(1) 51848

(3 Hours) [Total Marks: 100

Note: (*i*) All questions are compulsory.
(*ii*) Figures to the right indicate marks for respective parts.

Q.1	Choose correct alternative in each of the following:					(20)
	i.	Integrating factor of $(2x\log x - xy)dy + 2ydx = 0$ is				
		(a)	$\frac{1}{x}$	(b)	$\frac{1}{x^2}$	
		(c)	x	(d)	None of the above.	
		Ans	i.(a)			
	ii.	The equation of the orthogonal trajectories to the family of parabolas $y^2 = kx$ is				
		(a)	$\frac{y^2}{2} + x^2 = c$	(b)	$y = ce^{2x}$	
		(c)	$y^2 + x^2 = c$	(d)	None of these	
		Ans	ii(a)			
	iii.	The degree of the O.D.E. $y' + x = (y - xy')^{-2}$ is				
		(a)	1	(b)	2	
		(c)	3	(d)	None of these	
		Ans	iii(b)			
	iv.	Which of the following is a homogenous first order differential equation?				
		(a)	$\frac{dy}{dx} = \frac{2x+3y}{2x-3y}$	(b)	$\frac{dy}{dx} = \frac{2x+3y+1}{2x-3y}$	
		(c)	$\frac{dy}{dx} = \frac{2x+3y}{2x-3y+1}$	(d)	$\frac{dy}{dx} = \frac{2x+3y+1}{2x-3y-1}$	
		Ans	iv.(a)			
	v.	General solution of $y'' + 16y = 0$ is $y =$				
		(a)	$c_1e^{4x} + c_2e^{-4x}$	(b)	$(c_1 + c_2x)e^{4x}$	
		(c)	$c_1 \sin 4x + c_2 \cos 4x$	(d)	$c_1x^4 + c_2x^{-4}$	
		Ans	$v.(c)$			
	vi.	General solution for differential equation $y'' + y' = 0$ is				
		(a)	$y = c_1e^x + c_2e^{-x}$	(b)	$y = c_1 + c_2e^{-x}$	
		(c)	$y = c_1\cos x + c_2 sinx$	(d)	$y = c_1 + c_2e^x$	
		Ans	$vi.(b)$			
	vii.	Wronskian determinant $W(y_1, y_2)$ with usual symbols is equal to				
		(a)	$y_1y_2' - y_2y_1'$	(b)	$y_1y_2' + y_2y_1'$	
		(c)	$y_1y_1' - y_2y_2'$	(d)	$y_1y_1' + y_2y_2'$	
		Ans	$vii.(a)$			
	viii.	One of the solutions of the homogeneous linear system of differential equations $\begin{cases} \frac{dx}{dt} = 3x \\ \frac{dy}{dt} = 4y \end{cases}$ is				
		(a)	$x = 3e^{3t}$, $y = e^{4t}$	(b)	$\begin{cases} x = 3e^t \\ y = 4e^t \end{cases}$	

