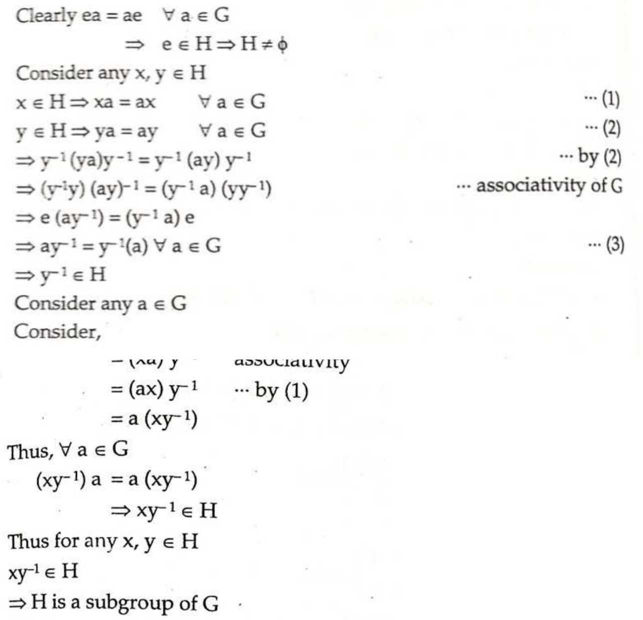
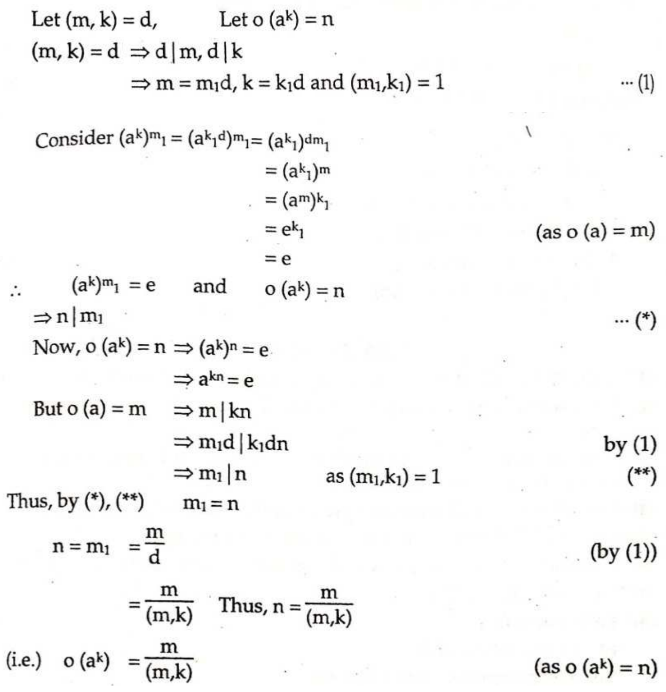
Examination : SYBA_Semester IV **Subject : Mathematics (Paper III)**
Exam Date : 03-05-2019 **Q.P.Code : 66044**

(3 Hours) [Total Marks: 100]

Note: (i) All questions are compulsory.
(ii)Figures to the right indicate marks for respective parts.

Q.1	Choose correct alternative in each of the following (20)			
$i.$	Let $(G,\cdot)$ be a group such that $\forall x, y \in G$, then which of the following is always true?			
