2/4$ contains a triangle. State clearly the theorem used.

(b) Prove that graph is bipartite if and only if it has no odd cycle.

2. (a) State and prove Kruskal's algorithm for finding a minimum weight spanning tree.

(b) State Erdos-Gallai conditions for existence of degree sequence to be graphic and show that these conditions are necessary.

3. (a) Prove that the matching in a graph G is maximum if and only if G contains no M augmenting path.

(b) Which is the Hall's matching condition for bipartite graph? Prove it.

4. (a) What is the Purfer code for a labeled tree? Draw a labeled tree with Purfer code 7,2,4,5,3,3,1 .

(b) State Menger's theorem and give one of its application.

5. (a) Define chromatic number of graph G. Prove that if G contains complete graph Kn then $\chi(G) \geq n$

(b) Show that there is no graph with chromatic polynomial $\lambda^3 - 4\lambda^2 + 3\lambda$.

6. (a) Prove that a connected graph is isomorphic to its line graph if and only if it is a cycle.

(b) If G is a (p, q) graph with at least three vertices and $\delta(G) \geq \frac{p}{2}$ then prove that G is hamiltonian

7. (a) Prove that every planar graph G with $p \geq 4$ has at least four points of degree not exceeding 5.

(b). Prove that edges in a plane graph G form a cycle in G if and only if the corresponding dual edges form a bond in G*(G* is planar dual).

8. (a) Define Ramsey Number R(p,q) for p, $q \geq 2$. Show that $R(p,q) \leq R(p-1,q) + R(p,q-1)$ if p, $q \geq 3$.

(b) If T is an m-vertex tree then prove that $R(T, K_n) = (m-1)(n-1) + 1$.

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M.Sc (Mathematics) (Part-II)
Numerical Analysis
(OCT-16)

QP Code : 74663

External (Scheme A) (3 Hours) [Total Marks:100
Internal (Scheme B) (2 Hours) [Total Marks:40

Note:

(1) External (Scheme A) students answer any five questions.

(2) Internal (Scheme B) students answer any three questions.

(3) All questions carry equal marks. Scientific calculator can be used.

(4) Write on top of your answer book the scheme under which you are appearing.

Que. 1 (a) Define: Absolute error, Relative error and Percentage error. Round-off the number 658394 upto four significant figures and find the absolute error, relative error and percentage error.

(b) Convert the hexadecimal number $(BBC.10)_{16}$ to the binary form and then convert to the octal form.

Que. 2 (a) Derive the Chebyshev iteration formula to find a root of the algebraic or transcendental equation $f(x) = 0$.

(b) Perform two iterations of the Bairstow method to extract a quadratic factor $x^2 + px + q$ correct upto four decimal places from the equation $x^4 + 5x^3 + 3x^2 - 5x - 9 = 0$. Use initial approximations $p_0 = 3,\ q_0 = -5$.

Que. 3 (a) Let $A = [a_{ij}]$ be a real matrix of order $m \times n$ with $m \geq n$. Derive a formula giving Singular Value Decomposition of a matrix A.

(b) Find the inverse of a following matrix by Gauss elimination method

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}.$$

Que. 4 (a) Derive Lagrange's interpolation formula for unequal intervals.

(b) Use Newton's divided difference formula to find the fourth degree curve passing through the points $(2, 3)$, $(4, 43)$, $(5, 138)$, $(7, 778)$ and $(8, 1515)$.

Que. 5 (a) Derive Newton-Cotes quadrature formula and use it to derive Simpson's rule for numerical integration.

(b) Evaluate $\int_0^1 \int_0^1 \frac{\sin xy}{1 + xy}\, dx\, dy$ using Trapezoidal rule with $h = k = 0.5$.

Que. 6 (a) Use Gram-Schmidt orthogonalizing process to determine first two orthogonal polynomials which are orthogonal on $[0, 1]$ with respect to the weight function $w(x) = 1$. Using these polynomials, obtain the least squares approximation of first degree for the function $f(x) = e^x$ on $[0, 1]$.

(b) Explain the term Discrete Fourier Transform (D.F.T.) and compute the (4-point) inverse D.F.T. of the sequence $X = (2.5, -0.5i, -0.5, 0.5i)$.

