**Exam Date: 25/4/2019** **SYBSc/Sem-IV/Maths-I (Revised)** **QP Code: 66046**

(3 Hours) [Total Marks: 100]

**Note:** $(i)$ All questions are compulsory. $(ii)$
Figures to the right indicate marks for respective parts.

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| Q.1 | Choose correct alternative in each of the following (20) | | | | |
| $i.$ | The norm of the partition P = { − 6, − 5.5, − 5, − 4.8, − 3} is | | | | |
| | (a) | 0.5 | (b) | 1.8 | |
| | (c) | 0.2 | (d) | None of these | |
| | Ans | (b) 1.8 | | | |
| | | | | | |
| $ii.$ | Let $f$: [0,1]→ℝ is defined as $f(x) = 1$ , $if\ x \epsilon [0,1] \cap \mathbb{Q}$ <br> $= 0$ , $o,w.$ <br> Then for any partition $P$ of [0,1] , | | | | |
| | (a) | $L(P,f) = \frac{1}{2}\ and\ \ U(P,f) = 1$ | | | |
| | (b) | $L(P,f) = 0\ \ and\ \ U(P,f) = 0$ | | | |
| | (c) | $L(P,f) = 0\ and\ \ U(P,f) = 1$ | | | |
| | (d) | None of these | | | |
| | Ans | (c) $L(P,f) = 0\ and\ \ U(P,f) = 1$ | | | |
| | | | | | |
| $iii.$ | Let $f\ and\ g$ be functions such that the function $f + g$ is integrable on $I$ , then | | | | |
| | (a) | Both $f$ and $g$ must be integrable on $I$ | (b) | At least one of $f$ and $g$ must be integrable on $I$ | |
| | (c) | $f$ and $g$ may or may not be integrable on $I$ | (d) | None of these | |
| | Ans | $f$ and $g$ may or may not be integrable on $I$ | | | |
| | | | | | |
| $iv.$ | If $F$:[0,1]→ℝ is defined by $F(x) = \int_0^x sin(t^2)dt$ ,then | | | | |
| | (a) | $F^{'}(x) = \sin(x^2)$ | (b) | $F^{'}(x) = \cos(x^2)$ | |
| | (c) | $F^{'}(x) = \sin x$ | (d) | None of these | |
| | Ans | $F^{'}(x) = \sin(x^2)$ | | | |
| | | | | | |
| $v.$ | The improper integral $\int_1^{\infty} \frac{x+1}{x^5+5} dx$ is | | | | |
| | (a) | Type I and Converges | (b) | Type I and diverges | |
| | (c) | Type II and Converges | (d) | None of these | |
| | Ans | (a) Type I and Converges | | | |

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| *vi.* | If $f:\left[0,\frac{\pi}{2}\right]\to\mathbb{R}$ is defined by $f(x)=\int_{sinx}^{cosx} e^{-t^2}dt$,then $f'\left(\pi/4\right)$ equals | | | |
| | (a) | $\sqrt{\frac{1}{e}}$ | (b) | $-\sqrt{\frac{2}{e}}$ |
| | (c) | $\sqrt{\frac{2}{e}}$ | (d) | $-\sqrt{\frac{1}{e}}$ |
| | Ans | (b) $-\sqrt{\frac{2}{e}}$ | | |
| | | | | |
| *vii.* | Length of the curve $y=2x$ between $x=0$ & $x=1$ is | | | |
| | (a) | $\sqrt{5}$ | (b) | $\sqrt{3}$ |
| | (c) | $\sqrt{2}$ | (d) | 1 |
| | Ans | (a) $\sqrt{5}$ | | |
| | | | | |
| *viii.* | Which of the following definite integral represents the area of region bounded by the graphs of $y=x^2$& $y=x$? | | | |
| | (a) | $\int_{-1}^{1}(x-x^2)dx$ | (b) | $\int_{0}^{1}(x-x^2)dx$ |
| | (c) | $\int_{0}^{2}(x^2-x)dx$ | (d) | $\int_{-1}^{0}(x^2-x)dx$ |
| | Ans | (b) $\int_0^1(x-x^2)dx$ | | |
| | | | | |
| *ix.* | $\beta(m,n)=$ __________ | | | |
| | (a) | $\frac{\Gamma(m)\Gamma(n)}{\Gamma(mn)}$ | (b) | $\frac{\Gamma(mn)}{\Gamma(m+n)}$ |
| | (c) | $\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ | (d) | None of these |
| | Ans | (c) $\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ | | |
| | | | | |
| *x.* | $\int_0^{\frac{\pi}{2}}(sin\theta)^5(cos\theta)^7d\theta.$ | | | |
| | (a) | $\frac{3!2!}{6!}$ | (b) | $\frac{4!3!}{7!}$ |
| | (c) | $\frac{1}{60}$ | (d) | $\frac{1}{120}$ |
| | Ans | (d) $\frac{1}{120}$ | | |
| | | | | |
| Q2. | Attempt any **ONE** question from the following: (08) | | | |