	(a)	$(xy)^{-1} = y^{-1}x^{-1}$	(b)	$(x^{-1})^{-1} = x \ \forall x, y \in G, xy \neq yx$
	(c)	$x^m x^n = x^{m+n}$	(d)	All of the above
	Ans	(c)		
$ii.$	The set $\mathbb{Q}^*\left(\mathbb{Q}\setminus\{0\}\right)$ is forms a group under the binary operation			
	(a)	‘+’	(b)	‘-’
	(c)	‘•’	(d)	None of the above
	Ans	(a)		
$iii.$	Let D_n denote the dihedral group. Then $\lvert D_5 \rvert =$			
	(a)	5	(b)	120
	(c)	10	(d)	None of the above
	Ans	(b)		
$iv.$	Let H be a subgroup of a group G. then			
	(a)	$\forall x, y \in H, xy^{-1} \in H$	(b)	$\forall x, y \in H, xy^{-1} \in G$
	(c)	$\forall x, y \in H, xy^{-1} \notin H$	(d)	None of the above
	Ans	(a)		
$v.$	Let G be a cyclic group of order n generated by ‘a’ then $\langle a^r \rangle = \langle a^s \rangle$ implies			
	(a)	$(r, s) = 1$	(b)	$s = (n, r)$
	(c)	$(n, r) = (n, s)$	(d)	$r/(n, s)$
	Ans	(c)		
$vi.$	The generators of $20\mathbb{Z} \cap 30\mathbb{Z}$ are			
	(a)	$60, -60$	(b)	$10, -10$
	(c)	$20, 30$	(d)	None of the above
	Ans	(a)		

$vii.$	Let H be a subgroup of G and $a,b \in G$. If $aH \neq bH$ then				
	(a)	$aH \cap bH = \emptyset$	(b)	$aH \cap bH \neq \emptyset$	
	(c)	$aH \subset bH$	(d)	None of these	
	Ans	(a) $aH \cap bH = \emptyset$			
$viii.$	Let G be a group of order 25. Then				
	(a)	G is cyclic	(b)	G is cyclic or $g^5 = e,\ \forall\, g \in G$	
	(c)	$g^5 = e, \quad \forall\, g \in G$	(d)	None of these	
	Ans	(b) G is cyclic or $g^5 = e,\ \forall\, g \in G$			
$ix.$	Let $\phi: \mathbb{C}^* \longrightarrow \mathbb{C}^*$ given by $\phi(x) = x^4$ be a homomorphism then $ker\phi =$				
	(a)	$\{1, -1\}$	(b)	$\{1, -1, i, -i\}$	
	(c)	$\{i, -i\}$	(d)	None of these	
	Ans	(b) $\{1, -1, i, -i\}$			
$x.$	Let $\phi: \mathbb{Z}_{12} \longrightarrow \mathbb{Z}_{10}$ given by $\phi(x) = 3x$ then				
	(a)	ϕ is not a group homomorphism.			
	(b)	ϕ is a group homomorphism which is not one $-$one.			
	(c)	ϕ is a group homomorphism which is not onto.			
	(d)	None of these			
	Ans	(a) ϕ is not a group homomorphism.			
Q2.	Attempt any **ONE** question from the following: (08)				
$a)$	i.	Define Centre of Group G. Hence or otherwise prove that the Centre of any group is a subgroup of the group.			
	Ans	 Clearly ea = ae $\forall$ a $\in$ G $\Rightarrow$ e $\in$ H $\Rightarrow$ H $\neq \phi$ Consider any x, y $\in$ H x $\in$ H $\Rightarrow$ xa = ax $\forall$ a $\in$ G ··· (1) y $\in$ H $\Rightarrow$ ya = ay $\forall$ a $\in$ G ··· (2) $\Rightarrow y^{-1}(ya)y^{-1} = y^{-1}(ay)\,y^{-1}$ ··· by (2) $\Rightarrow (y^{-1}y)\,(ay)^{-1} = (y^{-1}a)\,(yy^{-1})$ ··· associativity of G $\Rightarrow e\,(ay^{-1}) = (y^{-1}a)\,e$ $\Rightarrow ay^{-1} = y^{-1}(a)$ $\forall$ a $\in$ G ··· (3) $\Rightarrow y^{-1} \in$ H Consider any a $\in$ G Consider, $= (xa)\,y^{-1}$ associativity $= (ax)\,y^{-1}$ ··· by (1) $= a\,(xy^{-1})$ Thus, $\forall$ a $\in$ G $(xy^{-1})\,a = a\,(xy^{-1})$ $\Rightarrow xy^{-1} \in$ H Thus for any x, y $\in$ H $xy^{-1} \in$ H $\Rightarrow$ H is a subgroup of G			1 3 2

			2
	ii.	Prove that if for $a \in G$,$O(a)=m$ then $O\left(a^k\right)=\dfrac{m}{g.c.d.\left(m,k\right)}$.	