		(c)	$\begin{cases} x = 3e^{2t} \\ y = 5e^{3t} \end{cases}$	(d)	None of these	
		Ans	viii.(a)			
	ix.	Amongst the following, the pair of linearly independent solutions is ____				
		(a)	$\begin{cases} x = e^{t} \\ y = e^{t} \end{cases}$ and $\begin{cases} x = e^{-t} \\ y = 2e^{-t} \end{cases}$	(b)	$\begin{cases} x = e^{t} \\ y = e^{t} \end{cases}$ and $\begin{cases} x = 3e^{t} \\ y = 3e^{t} \end{cases}$	
		(c)	$\begin{cases} x = e^{t} \\ y = -e^{3t} \end{cases}$ and $\begin{cases} x = -e^{t} \\ y = e^{3t} \end{cases}$	(d)	None of these	
		Ans	ix.(a)			
	x.	The Wronskian of two solutions $\begin{cases} x = e^{2t} \\ y = 2e^{2t} \end{cases}$ and $\begin{cases} x = e^{3t} \\ y = 2e^{3t}, \end{cases}$ of a homogeneous linear system of differential equations, is equal to				
		(a)	$4e^{5t}$	(b)	$2e^{2t}$	
		(c)	0	(d)	None of these	
		Ans	x. (c)			
Q.2	*a)*	Attempt any **ONE** question from the following:				(08)
		i.	Show that the general solution of the linear first order O.D.E. $\frac{dy}{dx} + Py = Q$, P and Q are integrable functions of x, is $y = e^{-\int Pdx}\left(\int Qe^{\int Pdx}dx + c\right)$, c being an arbitrary constant. Hence solve the O.D.E. $\frac{dy}{dx} + \frac{y}{1-x} = x^2 - x$.			
			Consider $\frac{d}{dx}\left(ye^{\int Pdx}\right)$. We have by the product rule of differentiation $\frac{d}{dx}\left(ye^{\int Pdx}\right) = \frac{dy}{dx}e^{\int Pdx} + y\frac{d}{dx}\left(e^{\int Pdx}\right)$ $= \frac{dy}{dx}e^{\int Pdx} + ye^{\int Pdx}\frac{d}{dx}\left(\int P(x)dx\right)$ … by Chain Rule $= \frac{dy}{dx}e^{\int Pdx} + ye^{\int Pdx}P(x)$ $= e^{\int Pdx}\left(\frac{dy}{dx} + P(x)y\right)$ $= Q(x)e^{\int Pdx}$……. $(\because \frac{dy}{dx} + Py = Q)$ Hence $ye^{\int Pdx} = \int Q(x)e^{\int Pdx}dx + c$, upon integration. $\therefore y = e^{-\int Pdx}\left(\int Qe^{\int Pdx}dx + c\right)$ is the required solution. The given O.D.E. is linear with $P = \frac{1}{1-x}$ & $Q = x^2 - x$ $\therefore$ its solution is given by $y = e^{-\int Pdx}\left(\int Qe^{\int Pdx}dx + c\right)$ $= e^{-\log(1-x)}\left(\int (x^2 - x)e^{\log(1-x)}dx + c\right)$ $= \frac{1}{1-x}\left(\int (x^2 - x)(1-x)dx + c\right)$ $= \frac{1}{1-x}\left(\int (-x^3 + 2x^2 - x)dx + c\right)$ $= \frac{1}{1-x}\left(\left(-\frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^2}{2}\right) + c\right)$, is the required solution.			1 1 1 1 1 1 1 1
		ii.	Verify that the given differential equation is not exact. Further, find an			

			I.F. of the differential equation and solve: $(x^3 - 2y^3 - 3xy)dx + 3x(y^2 + x)dy = 0$.	
			Sol: Comparing the given differential equation with $M\,dx + N\,dy = 0$, $M = x^3 - 2y^3 - 3xy$ and $N = 3xy^2 + 3x^2$ $\therefore \frac{\partial M}{\partial y} = -6y^2 - 3x$ and $\frac{\partial N}{\partial x} = 3y^2 + 6x$ $\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, $\therefore$ the given differential equation is not exact. Now, $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{3xy^2+3x^2}(-6y^2 - 3x - 3y^2 - 6x) = \frac{-9(y^2+x)}{3x(y^2+x)}$ $= -\frac{3}{x}$, which is a function of x alone. $\therefore I.F = e^{\int \frac{-3}{x}dx} = e^{-3\log x} = \frac{1}{x^3}$ Multiplying the given differential equation by I.F= $\frac{1}{x^3}$ we get the exact differential equation $\frac{(x^3-2y^3-3xy)}{x^3}dx + \frac{3x(y^2+x)}{x^3}dy = 0$ Its solution is, $\int \frac{(x^3-2y^3-3xy)}{x^3}\partial x = c$ i.e. $\int dx - 2y^3\int \frac{1}{x^3}dx - 3y\int \frac{1}{x^2}dx = c$ or that $x + \frac{y^3}{x^2} + \frac{3y}{x} = c$	2 3 3
	b)	Attempt any **TWO** questions from the following:		(12)
		i.	By Substituting $x + y = v$, solve the differential equation: $(x+y)^2\frac{dy}{dx} = 1$	
			Sol:- $x + y = v$ $1 + \frac{dy}{dx} = \frac{dv}{dx}$ $\therefore \frac{dy}{dx} = \frac{dv}{dx} - 1$ $\therefore v^2\left(\frac{dv}{dx} - 1\right) = 1$ $\therefore \frac{dv}{dx} = \frac{1}{v^2} + 1$ $\therefore \frac{v^2}{v^2+1}dv = dx$ $\therefore \left(1 - \frac{1}{v^2+1}\right)dv = dx$ $\therefore \int\left(1 - \frac{1}{v^2+1}\right)dv = \int dx$ $\therefore v - \tan^{-1}v = x + c$ $\therefore x + y - \tan^{-1}(x+y) = x + c$ $\therefore y = \tan^{-1}(x+y) + c$	1 1 1 1 1 1
		ii.	Solve the following non-homogenous differential equation: $\frac{dy}{dx} = \frac{x+2y-1}{x+2y+1}$	
			Sol:	