| | | |
|---|---|---|
| $a)$ | i. | Let $f$: $[a, b]\to\mathbb{R}$ be a bounded function. Prove that $f$ is R-integrable on $[a, b]$ iff for any $\in\ > 0$, there exists a partition $P_{\in}$ of $[a, b]$ such that <br> $U(f,P_{\in}) - L(f,P_{\in}) < \in$. |
| | Ans | Proof: ($\Rightarrow$) Given $f$ is $R$ integrable on $[a, b]$. <br> T.P.T: $\forall\ \epsilon > 0, \exists$ apartition $P_\epsilon$of $[a,b]$ such that, $U(f,P_\epsilon)$ - $L(f, P_\epsilon) < \epsilon$. <br> Let, $\epsilon > 0$ be any real number, as $f$ is $R$ integrable, <br> $\therefore U(f) = L(f)$ <br> Where, $U(f) = \inf\{U(f, P)$:$P$ is any partion of $[a, b]\}$ <br> And $L(f) = \sup\{L(f, P)$:$P$ is any partion of $[a, b]\}$ <br> $\therefore$ for given $\epsilon > 0, \exists$ a partition $P_1$ of $[a, b]$ such that, <br> $U(f) \leq U(f, P_1) < U(f) + \frac{\epsilon}{2}$ (1) <br> Also, for given $\epsilon > 0, \exists$ a partition $P_2$ of $[a, b]$ such that, <br> $L(f)$ - $\frac{\epsilon}{2} < L(f, P_2) \leq L(f)$ <br> or $-L(f) \leq -L(f, P_2) < -L(f) + \frac{\epsilon}{2}$ (2) <br> from (1) and (2) <br> $U(f)$ - $L(f) \leq U(f, P_1)$ - $L(f, P_2) < U(f)$ - $L(f) + \epsilon$ <br> $\therefore 0 \leq U(f, P_1)$ - $L(f, P_2) < \epsilon$ (3) ($\because U(f) = L(f)$)… 3 marks <br> Now taking $P_\epsilon = P_1 \cup P_2$, <br> $\therefore U(f, P_\epsilon) \leq U(f, P_1)$ & $L(f, P_\epsilon) \geq L(f, P_2)$($\because P_1 \subseteq P_\epsilon$ & $P_2 \subseteq P_\epsilon$) <br> $\therefore U(f, P_\epsilon)$ - $L(f, P_\epsilon) \leq U(f, P_1)$ - $L(f, P_2) < \epsilon$ by (3) <br> $\therefore U(f, P_\epsilon)$ - $L(f, P_\epsilon) < \epsilon$ …… 3 marks <br> ($\Leftarrow$) Given: $\forall\ \epsilon > 0, \exists$ apartition $P_\epsilon$of $[a,b]$ such that, $U(f,P_\epsilon)$ - $L(f, P_\epsilon) < \epsilon$. <br> T.P.T: $f$ is $R$ integrable on $[a, b]$. <br> i.e. T.P.T: $L(f) = U(f)$ <br> $\because\ \forall\ \epsilon > 0, \exists$ apartition $P_\epsilon$of $[a,b]$ such that, $U(f,P_\epsilon)$ - $L(f, P_\epsilon) < \epsilon$ <br> We know that, $U(f) \leq U(f, P_\epsilon)$& $L(f) \geq L(f, P_\epsilon)$ <br> $\therefore 0 \leq U(f)$ - $L(f) \leq U(f, P_\epsilon)$ - $L(f, P_\epsilon) < \epsilon$( $\because U(f) \geq L(f)$) <br> $\therefore 0 \leq U(f)$ - $L(f) < \epsilon$ $\therefore U(f) = L(f)$ 2M |
| | ii. | . If $f$ is R-integrable on $[a,b]$ and $a < c < b$ then prove that <br> $\int_a^b f = \int_a^c f + \int_c^b f$ |