	Ans	 Let $(m, k) = d$, Let $o\,(a^k) = n$ $(m, k) = d \Rightarrow d \mid m, d \mid k$ $\Rightarrow m = m_1 d, k = k_1 d$ and $(m_1,k_1) = 1$ ··· (1) 2 Consider $(a^k)^{m_1} = (a^{k_1 d})^{m_1} = (a^{k_1})^{dm_1}$ 2 $= (a^{k_1})^m$ $= (a^m)^{k_1}$ $= e^{k_1}$ (as $o\,(a) = m$) $= e$ $\therefore$ $(a^k)^{m_1} = e$ and $o\,(a^k) = n$ 1 $\Rightarrow n \mid m_1$ ··· (*) Now, $o\,(a^k) = n \Rightarrow (a^k)^n = e$ $\Rightarrow a^{kn} = e$ But $o\,(a) = m$ $\Rightarrow m \mid kn$ $\Rightarrow m_1 d \mid k_1 dn$ by (1) $\Rightarrow m_1 \mid n$ as $(m_1,k_1) = 1$ (**) 2 Thus, by (*), (**) $m_1 = n$ $n = m_1 = \dfrac{m}{d}$ (by (1)) $= \dfrac{m}{(m,k)}$ Thus, $n = \dfrac{m}{(m,k)}$ 1 (i.e.) $o\,(a^k) = \dfrac{m}{(m,k)}$ (as $o\,(a^k) = n$)	
Q.2	Attempt any **TWO** questions from the following: (12)		
$b)$	i.	Show that $(\mathbb{Q}^*, o)$ is a group, where $a\ ob = \frac{ab}{3}$, for $a, b \in \mathbb{Q}^*$.	
	Ans	Closure Associative trivial identity $= e = 3$ Inverse of $a \in \mathbb{Q}^*$ is $b = \frac{9}{a}$	2 2 2
	ii.	Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 4 & 2 & 1 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 1 & 5 & 3 \end{pmatrix} \in S_6$. Find $\alpha\beta, \beta^{-1}, \alpha\beta^{-1}$ and their orders in the group S_6. Clearly state the result used to compute order.	
	Ans	$\alpha = (1\ 3\ 6)(2\ 5)$, $\beta = (1\ 2\ 6\ 3\ 4)$ $\alpha\beta = (1\ 3\ 6)(25)(1\ 2\ 6\ 3\ 4) = (1\ 5\ 2)(3\ 4)$ $O(\alpha\beta) = 3 \times 2 = 6(\because \alpha, \beta$ are disjoint cycles)	2

		$\beta^{-1} = (4\ 3\ 6\ 2\ 1), O(\beta^{-1}) = 5$ ($\because \beta$ is a cycle of length 5) $\alpha\beta^{-1} = (1\ 3\ 6)(2\ 5)(4\ 3\ 6\ 2\ 1) = (1\ 4\ 6\ 5\ 2\ 3)$ $O(\alpha\beta^{-1}) = 6$ ($\because \beta$ is a cycle of length 6)	2 2
	iii.	Let H and K be subgroups of group G. Prove that $H \cup K$ is also a subgroup of G if and only if either $H \subseteq K$ or $K \subseteq H$	
	Ans	If $H \subseteq K, H \cup K =$ K, which is a subgroup of G If $K \subseteq H, H \cup K =$ H, which is a subgroup of G $H \cup K$ is a subgroup of G T.P.T either $H \subseteq K$ or $K \subseteq H$ If $H \subseteq K$ then nothing to prove If not, $\exists\, x \in H\ s.t\ x \notin K$ Let $y \in K$ $x, y \in H \cup K$ $\therefore xy \in H \cup K$ If $xy \in K$ then $xy.y^{-1} = x \in K$, which is a contradiction. $\therefore xy \in H$ $x^{-1}xy = y \in H$ $\therefore K \subseteq H$	2 4
	iv.	Let G be a group. Then prove that, $(aba^{-1})^n = ab^na^{-1}, \forall\, a, b \in G$ and $\forall\, n \in \mathbb{Z}$.	