			$$\frac{dy}{dx}=\frac{x+2y-1}{x+2y+1}\ldots.(1)$$ Put $x+2y=v$ $$\therefore \frac{dy}{dx}=\frac{1}{2}\frac{dv}{dx}-\frac{1}{2}$$ Substituting in (1), we get, $$\frac{dv}{dx}=\frac{3v-1}{v+1}$$ $$\therefore \frac{v+1}{3v-1}dv=dx$$ $$\therefore \frac{1}{3}\left(1+\frac{4}{3v-1}\right)dv=dx$$ Integrating, we get, $\int\left(1+\frac{4}{3}\frac{3}{3v-1}\right)dv=3\int dx+c$ $$\therefore v+\frac{4}{3}\log(3v-1)=3x+c$$ Substituting back $v=x+2y$, we get, $$3x-3y-2\log(3x+6y-1)=k$$	1 1 1 1 1 1
		iii.	Solve the differential equation: $$\frac{dy}{dx}+\frac{y}{x}=x^2y^6$$	
			Sol: - The given differential equation is a Bernoulli's equation Now, putting $v=y^{-5}$ the equation transforms to $\frac{dv}{dx}-5\frac{v}{x}=-5x^2$, which is linear and has solution $$v\,e^{\int-\frac{5}{x}dx})=-\int 5x^2e^{\int-\frac{5}{x}dx}\,dx+c$$ i.e. $\frac{1}{y^5}\frac{1}{x^5}=-\int 5x^2\left(\frac{1}{x^5}\right)dx+c$ i.e. $\frac{1}{x^5y^5}=\frac{5}{2x^2}+c$	1 2 1 2
		iv.	An RL circuit has an emf of 15V, a resistance of 90Ω and inductance of 3H, and no initial current. Find the current in the circuit at any time t.	
			Sol: $V=15\ ,R=90,\ L=3$ We have $\frac{di}{dt}+\frac{R}{L}i=\frac{V(t)}{L}$ $$\therefore \frac{di}{dt}+\frac{90}{3}i=\frac{15}{3}$$ $$\therefore \frac{di}{dt}+30i=5$$ $$\therefore \frac{di}{dt}=5-30i$$ $$\therefore \frac{di}{dt}=5(1-6i)$$ $$\therefore \frac{di}{1-6i}=5\,dt$$ $$\therefore \int\frac{di}{1-6i}=5\int dt$$ $\therefore \frac{\log(1-6i)}{-6}=5t+c$ (1) At $t=0, i=0$ given, $\therefore \frac{\log 1}{-6}=0+c$ from (1)	1 1 1 1