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| | Ans | Claim : $f$ is integrable on $[a, c]$ and $[c, b]$<br>Given $f$ is integrable on $[a, b]$<br>for any $\in\ > 0$ $\exists$ a partition $P'$ of $[a, b]$ such that $U(f, P') - L(f, P') < \ \in$<br>take $P = P' \cup \{c\} \Longrightarrow U(f, P) \le U(f,P')$ $and$ $L(f, P') \le L(f, P)$<br>$\therefore\ U(f, P) - L(f, P) \le U(f,P') - L(f, P') < \ \in$ 1M<br>Let $P = \{a = x_0, x_1, \cdots\cdots, x_j = c\ ,\ x_{j+1}, \cdots\cdots, x_n = b\}$<br>Let $P_1 = \{a = x_0, x_1, \cdots\cdots, x_j = c\}$ $and$ $P_2 = \{x_j = c\ ,\ x_{j+1}, \cdots\cdots, x_n = b\}$<br>$U(f, P) = U(f, P_1) + U(f, P_2)$ $and$ $L(f, P) = L(f, P_1) + L(f, P_2)$ 1M<br>$\therefore\ U(f, P) - L(f, P) = [U(f, P_1) - L(f, P_1)] + [U(f, P_2) - L(f, P_2)]$<br>$\therefore\ U(f, P_1) - L(f, P_1) < \ \in$ and $U(f, P_2) - L(f, P_2) < \ \in$<br>Hence $f$ is integrable on $[a, c]$ and $[c, b]$. 2M<br>Claim : $\int_a^b f = \int_a^c f + \int_c^b f$<br>LHS = $\int_a^b f = \overline{\int_c^b} f \le U(f, P) < \ \in + L(f, P) < \ \in + L(f, P_1) + L(f, P_2) < \ \in + \int_a^c f + \int_c^b f$<br>therefore $\int_a^b f \le \int_a^c f + \int_c^b f$ 2M<br>Similarly one can show $\int_a^b f \ge \int_a^c f + \int_c^b f$ 1M<br>Hence $\int_a^b f = \int_a^c f + \int_c^b f$ 1M |
| | | |
| Q.2 | Attempt any **TWO** questions from the following: (12) | |
| *b*) | i. | Let $f$ be a bounded function on [a, b]. Let $P$ and $P'$ are two partitions of [a, b] with $P \subseteq P'$. Show that U (f, $P$) $\ge$ U (f, $P'$ ) |
| | Ans | Let P={x0, x1, …, xn ] be a partition of [a,b]. Given P is subset of Q<br>Let y1,y2 ,….ym are extra points which are in Q but not in P.<br>Let $P_1$ = P $\cup$ {y1}.let y1 $\in [xj - 1, xj]$ …..2 marks<br>U(P,f) – U($P_1$ ,f)=($M_j$ –$M'_j$ )(y1-xj-1)+ ($M_j$ –$M''_j$ )(xj- y1) $\ge 0$<br>Where $M_j$ = sup{f(x)/x $\in [x_{j-1}, x_j]$ }<br>$M'_j$ = sup{f(x)/x $\in [x_{j-1}, y1]$ }<br>$M''_j$ = sup{f(x)/x $\in [y1, x_j]$ }<br>As $M_j \ \ge$ M’j and $M_j \ \ge$ M’’j ……. 3 marks<br><br>Therefore U(P,f) $\ge$ U($P_1$ ,f)<br>Similarly , U($P_1$,f) $\ge$ U($P_2$ ,f) $\ge$ U($P_3$,f) $\ge$ …… U($P_m$ ,f). but $P_m$ =Q ….1mark<br>U(P,f) $\ge$ U(Q ,f) |
| | ii. | Let $f$:$[a, b] \rightarrow \mathbb{R}$ be a monotonic increasing function. Prove that $f$ is Riemann integrable on $[a,b]$ |