	Ans	By induction $n = 0$ $(aba^{-1})^0 = e = ab^0a^{-1}$ Assume for $n > 0$ $(aba^{-1})^{n+1} = (aba^{-1})^n.aba^{-1}$ $= ab^na^{-1}.aba^{-1}$ $= ab^{n+1}a^{-1}$ For $n < 0,\ -n > 0$ $(aba^{-1})^n(aba^{-1})^{-n} = e$ $(aba^{-1})^n ab^{-n}a^{-1} = e$ $(aba^{-1})^n = ab^na^{-1}$	2 2 2
Q3.	Attempt any **ONE** question from the following: (08)		
$a)$	i.	Let G be a finite cyclic group of order 'n' then prove that G has unique subgroup of	

		order $'d'$ for each divisor d of n.	
	Ans	G is a finite cyclic group of order ‘n’ generated by $'a'$ $G =< a >, O(a) = n$ Let $d \mid n \therefore n = dd_1$ Consider $H =< a^{\frac{n}{d}} > = < a^{d_1} >$ $O(a^{d_1}) = \frac{n}{(n, d_1)} = \frac{n}{d_1} = d$ $\therefore H =< a^{\frac{n}{d}} >$ is a subgroup of order d Uniqueness: Let H' be any other subgroup of order d We know that H' is generated by a^m where m is the smallest positive integer such that $a^m \in H'$ $H' =< a^m >$ $\exists!\, q, r\ s.t\ n = mq + r, where\ r = 0\ or\ r < m$ If $r < m$ then $a^r = (a^m)^{-q} \in H'$ Which is a contradiction because m is the smallest positive integer such that $a^m \in H'$ $n = mq$ $O(H') = d$ $O(a^m) = d$ $\frac{n}{(n, m)} = d$ $\frac{n}{m} = d$ $\therefore H' =< a^m >=< a^{\frac{n}{d}} >= H$	4 4
	ii.	List all generators and all subgroups of the cyclic group $G = \langle a \rangle$ of order 18.	
	Ans	$G =< a >, O(G) = O(a) = 18$ Generators $a^1, a^5, a^7, a^{11}, a^{13}, a^{17}$ Subgroups $1 \mid 18$ $\exists!$ subgroup of order 1 namely $H_1 =< a^{\frac{18}{1}} >= \{e\}$ $2 \mid 18$ $\exists!$ subgroup of order 2 namely $H_2 =< a^{\frac{18}{2}} >=< a^9 >= \{a^9, e\}$	2

		$3\mid 18$ ∃! subgroup of order 3 namely $H_3 =< a^{\frac{18}{3}} >=< a^6 >= \{a^6, a^{12}, e\}$ $6\mid 18$ ∃! subgroup of order 6 namely $H_4 =< a^{\frac{18}{6}} >=< a^3 >= \{a^3, a^6, a^9, a^{12}, a^{15}, e\}$ $9\mid 18$ ∃! subgroup of order 9 namely $H_5 =< a^{\frac{18}{9}} >=< a^2 >= \{a^2, a^4, a^6, a^8, a^{10}, a^{12}, a^{14}, a^{16}, e\}$ $18\mid 18$ ∃! subgroup of order 18 namely $H_6 =< a^{\frac{18}{18}} >=< a^1 >= G$	6
Q3.	Attempt any **TWO** questions from the following: (12)		
$b)$	i.	If G is infinite cyclic group generated by a then show that G has exactly two generators $'a'$ and $'a^{-1}'$	
	Ans	G is infinite cyclic group generated by a Let b be a generator of G $\therefore < a >=< b >$ $a \in < a > \subseteq < b > \ \therefore a = b^n$, for some $n \in \mathbb{Z}$ $b \in < b > \subseteq < a > \ \therefore b = a^m$, for some $m \in \mathbb{Z}$ $a = b^n = (a^m)^n = a^{mn}$ $\therefore mn = 1$ ($\because$ a is of infinite order) $m, n = 1 \ or - 1$ $b = a^1 \quad or \quad a^{-1}$ $\therefore$ G has exactly two generators $'a'$ and $'a^{-1}'$	6