			$\therefore c = 0$ $\therefore$ (1) becomes $\therefore \dfrac{\log(1-6i)}{-6} = 5t$ $\therefore \log(1-6i) = -30t$ $\therefore (1-6i) = e^{-30t}$ $\therefore 6i = 1 - e^{-30t}$ $\therefore i = \dfrac{1-e^{-30t}}{6}$	1 1
Q.3	$a)$	Attempt any **ONE** question from the following:		(08)
		i.	Let $y_1(x)$ be a non-zero solution to the differential equation $y'' + P(x)y' + Q(x)y = 0$ on $[a,b]$then show that another linearly independent solution $y_2(x) = y_1(x)\int \frac{e^{\int -p(x)dx}}{{y_1}^2(x)} dx$	
		Ans	Let $y_1(x)$and$y_2(x)$ are two independent solution of the differential equation $y'' + P(x)y' + Q(x)y = 0$ $\therefore \exists$ non constant function $V(x)$ such that, $y_2(x) = V(x)y_1(x)$ (1) $\therefore y_2' = Vy_1' + y_1V'$ $y_2'' = Vy_1'' + 2V'y_1' + V''y_1$ Substituting these values in $y_2'' + P(x)y_2' + Q(x)y_2 = 0$ $\therefore Vy_1'' + 2V'y_1' + V''y_1 + P(Vy_1' + y_1V') + Q\,V(x)y_1(x) = 0$ $\therefore V(y_1'' + Py_1' + Q) + V''y_1 + V'(2y_1' + Py_1) = 0$ (2) As y_1 is a solution to $y'' + P(x)y' + Q(x)y = 0$ $\therefore y_1'' + P(x)y_1' + Q(x)y_1 = 0$ hence equation (2) becomes $V''y_1 + V'(2y_1' + Py_1) = 0$ $\therefore V''y_1 = -V'(2y_1' + Py_1)$ $\therefore \dfrac{V''}{V'} = -\dfrac{(2y_1' + Py_1)}{y_1}$ $\therefore \dfrac{V''}{V'} = -\dfrac{2{y_1}'}{y_1} - P$ Integrating both sides we get, $\log V' = -2\log y_1 - \int P dx$ $\therefore \log V' + 2\log y_1 = -\int P dx$ $\therefore \log V' + \log y_1^2 = -\int P dx$ $\therefore \log(V'y_1^2) = -\int P dx$ $\therefore V'y_1^2 = e^{-\int P dx}$ $\therefore V' = \dfrac{1}{y_1^2} e^{-\int P dx}$ $\therefore V = \displaystyle\int \frac{1}{y_1^2} e^{-\int P dx}\, dx$ $\therefore$ (2) becomes $y_2(x) = y_1(x)\displaystyle\int \frac{1}{y_1^2} e^{-\int P\, dx}\, dx$ Now we shall show that, $y_1(x)$ and $y_2(x)$ are linearly independent. $\therefore W(y_1, y_2) = y_1y_2' - y_2{y_1}'$	1 ($y_2(x) = V(x)y_1(x)$) 1 (y_2'') 1 (eq. 2) 1 ($V''y_1 + V'(2y_1'+Py_1)=0$) 1 ($\frac{V''}{V'}$) 1 ($\log V' + 2\log y_1$) 1 ($y_2(x)$)

$$= y_1(Vy_1 + V'y_1) - Vy_1y_1'$$

$= V'y_1^2$ as $V' = y_1(x)\int \frac{1}{y_1^2} e^{-\int P\,dx}\,dx$ is never zero,

and $y_1^2 \neq 0$

$\therefore W(y_1, y_2)$ is never zero.

$\therefore y_1$ and y_2 are linearly independent — 1

ii. Describe the method of variation of parameters to solve a non-homogenous differential equation of the second order.

Ans . Let $y_1(x)$and$y_2(x)$ are solutions of the differential equation

$y'' + py' + qy = 0$

$\therefore$ complementary function is

$$y_c = c_1y_1 + c_2y_2 \qquad 1$$

Let particular integral for $y'' + py' + qy = R(x)$ be

$y_p = uy_1 + vy_2$ ... (1) — 1

$\therefore y_p' = u'y_1 + uy_1' + v'y_2 + vy_2'$ ... (2) — 1

Let u, v satisfy the equation — 1

$u'y_1 + v'y_2 = 0$ ... (3)

$\therefore$ equation (2) becomes

$$\therefore y_p' = uy_1' + vy_2' \qquad 1$$

$$\therefore y_p'' = u'y_1' + uy_1'' + v'y_2' + vy_2''$$

Substituting these values in $y_p'' + py_p' + qy_p = R(x)$,

$\therefore u'y_1' + uy_1'' + v'y_2' + vy_2'' + p(uy_1' + vy_2') + q(uy_1 + vy_2) = R(x)$