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Que. 7 (a) Derive the Milne's corrector formula to solve the differential equation $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.

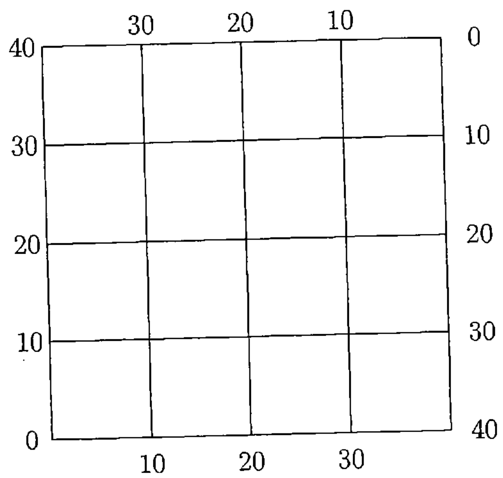
(b) Solve

$$\frac{dy}{dx} = yz + x$$
$$\frac{dz}{dx} = xz + y$$

given that $y(0) = 1, z(0) = -1$ for $y(0.1), z(0.1)$ by Runge-Kutta method.

Que. 8 (a) Derive a Bender-Schmidt numerical method to obtain the numerical solution of one dimensional heat equation with initial and boundary conditions.

(b) Use Liebmann's method to solve the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ at the interior mesh points of the square region with boundary values given in the following figure.



[Take 2 iterations and obtain result correct upto three decimal places.]

M.Sc (Mathematics) (Part-II)
Functional Analysis
(OCT-16)

QP Code : 74768

(3 Hours) **[Total Marks : 100**

N.B. 1) Solve any **Five** questions from question number 1 to 8.
2) All questions carry equal marks.
3) K denote either $\mathbb{R}$, the set of real numbers or $\mathbb{C}$, the set of complex numbers.

1. (a) (i) Define the normed linear space, Banach Space. Verify that $\mathbb{R}^n$ is a Banach space with norm defined by: (5)

$$x = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n \quad \text{and} \quad ||x|| = \Big(\sum_{j=1}^{n} |\xi_j|^2\Big)^{1/2}.$$

(ii) Let X be a normed space, Y be a closed subspace of X and $Y \neq X$. Let r be a real number such that $0 < r < 1$. Then show that there exists some $x_r \in X$ such that $||x_r|| = 1$ and $r \leq d(x_r, Y) \leq 1$. (5)

(b) Let X be a normed space. Prove that the following conditions are equivalent. (10)
(i) Every closed and bounded subset of X is compact.
(ii) The subset $\{x \in X : ||x|| \leq 1\}$ of X is compact.
(iii) X is finite dimensional.

2. (a) (i) Prove that every finite dimensional subspace of Y of normed space X is complete. (5)

(ii) Define equivalent norms. Hence prove that on a finite dimensional vector space X, any norm $||.||$ is equivalent to any other norm $||.||_0$. (5)

(b) (i) Give example of subspaces of l_∞ and l_2 which are not closed. (5)
(ii) If $||.||$ and $||.||_0$ are equivalent norms on X, show that the Cauchy sequences in $(X, ||.||)$ and $(X, ||.||_0)$ are the same. (5)

3. (a) (i) Let Y and Z be subspaces of normed space X, and suppose that Y is a closed and is a proper subset of Z. Then show that for every real number θ in the interval $(0,1)$ there is a $z \in Z$ such that $||z|| = 1$, $||z - y|| \geq \theta$ for all $y \in X$. (6)

(ii) Let $X = \mathbb{R}^3$. For $x = (x(1), x(2), x(3)) \in X$, let (4)

$$||x|| = \Big[\big(|x(1)|^2 + |x(2)|^2\big)^{3/2} + |x(3)|^3\Big]^{1/3}.$$

Then show that $||.||$ is a norm on $\mathbb{R}^3$.