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| | Ans | Claim: if f is increasing function on [a,b] then f is R integrable.<br>Let P={x0, x1, …, xn ] be a partition of [a,b]<br>As f is increasing on $[x_{i-1}, x_i]$ such that $M_i=f(x_i)$ and $m_i=f(x_{i-1})$ where $M_i=\sup\{f(x)/x \in [x_{i-1},x_i]\}$ & $m_i=\inf\{f(x)/x \in [x_{i-1},x_i]\}$<br>……….2Marks<br>U(P,f)—L(P,f)=$\sum_{i=1}^{n}(Mi-mi)(xi-\mathrm{x}i-1) \le \sum_{i=1}^{n}(\mathrm{f}(\mathrm{x}i)-\mathrm{f}(xi-1))\|P\|$<br>$= (\mathrm{f(b)}-\mathrm{f(a)})\ \|P\|$<br>Select P such that $\|P\| < \frac{\epsilon}{\mathrm{f(b)}-\mathrm{f(a)}+1}$<br>…………. 2 marks<br>Hence U(P,f)—L(P,f) $< \frac{\epsilon}{\mathrm{f(b)}-\mathrm{f(a)}+1}(\mathrm{f(b)}-\mathrm{f(a)}) < \epsilon$ ……….2marks |
| | iii. | Using Riemann Criterion, show that the function $f:[0,1]\longrightarrow\mathbb{R}$ defined by $f(x)=x^2$ is Riemann integrable. |
| | Ans | For any $\in\ > 0$ Claim : $U(P,f) - L(P,f) < \in$<br>By Archimedean property, $\exists\ n \in \mathbb{N}$ such that $n > {}^{2}/_{\in} \Rightarrow {}^{2}/_{n} < \in$ 1M<br>Let $P = \{0, {}^{1}/_{n}, {}^{2}/_{n}, \cdots\cdots\cdots, 1\}$ be a partition of [0, 1].<br>$x_k - x_{k-1} = {}^{1}/_{n}$ and $x_k = {}^{k}/_{n}$ 1M<br>Since $f$ is increasing, hence $M_k = x_k^2$ and $m_k = x_{k-1}^2$ 1M<br>$U(P,f) - L(P,f) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) - \sum_{k=1}^{n} m_k(x_k - x_{k-1})$<br>$= \sum_{k=1}^{n}(x_k^2 - x_{k-1}^2)(x_k - x_{k-1})$<br>$= \sum_{k=1}^{n}(x_k + x_{k-1})\frac{1}{n}\frac{1}{n} < \sum_{k=1}^{n}(1+1)\frac{1}{n^2}$ as $0 < x_k,\ x_{k-1} < 1$ 2M<br>$< 2\ \frac{1}{n^2}\times n < \frac{2}{n} < \in$<br>$\therefore$ $f$ is R-integrable. 1M |
| | iv. | Let $f:[a,b]\to\mathbb{R}$ be a integrable function such that<br>$f(x) \ge 0, \forall\ x \in [a,b]$. Then prove that $\int_a^b f(x)\,dx \ge 0$.<br>Further if f, $g:[a,b]\to\mathbb{R}$ are integrable functions such that<br>$f(x) \le g(x), \forall\ x \in [a,b]$ then prove that $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$. |
| | Ans | Let f be a bounded function on [a,b] .<br>Let P={x0, x1, …, xn ] be a partition of [a,b]<br>Given $f(x) \ge 0, \forall\ x \in [a,b]$ |