$\therefore u(y_1'' + py'_1 + qy_1) + v(y_2'' + py'_2 + qy_2) + (u'y_1' + v'y_2') = R(x)$ — 1

[$\because y_1$andy_2 are solutions for $y'' + py' + qy = 0$

$$\therefore u(0) + v(0) + (u'y_1' + v'y_2') = R(x)$$

$\therefore (u'y_1' + v'y_2') - R = 0$ ... (4)

Using Crammer's Rule for equations (3) and (4) we get,

$$\frac{u'}{\begin{vmatrix} y_2 & 0 \\ y_2' & -R \end{vmatrix}} = \frac{-v'}{\begin{vmatrix} y_1 & 0 \\ y_1' & -R \end{vmatrix}} = \frac{1}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \qquad 1$$

$$\frac{u'}{\begin{vmatrix} y_2 & 0 \\ y_2' & -R \end{vmatrix}} = \frac{-v'}{\begin{vmatrix} y_1 & 0 \\ y_1' & -R \end{vmatrix}} = \frac{1}{W}$$

$$\therefore \frac{u'}{-y_2R} = -\frac{v'}{-y_1R} = \frac{1}{W}$$

$$\therefore u' = -\frac{y_2R}{W} \text{and} v' = \frac{y_1R}{W} \qquad 1$$

$$\therefore u = -\int \frac{y_2R}{W}dx \text{ and } v = \int \frac{y_1R}{W}dx$$

	b)	Attempt any **TWO** questions from the following:		(12)
		i.	Find the general solution for the differential equation $y'' - 7y' + 12y = 0$	
		Ans	Associated auxiliary equation $x^2 - 7x + 12 = 0 \therefore x = 3$ and $x = 4$ are the two roots. $\therefore y_1(x) = e^{3x}$ is one solution. $y_2(x) = e^{4x}$ are two linearly independent solution. $\therefore y(x) = c_1 e^{3x} + c_2 e^{4x}$ is general solution.	2 2 2
		ii.	Solve the non-homogenous differential equation $$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = x + 12.$$	
		Ans	Auxiliary equation is $m^2 + 4m + 5 = 0$ $\therefore (m+1)(m+4) = 0$ $\therefore m = -1, -4$ $\therefore y_c = c_1 e^{-x} + c_2 e^{-4x}$ Let $y_p = Ax + B$ $\therefore y_p' = A$ $y_p'' = 0$ $\therefore y_p'' + 4y_p' + 5y_p = x + 12$ $\therefore 0 + 4A + 5(Ax + B) = x + 12$ $\therefore A = \frac{1}{5}, \quad 4A + 5B = 12$ $\therefore B = \frac{56}{25}$ $\therefore y_p = \frac{1}{5}x + \frac{56}{25}$ $\therefore$ general solution is $y = y_c + y_p$ $\therefore y = c_1 e^{-x} + c_2 e^{-4x} + \frac{1}{5}x + \frac{56}{25}$	1 1 1 1 1 1
		iii.	Solve the differential equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 10y = 6e^{4x}$	
		Ans	Associated auxiliary equation of homogeneous system is $x^2 + 3x - 10 = 0 \therefore x = -5 \;\&\; x = 2 \;\therefore y_1(x) = e^{-5x} \;\&\; y_2(x) = e^{2x}$ Are linearly independent solutions of associated homogeneous system Take $y_P(x) = Ae^{4x}$ then $A = \frac{1}{3} \therefore y(x) = c_1 e^{-5x} + c_2 e^{2x} + \frac{1}{3}e^{4x}$ is the general solution of given D.E.	3 1 2
		iv.	Using the method of variation of parameters solve $$\frac{d^2y}{dx^2} + 4y = \sin x$$	
		Ans	Auxiliary equation for the differential equation is $m^2 + 4 = 0$ $\therefore m = \pm 2i$ $\therefore y_c = c_1 \cos 2x + c_2 \sin 2x$ $\therefore$ two independent solution for $\frac{d^2y}{dx^2} + 4y = 0$ are	1

$$y_1(x) = \cos 2x \text{ and} y_2(x) = \sin 2x$$
$$\therefore y_1'(x) = -2\cos 2x \text{ and} y_2'(x) = 2\sin 2x$$ **1**

$$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$
$$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$ **1**
$$= 2$$