(b) Let X be a linear space over $\mathbb{R}$ and Y be a subspace of X which is not a hyperspace in X. If x_1 and x_2 are in X but not in Y, then prove that there is some x in X such that for all $t \in [0,1]$, $tx_1 + (1-t)x \notin Y$ and $tx_2 + (1-t)x \notin Y$. Hence prove that if X is normed space, then compliment Y^c is connected. (10)

4. (a) Let E be a non empty convex subset of a normed space X over K. Prove that:
   (i) If $a \in X$ but $a \notin \bar{E}$, then there are $f \in X'$ (dual of a normed space X) and $t \in I\!R$ such that $\mathrm{Re}f(x) \leq t < \mathrm{Re}f(a)$ for all $x \in \bar{E}$. (5)
   (ii) If $E^0 \neq \phi$ (Interior of E) and b belongs to the boundary of E in X, then there is non zero $f \in X'$ such that $\mathrm{Re}f(x) \leq \mathrm{Re}f(b)$ for all $x \in \bar{E}$. (5)

   (b) (i) Let $X = C([a,b])$ with sup norm, Y be the subspace of X consisting of all constant functions and $g(y) = y(a)$ for $y \in Y$. For a nondecreasing function on $[a,b]$ such that $z(b) - z(a) = 1$, define (6)

$$f_z(x) = \int_a^b x dz, \quad x \in X.$$

   Then show that f_z is a Hahn-Banach extension of g.
   (ii) Prove that a normed space Y, $BL(X,Y)$ space of bounded linear maps from a normed space X to a normed space Y, $BL(X,Y) = \{0\}$ if and only if $Y = \{0\}$. (4)

5. (a) State and prove Uniform Boundedeness Principle. (10)
   (b) (i) Let X be a normed space, E be the subset of X. Then prove that E is bounded in X if and only if $f(E)$ is bounded in K for every $f \in X'$ (dual of normed space X). (6)
   (ii) Let X and Y be normed spaces and $F : X \to Y$ be a linear. Then prove that F is continuous if and only if $g \circ F$ is continuous for every $g \in Y'$ (dual of normed space Y). (4)

6. (a) (i) Define the terms continuous map and closed map. Hence prove that continuous map is closed. Does the converse is true? Justify your answer. (6)
   (ii) Let X be a linear space over K. Consider subsets U and V of X, and $k \in K$ such that $U \subset V + kV$. Then prove that for every $x \in U$, there is a sequence (v_n) in V such that (4)

$$x - (v_1 + kv_2 + \ldots + k^{n-1}v_n) \in k^n U, \quad n = 1, 2, 3, \ldots$$

   (b) (i) Let X and Y be normed spaces and $F : X \to Y$ be a linear. Then prove that F is an open map if and only if there exists some $\gamma > 0$ such that for every $y \in Y$, there is some $x \in X$ with $F(x) = y$ and $||x|| \leq \gamma ||y||$. (6)
   (ii) Let X and Y be normed spaces. Prove that if Z is closed subspace of X, then the quotient map Q from X to $X|Z$ is continuous and open. (4)

7. (a) (i) Define an inner product space and Hilbert space. Show that the Unitary space $\mathbb{C}^n$ is a Hilbert space with inner product given by (6)

$$\langle x, y \rangle = \xi_1\bar{\eta}_1 + \xi_2\bar{\eta}_2 + \ldots + \xi_n\bar{\eta}_n$$

   where $x = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{C}^n$, $y = (\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{C}^n$.

(ii) Give an example of Banach Space which is not a Hilbert space. Verify your answer. (4)

(b) (i) If a linear operator T is defined on all of a complex Hilbert Space H and satisfies $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$, then show that T is bounded. (6)

(ii) Let S and T be linear operators which are defined on all of Hilbert space H and satisfy $\langle Tx, y\rangle = \langle y, Sx\rangle$ for all $x, y \in H$, then show that T is bounded and S is its Hilbert adjoint operator. (4)

8. (a) (i) Define Fredholm alternative. Let $T : X \to X$ be a compact linear operator on a normed space X,and let $\lambda \neq 0$. Then show that $T_\lambda = T - \lambda I$ satisfies the Fredholm alternatives. (6)

(ii) Formulate the Fredholm alternative for a system of n linear algebraic equations in n unknowns. (4)

(b) Solve the following linear integral equation. (10)

$$x(s) - \mu \int_0^1 x(t)dt = 1$$

Find all solutions of the corresponding homogeneous equation.

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