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| | | Hence $m_i \geq 0$, for i= 1,2,3...,n where $m_i = \inf\{f(x)/x \in [x_{i-1}, x_i]\}$<br>Therefore L(P,f) ≥ 0 => L(f) ≥ 0<br>But $\int_a^b f(x)\,dx$ =L(f) ≥ 0 ........3 marks<br>Let h(x)=g(x)-f(x) on [a,b] .<br>But g and f are R integrable on [a,b] hence h is R integrable on[a,b] and $\int_a^b h(x)\,dx = \int_a^b g(x)dx - \int_a^b f(x)\,dx$<br>As $h(x) \geq 0, \forall x \in [a, b] \Rightarrow \int_a^b h(x)\,dx \geq 0 \Rightarrow \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$......3 marks. |
| Q3. | Attempt any **ONE** question from the following: (08) | |
| $a)$ | i. | State and prove the Fundamental Theorem of Calculus. |
| | Ans | Statement: Let $f:[a,b]\to\mathbb{R}$ be R-integrable on $[a,b]$ and $F(x) = \int_a^x f(t)dt, \forall x \in [a,b]$. If $f$ is continuous on $[a,b]$ then $F$ is differentiable and $F'(x) = f(x)$.<br>Proof: Let $h > 0$ such that $x + h \in [a,b]$. Then<br>$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}[\int_x^{x+h} f(t)dt]$.since $f$ is continuous on $[x,x+h]$ hence bounded.<br>Let sup($f$)=$M$ and inf($f$)=$m \Rightarrow m \leq \frac{1}{h}\int_x^{x+h} f(t)dt \leq M \Rightarrow \exists c \in [x,x+h]$ such that $\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c(h))$.since $x \leq c(h) \leq x+h \Rightarrow c(h) = x$ $as$ $h\to 0$.hence the proof. |
| | ii. | If $\lim_{x\to\infty}\left\|\frac{f(x)}{g(x)}\right\| = l$ where $l$ is a non-zero finite number, then show that the integrals $\int_a^\infty \|f(x)\|\,dx$ and $\int_a^\infty \|g(x)\|\,dx$ either both converge or both diverge. |
| | Ans | Choose $0 < \epsilon < l$ (1M)<br>since $\lim_{x\to\infty}\left\|\frac{f(x)}{g(x)}\right\| = l$ ,<br>there is $x_1$ >0 s.t for all x ≥ $x_1$ $l - \varepsilon < \frac{\|f(x)\|}{\|g(x)\|} < l + \varepsilon$<br>$\|f(x)\| < K\|g(x)\|$ ,K= $l + \varepsilon$ for all x ≥ $x_1$ (*)<br>$\|g(x)\| < M\|g(x)\|$ ,M= $\frac{1}{l-\varepsilon}$ for all x ≥ $x_1$ (**) (2M)<br>By First comp. Test and (*)<br>Cgce of $\int_a^\infty \|g(x)\|dx$ implies Cgce of $\int_a^\infty \|f(x)\|\,dx$<br>dgce of $\int_a^\infty \|f(x)\|dx$ implies dgce of $\int_a^\infty \|g(x)\|dx$<br><br>by (**)<br>Cgce of $\int_a^\infty \|f(x)\|dx$ implies Cgce of $\int_a^\infty \|g(x)\|dx$<br><br>dgce of $\int_a^\infty \|g(x)\|dx$ implies dgce of $\int_a^\infty \|f(x)\|\,dx$ (3M) |

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| Q3. | Attempt any **TWO** questions from the following: (12) | |
| *b*) | i. | Let $f$:[0,1]→ℝ be defined by $f(x) = \begin{cases} 1 - 2x \ \ if\ 0 \le x \le {}^{1}/_{2} \\ 2x - 1 \ \ if\ {}^{1}/_{2} < x \le 1 \end{cases}$ And $F$:[0,1]→ℝ be defined by $F(x) = \int_0^x f(t)dt$ then discuss the continuity of $f$ at $x = \frac{1}{2}$ and differentiability of $x = \frac{1}{2}$.. |
| | Ans | $\lim_{x\to\frac{1}{2}^+} f(x) = \lim_{x\to\frac{1}{2}^-} f(x) = 0 = f(\frac{1}{2})$ ⇒ f is continuous. $F'\left(\frac{1}{2}\right) = \lim_{h\to 0} \frac{1}{h}\int_{\frac{1}{2}}^{\frac{1}{2}+h}[x^2 - x]dx.$ $\Rightarrow F'\left(\frac{1}{2}\right) = 0.$ |
| | ii. | If $x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$ ,$y \ge 0$.Find $\frac{d^2y}{dx^2}$. |
| | Ans | If $x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$ ,$y \ge 0$. Find $\frac{d^2y}{dx^2}$. Differentiate equation by chain rule ,we get $1 = \frac{1}{\sqrt{1+y^2}}\frac{dy}{dx} \Rightarrow \frac{d^2y}{dx^2} = y.$ |
| | iii. | State Abel's and Dirichlet's Tests for the conditional convergence of type 1 improper integral and discuss convergence of $I = \int_0^\infty \frac{\sin x}{\sqrt{x}}dx$ |
| | Ans | Statements (2M) Let f(x)= $\frac{\cos x}{x^2}$ β(x) = $1 - e^{-x}$ $\lvert \frac{\cos x}{x^2} \rvert \le \frac{1}{x^2}$ By comparison Test $\int_a^\infty \frac{\cos x}{x^2}dx$ is cgt (2M) β(x) = $1 - e^{-x}$ is monotonic increasing (by First deri. Test) and bounded By Abel's Test I is convergent. (2M) |
| | iv. | Prove that $\int_a^\infty \frac{1}{x^p}dx$ ,a> 0,converges if and only if $p > 1$. |
| | Ans | P=1 dgt (2M) p ≠ 1 $\int_a^\infty \frac{dt}{x^p} = \lim_{x\to\infty}\int_0^x \frac{1}{t^p}dt =$ .................(2M) For p<1 dgt(1M) For p>1 cgt(1M) |