Here $R = e^x \sin x$

$$\therefore u = -\int \frac{y_2 R}{W} dx$$
$$= -\int \frac{\sin 2x \sin x}{2} dx$$
$$= -\frac{1}{4}\int 2\sin 2x \sin x \; dx$$
$$= \frac{1}{4}\int (\cos x - \cos 3x)\, dx$$
$$= \frac{1}{4}\left(-\sin x + \frac{1}{3}\sin 3x\right)$$

and **1**

$$v = \int \frac{y_1 R}{W} dx$$
$$= \int \frac{\cos 2x \sin x}{2} dx$$
$$= \frac{1}{4}\int 2\cos 2x \sin x \; dx$$
$$= \frac{1}{4}\int (\sin 3x - \sin x)\, dx$$
$$= \frac{1}{4}\left(-\frac{1}{3}\cos 3x + \sin x\right)$$

$\therefore$ particular integral is $y_p = uy_1 + vy_2$ **1**

$$\therefore y_p = \frac{1}{4}\sin 2x\left(-\frac{1}{3}\cos 3x + \sin x\right)$$

$\therefore$ particular solution is $y = y_c + y_p$

$$\therefore y = \frac{1}{4}\cos 2x\left(-\sin x + \frac{1}{3}\sin 3x\right) + \frac{1}{4}\sin 2x\left(-\frac{1}{3}\cos 3x + \sin x\right)$$ **1**

Q.4	*a)*	Attempt any **ONE** question from the following:		(08)
		i.	Prove that the two solutions $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ of the homogeneous linear system $\begin{cases} \frac{dx}{dt} = a_1(t)x + b_1(t)y \\ \frac{dy}{dt} = a_2(t)x + b_2(t)y \end{cases}$ are linearly dependent on $[a, b]$ iff their Wronskian is identically zero on $[a, b]$.	
			Solution: If the solutions are linearly dependent, then it means there exists a real number k, such that $x_1(t) = kx_2(t)$ and $y_1(t) = ky_2(t)$. Then,	

(9)

			$W[T]=\begin{vmatrix}x_1(t) & x_2(t)\\ y_1(t) & y_2(t)\end{vmatrix}=\begin{vmatrix}x_1(t) & x_2(t)\\ kx_1(t) & kx_2(t)\end{vmatrix}=kx_1(t)x_2(t)-kx_2(t)x_1(t)=0$. It should be clear that $W[T]=0$, if $kx_1(t)=x_2(t)$ and $ky_1(t)=y_2(t)$ Now suppose, the Wronskian is identically zero on $[a,b]$. We will show that the solutions are dependent. That is, we will show that there exist real numbers c_1 and c_2, not both zero, such that $c_1x_1(t)+c_2x_2(t)=0$ $c_1y_1(t)+c_2y_2(t)=0$	2 3
			Let $t_0\in[a,b]$. Then $W(t_0)=0$. Hence, the following system of linear algebraic equations has a solution c_1, c_2, in which these numbers are not both zero. $c_1x_1(t_0)+c_2x_2(t_0)=0$ $c_1y_1(t_0)+c_2y_2(t_0)=0$ Thus, the solution of the system, given by $\begin{matrix}x=c_1x_1(t_0)+c_2x_2(t_0)\\ y=c_1y_1(t_0)+c_2y_2(t_0)\end{matrix}$ ....(*) equals the trivial solution at t_0. Now from the uniqueness part of the Existence and Uniqueness theorem, it follows that (*) must equal the trivial solution throughout the interval $[a,b]$.	3
		ii.	State the theorem for existence and uniqueness of first order homogeneous linear system of O.D.E.in two variables. Let $\begin{cases}x=x_1(t)\\ y=y_1(t)\end{cases}$ and $\begin{cases}x=x_2(t)\\ y=y_2(t)\end{cases}$ be two solutions of the homogeneous system $\begin{cases}\frac{dx}{dt}=a_1(t)x+b_1(t)y\\ \frac{dy}{dt}=a_2(t)x+b_2(t)y\end{cases}$ where a_1,a_2,b_1,b_2 are continuous functions on $[a,b]$. Show that the linear independence and continuity of the solutions on $[a,b]$ implies $\begin{cases}x=c_1x_1(t)+c_2x_2(t)\\ y=c_1y_1(t)+c_2y_2(t)\end{cases}$ to be the general solution of the homogeneous system on $[a,b]$.	
			Solution: The linear system of homogeneous first order ordinary differential	