	ii.	Let $U(n) = \{\bar{x} \mid x \in \mathbb{N}, (x, n) = 1, 1 \leq x \leq n\}$ under multiplication modulo n. Determine which of the following groups are cyclic. Justify your answer. (p) $U(4)$ (q) $U(8)$	
	Ans	$U(4) = \{1,3\}$ $U(8) = \{1,3,5,7\}$ As $O(3) = 2 \ \therefore U(4) =< 3 >$ As $O(3) = O(5) = O(7) = 2 \ \therefore U(8)$ is not cyclic	2 2 2
	iii.	Prove that every cyclic group is abelian. Is the converse true? Justify	
	Ans	Let G be a cyclic group generated by $'a'$ Let $x, y \in G$ $\therefore x = a^i$,for some $i \in \mathbb{Z}$ and $y = a^j$,for some $j \in \mathbb{Z}$ $xy = a^i a^j = a^{i+j} = a^{j+i} = a^j a^i = yx$ $\therefore$ G is abelian Converse is not true As we have kliens 4 group of order 4 as abelian group where order of each element is 2 hence its not cyclic.	5 1

	iv.	Let $G = \langle a \rangle$ be a cyclic group of order 20. Find all distinct elements of the subgroups $\langle a^4 \rangle$ and $\langle a^7 \rangle$.	
	Ans	$O(a^4) = \frac{20}{(4,20)} = 5$, $\langle a^4 \rangle = \{a^4, a^8, a^{12}, a^{16}, e\}$ $O(a^7) = \frac{20}{(7,20)} = 20$, $\langle a^7 \rangle = G$	3 3
Q4.	Attempt any **ONE** question from the following: (08)		
a)	i.	Let H is a subgroup of a group G then $aH = H$ if and only if $a \in H$. Further aH is subgroup of G if and only if $a \in H$.	
	Ans	For $e \in H \Longrightarrow ae \in aH = H \Longrightarrow a \in H$ Conversely, Let $x \in aH \Longrightarrow x = ah$, $h \in H \Longrightarrow x = ah \in H$ as $a \in H \Longrightarrow aH \subseteq H$ For $a \in H$ and $e \in H \Longrightarrow a = ae \in aH \Longrightarrow H \subseteq aH$. Hence $aH = H$ Further aH is subgroup of $G \Longrightarrow e \in aH \Longrightarrow e = ah$, $h \in H \Longrightarrow eh^{-1} = a$ As $e, h^{-1} \in H \Longrightarrow eh^{-1} = a \in H$ Conversely, as $a \in H \Longrightarrow aH = H$ Hence aH is subgroup of G as H is subgroup of G.	4 4
	ii.	Let $f: G \longrightarrow G'$ is onto group homomorphism. then show that (p) $f(e) = e'$, where e and e' are identities of G and G' respectively. (q) $f(a^{-1}) = [f(a)]^{-1}, \forall\, a \in G$ (r) $f(a^m) = [f(a)]^m, \forall\, a \in G\,, m \in \mathbb{N}$	
	Ans	(p) Since $e \cdot e = e \Longrightarrow f(e \cdot e) = f(e) \Longrightarrow f(e) \cdot f(e) = f(e) \cdot e'$ using LCL we get $f(e) = e'$ (q) $\because\ a\,a^{-1} = e \Longrightarrow f(a \cdot a^{-1}) = f(e)$ $\Longrightarrow f(a) \cdot f(a^{-1}) = e' \Longrightarrow f(a^{-1}) = [f(a)]^{-1}$ (r) using induction, For $m = 1$, $f(a^1) = f(a)$ and $[f(a)]^1 = f(a)$ Suppose for $m = k$, $f(a^k) = [f(a)]^k$ Consider $f(a^{k+1}) = f(a^k)\, f(a) = [f(a)]^k\, f(a) = [f(a)]^{k+1}$ Hence $f(a^m) = [f(a)]^m, \ \forall\, a \in G\,, m \in \mathbb{N}$	2 2 4
Q4.	Attempt any **TWO** questions from the following: (12)		
b)	i.	Let G be a group and H and K are subgroups of G. Show that $(H \cap K)a = Ha \cap Ka\,, \forall a \in G$.	