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| Q4. | Attempt any **ONE** question from the following: (08) | | |
| *a)* | i. | Using induction on $n$, show that $\beta(x,y) = \frac{\beta(x,y+n)\beta(y,n)}{\beta(x+y,n)}$. | |
| | Ans | | |

Q.4 (a) (i) To show that $\beta(x,y) = \frac{\beta(x,y+n)\,\beta(y,n)}{\beta(x+y,n)}$

let $n = 1$

$\therefore \beta(x+y,1) = \int_0^1 t^{x+y-1}(1-t)^{1-1}dt$

$= \int_0^1 t^{x+y-1}dt$

$= \frac{1}{x+y}$

$\beta(x,y+1) = \int_0^1 t^{x-1}(1-t)^{y}dt$

$\beta(y,1) = \int_0^1 t^{y-1}(1-t)^{1-1}dt = \frac{1}{y}$

$\therefore \frac{\beta(x,y+1)\,\beta(y,1)}{\beta(x+y,1)} = \frac{1}{y} \times \frac{x+y}{1} \times \int_0^1 t^{x-1}(1-t)^{y}dy$

$= \frac{x+y}{y}\beta(x,y+1)$

$\therefore \frac{\beta(x,y+1)\beta(y,1)}{\beta(x+y,1)} = \beta(x,y) \quad \left(\because \frac{\beta(x,y+1)}{y} = \frac{\beta(x,y)}{x+y}\right)$

Assume result is true for $n=k$

i.e. $\beta(x,y) = \frac{\beta(x,y+k)\,\beta(y,k)}{\beta(x+y,k)}$ —— $(*)$

tpt result is true for $n=k+1$

~~Consider LHS =~~

WKT $\frac{\beta(x,y+(k+1))}{y+k} = \frac{\beta(x,y+k)}{x+y+k}$

$\therefore \beta(x,y+(k+1)) = \frac{y+k}{x+y+k}\beta(x,y+k)$

Also $\frac{\beta(y,k+1)}{k} = \frac{\beta(y,k)}{y+k}$

| | | |
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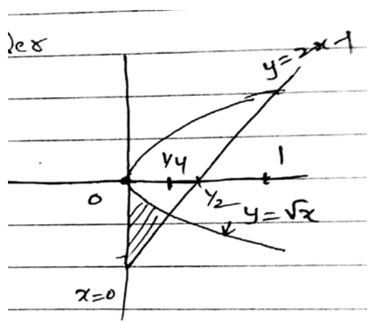
| | | $\Rightarrow \beta(y, k+1) = \frac{k}{y+k}\beta(y, k)$<br>Again $\frac{\beta(x+y, k+1)}{k} = \frac{\beta(x+y, k)}{x+y+k}$<br>$\Rightarrow \beta(x+y, k+1) = \frac{k}{x+y+k}\beta(x+y, k)$<br>Now consider RHS $= \frac{\beta(x, y+k+1)\,\beta(y, k+1)}{\beta(x+y, k+1)}$<br>$= \frac{y+k}{x+y+k}\beta(x, y+k) \times \frac{k}{y+k}\beta(y, k) \times \frac{x+y+k}{k} \times \frac{1}{\beta(x+y, k)}$<br>$= \frac{\beta(x, y+k)\,\beta(y, k)}{\beta(x+y, k)}$<br>$= \beta(x, y)$ (from (*))<br>∴ RHS = LHS<br>∴ result is true for $n = k+1$<br>∴ By induct$^n$, result is true $\forall\, n \in N$. |
| | ii. | State Cylindrical shell method for finding volume of solid generated by revolving a region bounded by $y = f(x)$& $y = g(x)$ with $f(x) \geq g(x), x \in [a,b]$ about Y-axis. Hence find volume of solid of revolution of region bounded by the curves $y = 2x - 1, x = 0$ & $y = \sqrt{x}$ about Y-axis. |
| | Ans | Cylindrical shell method:<br>Volume of solid generated by revolving a region bounded by $y = f(x)$& $y = g(x)$ with $f(x) \geq g(x), x \in [a,b]$ about Y-axis is given by<br>$$V = \int_a^b 2\pi x(f(x) - g(x))dx$$<br>To find volume of solid obtained by revolving region bounded by $y = 2x - 1, x = 0$ & $y = \sqrt{x}$ about Y-axis:<br>$f(x) = \sqrt{x},\ g(x) = 2x - 1$<br>To find interval of integration, consider<br>$\sqrt{x} = 2x - 1 \Longrightarrow x = 1,\ x = \frac{1}{4}$<br>Using cylindrical shell method volume is given by<br>$V = \int_0^{1/4} 2\pi x(\sqrt{x} - 2x + 1)dx = \pi/15\ cu.units$<br><br> |
| Q4. | Attempt any **TWO** questions from the following: (12) | |
| $b)$ | i. | With the usual notation of beta function, show that $\beta(m,n) = 2\int_0^{\frac{\pi}{2}}(sin\theta)^{2m-1}(cos\theta)^{2n-1}\, d\theta$. |