equations given as $\begin{cases} \frac{dx}{dt} = a_1(t)x(t) + b_1(t)y(t) \\ \frac{dy}{dt} = a_2(t)x(t) + b_2(t)y(t) \end{cases}$ ----------- * 2

where $a_1(t)$, $a_2(t)$, $b_1(t)$, $b_2(t)$ are continuous functions defined on interval $[a, b]$ has a unique solution $((x(t), y(t))$ satisfying initial conditions $x(t_0) = x_0$ & $y(t_0) = y_0$ for some $t_0 \in [a, b]$.

Let $(x_1(t), y_1(t))$ & $(x_2(t), y_2(t))$ where $x_1(t), y_1(t), x_2(t), y_2(t)$ are continuously differentiable on [a,b], be two linearly independent solutions of $*$, i.e.,$W(t) \neq 0, \forall t \in [a, b]$.

Then ,

$\begin{cases} x_1'(t) = a_1(t)x_1(t) + b_1(t)y_1(t) \\ y_1'(t) = a_2(t)x_1(t) + b_2(t)y_1(t) \end{cases}$ and

$$\begin{cases} x_2'(t) = a_1(t)x_2(t) + b_1(t)y_2(t) \\ y_2'(t) = a_2(t)x_2(t) + b_2(t)y_2(t) \end{cases}$$ 2

Now $(c_1x_1)'(t) = c_1x_1'(t)$, $(c_1y_1)'(t) = c_1y_1'(t)$,

$(c_2x_2)'(t) = c_2x_2'(t)$, $(c_2y_2)'(t) = c_2y_2'(t)$

$$a_1(t)(c_1x_1)(t) + b_1(t)(c_1y_1)(t) = c_1(a_1(t)x_1(t) + b_1(t)y_1(t))$$

$$a_2(t)(c_1x_1)(t) + b_2(t)(c_1y_1)(t) = c_1(a_1(t)x_2(t) + b_1(t)y_2(t))$$

Similarly for c_2x_2 & c_2y_2

Thus verifying that $(c_1x_1 + c_2x_2, c_1y_1 + c_2y_2)$ satisfies $*$

Now let $((x(t), y(t))$ satisfying initial conditions $x(t_0) = x_0$ & $y(t_0) = y_0$ for some $t_0 \in [a, b]$ be a particular solution to $*$ then consider the equations

$$c_1x_1(t_0) + c_2x_2(t_0) = x_0$$

$$c_1y_1(t_0) + c_2y_2(t_0) = y_0$$

$$\therefore \begin{bmatrix} x_1(t_0) & x_2(t_0) \\ y_1(t_0) & y_2(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$ 2

The above matrix equation has a unique solution provided

$$det \begin{bmatrix} x_1(t_0) & x_2(t_0) \\ y_1(t_0) & y_2(t_0) \end{bmatrix} \neq 0$$

But as $W(t) \neq 0, \forall t \in [a, b]$ hence $W(t_0) \neq 0$

$\therefore$ For above values of c_1, c_2, $(c_1x_1 + c_2x_2, c_1y_1 + c_2y_2)$ Satisfies $*$ but due to uniqueness of solution satisfying initial conditions hence

$c_1x_1 + c_2x_2 = x$ & $c_1y_1 + c_2y_2 = y$.

Thus $(c_1x_1 + c_2x_2, c_1y_1 + c_2y_2)$ is general solution to $*$ for arbitrary constants c_1 & c_2. **2**

b) Attempt any **TWO** questions from the following: **(12)**

i. Show that both $\begin{cases} x = e^{3t} \\ y = e^{3t} \end{cases}$ and $\begin{cases} x = e^{2t} \\ y = 2e^{2t} \end{cases}$ are solutions of the system $\begin{cases} \frac{dx}{dt} = 4x - y \\ \frac{dy}{dt} = 2x + y \end{cases}$.