	Ans	Let $x \in (H \cap K)a \Longrightarrow x = ga$, where $g \in H \cap K$ $\therefore g \in H$ and $g \in K \Longrightarrow x = ga \in Ha$ and $x = ga \in Ka \Longrightarrow x \in Ha \cap Ka$ $(H \cap K)a \subseteq Ha \cap Ka$ ------------------ (1) Let $y \in Ha \cap Ka \Longrightarrow y = ha$, $y = ka$ for $h \in H$ and $k \in K$ $\therefore ya^{-1} \in H$ and $K \Longrightarrow ya^{-1} \in H \cap K \Longrightarrow y \in (H \cap K)a \Longrightarrow Ha \cap Ka \subseteq (H \cap K)a$ --(2)	6

<table>
<tr><td></td><td></td><td>(1) and (2) gives $Ha \cap Ka \subseteq (H \cap K)a$</td><td></td></tr>
<tr><td></td><td>ii.</td><td colspan="2">Let H and K be two subgroups of G. If $o(H) = p$, a prime integer, then show that either $H \cap K = \{e\}$ or $H \subseteq K$.</td></tr>
<tr><td></td><td>Ans</td><td>Since H and K be two subgroups of $G \Longrightarrow H \cap K$ is also subgroup of G.
Further $H \cap K \subseteq H \Longrightarrow H \cap K$ is also subgroup of H.
By Lagrange's theorem , $o(H \cap K) | \, o(H) \Longrightarrow o(H \cap K) | \, p$
$o(H \cap K) = 1 \; or \; p$
If $o(H \cap K) = 1 \Longrightarrow H \cap K = \{e\}$
If $o(H \cap K) = p = o(H)$, also $H \cap K \subseteq H$ gives $H \cap K = H$
Hence $H \subseteq K$.</td><td>6</td></tr>
<tr><td></td><td>iii.</td><td colspan="2">Let $f: G \longrightarrow G^{'}$ is onto group homomorphism. then show that
(p) $o(f(a)) \,|\, o(a) \,, \forall a \in G$
(q) If G is abelian then $G^{'}$ is also abelian.</td></tr>
<tr><td></td><td>Ans</td><td>(p) Let $o(a) = n$ then $a^n = e$
Since f is homomorphism $\Longrightarrow [f(a)]^n = f(a^n) = f(e) = e'$
$\therefore o\big(f(a)\big) | \, n \Longrightarrow o(f(a)) \,|\, o(a) \,, \forall a \in G$
(q) Claim : $G^{'}$ is abelian (i.e.) $xy = yx$, $\forall \, x, y \in G^{'}$
Since f is onto $\exists \, a, b \in G$ such that $f(a) = x \,, f(b) = y$
Also G is abelian $\Longrightarrow ab = ba$
Now $xy = f(a)f(b) = f(ab) = f(ba) = f(b)f(a) = yx \Longrightarrow G^{'}$ is abelian.</td><td>3
3</td></tr>
<tr><td></td><td>iv.</td><td colspan="2">Show that $f: GL_2(\mathbb{R}) \longrightarrow (\mathbb{R}^*,\cdot)$ defined by $f(A) = detA$ is a group homomorphism. Also find $kerf$. Is f an isomorphism? Justify.</td></tr>
<tr><td></td><td>Ans</td><td>Now $f(AB) = \det(AB) = \det(A)\det(B) = f(A)\,f(B)$
$\Longrightarrow f$ is homomorphism
$kerf = SL_2(\mathbb{R})$,
since $kerf \neq \{e\} \Longrightarrow f$ an not isomorphism.</td><td>2
2
2</td></tr>
<tr><td></td><td></td><td colspan="2"></td></tr>
<tr><td>Q5.</td><td colspan="3">Attempt any **FOUR** questions from the following: (20)</td></tr>
<tr><td>$a)$</td><td colspan="3">Construct composition table of $\mathbb{Z}_5^*$ under multiplication modulo 5. Also find the order of each of its elements.</td></tr>