| | | |
|---|---|---|
| | Ans | Q.4 (b) (i) To show that $\beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} d\theta$<br><br>Beta $f^n$ is given by<br>$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$<br><br>sub $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$<br>& $1-x = \cos^2\theta$<br>when $x=0,\ \theta=0$ & $x=1,\ \theta=\pi/2$<br>$\therefore \beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (\cos^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta\, d\theta$<br>$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$ |
| | ii. | Define gamma function and show that $\Gamma(1) = 1$. |
| | Ans | Q.4 (b) (ii) Def$^n$ of Gamma $f^n$.<br><br>tst $\Gamma(1) = 1$<br><br>$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$<br><br>$\therefore \Gamma(1) = \int_0^\infty e^{-x} dx$<br><br>$= \lim_{t\to\infty} \int_0^t e^{-x} dx$<br>$= \lim_{t\to\infty} \left[(-1)\left[\frac{1}{e^t} + 1\right]\right]$<br>$\therefore \Gamma(1) = 1$ |
| | iii. | Find surface area of the solid obtained by revolving the curve $x = 3t - t^3, y = 3t^2, 0 \le t \le 1$ about X-axis. |

Ans

Q.4 (b)(iii) $x = 3t - t^3$, $y = 3t^2$, $0 \le t \le 1$, abt X-axis

$f(t) = 3t - t^3$, $g(t) = 3t^2$

$\therefore f'(t) = 3 - 3t^2$, $g'(t) = 9t$

$\therefore (f'(t))^2 + (g'(t))^2 = 9t^4 + 18t^2 + 9$

$\therefore \sqrt{(f'(t))^2 + (g'(t))^2} = 3(t^2+1)$

Surface area $= \int_0^1 2\pi g(t) \sqrt{(f'(t))^2 + (g'(t))^2}$

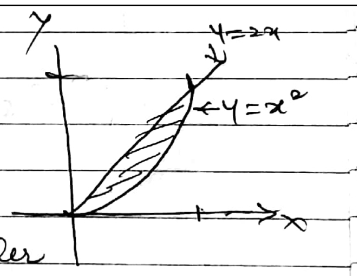
$= \int_0^1 2\pi \, 3t^2 \times 3(t^2+1) \, dt$

$= \frac{48\pi}{5}$

iv. Using disk method, find volume of solid generated by revolving the region bounded by parabola $y = x^2$ and the line $y = 2x$ in the 1st quadrant about the Y-axis.

Ans

Q.4 (b)(iv) $y = x^2$, $y = 2x$




Volume $= \int_c^d \pi \left( \left(\text{Outer rad}\right)^2 - \left(\text{Inner rad}\right)^2 \right) dy$

To find interval of integrn, consider

$x^2 = 2x \Rightarrow x = 0, x = 2$

$\Rightarrow y = 0, y = 4$

Outer rad $= \sqrt{y}$, Inner rad $= y/2$

$\therefore$ Volume $= \int_0^4 \pi \left[ (\sqrt{y})^2 - \left(\frac{y}{2}\right)^2 \right] dy$

$= \pi \int_0^4 y - \frac{y^2}{4} \, dy$

$= \frac{8\pi}{3}$

Q5. Attempt any **FOUR** questions from the following: (20)

*a)* Let P = { 0, 0.5, 1, 1.75, 2.1,3 } be a partition of [0, 3] and $f : [0, 3] \longrightarrow \mathbb{R}$ is a function such that $f(x) = 1 - x$ then find the lower sum $L(P, f)$ and upper sum $U(P, f)$.