<tr><td>Ans</td><td colspan="2"><table>
<tr><td></td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>1</td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>2</td><td>2</td><td>4</td><td>1</td><td>3</td></tr>
<tr><td>3</td><td>3</td><td>1</td><td>4</td><td>2</td></tr>
<tr><td>4</td><td>4</td><td>3</td><td>2</td><td>1</td></tr>
</table>$O(1) = 1, O(2) = 4, O(3) = 4, O(4) = 2$</td><td>3
2</td></tr>
<tr><td>$b)$</td><td colspan="3">Let G be a group. Prove that for $a \in G$,if $O(a) = nm$ then $O(a^n) = m$</td></tr>
<tr><td>Ans</td><td colspan="2">Let $O(a^n) = t$
$(a^n)^t = e$
$a^{nt} = e$</td><td></td></tr>
</table>

	$nm\mid nt$ $\therefore m\mid t$…..(1) $(a^n)^m = a^{nm} = e$ But $O(a^n) = t$ $\therefore t\mid m$…..(2) From (1) and (2) $t = m$	5
$c)$	Show that every group of prime order p is cyclic.	
Ans	Let G be a cyclic group of order p Let $a \neq e, a \in G$(Note : Such a choice is always possible) $O(a)\mid O(G)$ $O(a)\mid p$ $O(a) = 1or\ p$ $If\ O(a) = 1, then\ a = e,$ but e was non identity element If $O(a) = p, then\ G =< a >$	5
$d)$	Let G be a cyclic group of order 40. Find the number of elements of order 4 and the number of elements of order 10 in G. Clearly state the result used.	
Ans	Result : If G is a cyclic group of order n generated by a then for every divisor d of n there are $\varphi(d)$ elements of order d $4\mid 40$ ∴number of elements of order 4 =$\varphi(4) = 2$ $10\mid 40$ ∴number of elements of order 10 =$\varphi(10) = 4$	2 3
$e)$	Prove that every group of order 49 contains a subgroup of order 7.	
Ans	Let G be a group of order 49. If G is cyclic then $\exists\ a \in G$ such that $o(a) = 49$ Now $o(a^7) = \frac{o(a)}{(o(a)\,,\ 7)} = \frac{49}{(49,7)} = 7$ then $H = \langle a^7 \rangle$ is subgroup of G of order 7. If G is not cyclic then G doesn't have any element of order 49. Also $o(a)\mid o(G),\ \forall\ a \in G$ then $\exists\ b \in G\ ,\ b \neq e$ such that $o(b) = 7$ Hence $K = \langle b \rangle$ is a subgroup of G of order 7.	6
$f)$	Check whether $(\mathbb{Q}\,,+)$ and $(\mathbb{Q}^+,\cdot)$ are isomorphic.	
Ans	Suppose $\phi: (\mathbb{Q}\,,+) \to (\mathbb{Q}^*,\cdot)$ is isomorphism. As $-1 \in \mathbb{Q}^*\ \exists\ a \in \mathbb{Q}$ such that $\phi(a) = -1$ $-1 = \phi(a) = \phi\left(\frac{1}{2}a + \frac{1}{2}a\right) = \phi\left(\frac{1}{2}a\right)\ \phi\left(\frac{1}{2}a\right) = \left[\phi\left(\frac{1}{2}a\right)\right]^2$ Let $\phi\left(\frac{1}{2}a\right) = b \in \mathbb{Q}^*$ then $b^2 = -1$ Which is contradiction, square of no rational number is -1. (any other example may be taken)	5

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