| | |
|---|---|
| Ans | $m_1 = 0.5, m_2 = 0, m_3 =- 0.75, m_4 =- 1.1, m_5 =- 2$<br>$\therefore L(P,f) = 0.5 \times 0.5 + 0 \times 0.5 + (-0.75) \times 0.75 + (-1.1) \times 0.35 + (-2) \times 0.9 = -2.4975$<br>$M_1 = 1, M_2 = 0.5, M_3 = 0, M_4 =- 0.75, M_5 =- 1.1$<br>$\therefore U(P,f) = 1 \times 0.5 + 0.5 \times 0.5 + 0 \times 0.75 + (-0.75) \times 0.35 + (-1.1) \times 0.9 = -0.5025$ |
| $b$) | Show that every constant function is Riemann integrable on [a, b]. |
| Ans | Let $P = \{x_0, x_1, \cdots\cdots, x_n\}$ be a partition of [a,b] ……3marks<br>let $M_i = sup$ {f(x) / x ∈ $[x_{i-1}, x_i]$ }=c & $m_i = inf$ {f(x)/x ∈ $[x_{i-1}, x_i]$ }=c<br>∴ U(P.f)=c(b-a) & L(P.f)=c(b-a)<br>∴ U(f)=L(f)=c(b-a) 2 Marks<br>∴ $f$ is Riemann integrable. |
| $c$) | Evaluate $\lim_{x\to\infty} \frac{1}{x^3}\int_0^x \frac{t^2}{1+t^4}dt$. |
| Ans | Let $F(x) = \int_0^x \frac{t^2}{1+t^4}$. Since $f(t) = \frac{t^2}{1+t^4}$ is continuous ∴ by FTC $F$is differentiable and ⇒$F'(x) = f(x)$.<br>$\therefore \lim_{x\to\infty} \frac{1}{x^3}\int_0^x \frac{t^2}{1+t^4}dt = \lim_{x\to\infty} \frac{F(x)}{x^3} = \lim_{x\to\infty} \frac{F'(x)}{3x^2} = 0$.(by L Hospitals rule) |
| $d$) | Discuss convergence of each of the following:<br>$I)\int_1^\infty \frac{1}{x^{1.001}}dx$. (II) $\int_0^1 \frac{dx}{x^{\frac{1}{2}}(1+x^2)}$ |
| Ans | I)g(x) =$\frac{1}{x^2}$ $\lim_{x\to 0+} \frac{f}{g}$ =1, finite non zero<br>since $\int_2^\infty g(x)dx$ is cgt ,(p=2) by limit comparison Test $\int_2^\infty \frac{x^2+x+1}{x^4+3x+1}dx$ is convergent<br>(II) g(x) =$\frac{1}{x^{\frac{1}{2}}}$ $\lim_{x\to 0+} \frac{f}{g}$ =1, finite non zero<br>since $\int_0^1 g(x)dx$ is cgt ,(p=1/2) by limit comparison Test $\int_0^1 \frac{dx}{x^{\frac{1}{2}}(1+x^2)}$ is convergent |
| $e$) | The solid lies between planes perpendicular to X-axis at $x = 0$ & $x = 4$. The cross-section perpendicular to X-axis between these planes run from the parabola $y =- \sqrt{x}$ to $y = \sqrt{x}$. Using Slicing method, find volume of solid if cross-sections are squares with bases in XY-plane. |

| | |
|---|---|
| Ans | Q.5 (e) cross-sect$^n$s are squares with base in XY plane.<br>length of side of square $= 2\sqrt{x}$<br>$\therefore$ Area of cross sect$^n$ $= 4x = (2\sqrt{x})^2$<br>$\therefore$ Volume $= \int_0^4 A(x)\,dx$<br>$= \int_0^4 4x\,dx = 32$ |
| $f)$ | Evaluate: $\int_0^\infty x^3 e^{-4x} dx$. |
| Ans | (f) $\int_0^\infty x^3 e^{-4x}\,dx$<br>$\int_0^\infty$ sub $4x = t \Rightarrow dx = \frac{1}{4}dt$<br>$\therefore \int_0^\infty x^3 e^{-4x} dx = \int_0^\infty \left(\frac{t}{4}\right)^3 e^{-t} \frac{dt}{4}$<br>$= \frac{1}{4^4} \int_0^\infty t^3 e^{-t} dt$<br>$= \frac{1}{256} \int_0^\infty t^{4-1} e^{-t} dt$<br>$= \frac{1}{256} \Gamma(4)$<br>$= \frac{3!}{256}$<br>$= \frac{3}{128}$ |

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