Hence or otherwise, write general solution of the system.

Solution:

Consider $\begin{cases} x = e^{3t} \\ y = e^{3t} \end{cases}$. Therefore,

$$\frac{dx}{dt} = 3e^{3t} = 4e^{3t} - e^{3t} = 4x - y \quad \text{and} \quad \frac{dy}{dt} = 3e^{3t} = 2e^{3t} + e^{3t} = 2x + y$$

Thus, $\begin{cases} x = e^{3t} \\ y = e^{3t} \end{cases}$ satisfies the given system. **2**

Similarly, consider $\begin{cases} x = e^{2t} \\ y = 2e^{2t} \end{cases}$. Therefore,

$$\frac{dx}{dt} = 2e^{2t} = 4e^{2t} - 2e^{2t} = 4x - y \quad \text{and} \quad \frac{dy}{dt} = 4e^{2t} = 2e^{2t} + 2e^{2t} = 2x + y$$

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Thus, $\begin{cases} x = e^{2t} \\ y = 2e^{2t} \end{cases}$ too satisfies the given system.

Consider the Wronskian of the two solutions:

$$W = \begin{vmatrix} e^{3t} & e^{2t} \\ e^{3t} & 2e^{2t} \end{vmatrix} = 2e^{5t} - e^{5t} = e^{5t}$$

Since, exponential function never vanishes, $W \neq 0$. Therefore, the two solutions $\begin{cases} x = e^{3t} \\ y = e^{3t} \end{cases}$ and $\begin{cases} x = e^{2t} \\ y = 2e^{2t} \end{cases}$ are linearly independent.

Thus, the general solution of the system is

12

			$\begin{cases} x = c_1e^{3t} + c_2e^{2t} \\ y = c_1e^{3t} + 2c_2e^{2t} \end{cases}$	2
		ii.	Solve the system: $$\begin{cases} \frac{dx}{dt} = 4x - 2y \\ \frac{dy}{dt} = 5x + 2y \end{cases}$$	
			Solution: Auxiliary equation is $m^2 - (4+2)m + 8 + 10 = 0$ Roots are $m = 3 \pm 3i$ Taking $\begin{cases} x(t) = Ae^{mt} \\ y(t) = Be^{mt} \end{cases}$ & $m = 3 + 3i \therefore x'(t) = 4x(t) - 2y(t)$ $\therefore\ mAe^{mt} = 4Ae^{mt} - 2Be^{mt}$	2
			$\therefore (4 - m)A - 2B = 0$ as $e^{mt} \neq 0$ $\therefore$ using $m = 3 + 3i$ we get $(1 - 3i)A - 2B = 0$	1
			Take $A = 1 + 3i\ then\ (1 - 3i)(1 + 3i) = 2B \therefore B = 5$ $\therefore \begin{cases} x(t) = (1+3i)e^{(3+3i)t} \\ y(t) = 5e^{(3+3i)t} \end{cases}$ $\therefore \begin{cases} x(t) = e^{3t}[cos3t - 3sin3t + i(3cos3t + sin3t)] \\ y(t) = e^{3t}[5cos3t + i(5sin3t)] \end{cases}$	1
			Consider, $\begin{cases} x_1(t) = e^{3t}[cos3t - 3sin3t] \\ y_1(t) = e^{3t}[5cos3t] \end{cases}$ and $\begin{cases} x_2(t) = e^{3t}[3cos3t + sin3t] \\ y_2(t) = e^{3t}[5sin3t] \end{cases}$ Then check these solve $\begin{cases} \frac{dx}{dt} = 4x - 2y \\ \frac{dy}{dt} = 5x + 2y \end{cases}$	1
			Also Wronskian of these solutions $W(t) = \begin{vmatrix} e^{3t}(cos3t - sin3t) & 5e^{3t}cos3t \\ e^{3t}(3cos3t + sin3t) & 5e^{3t}sin3t \end{vmatrix} = -15e^{6t} \neq 0$ Hence these are linearly independent solutions $\therefore \begin{cases} x(t) = c_1e^{3t}[cos3t - 3sin3t] + c_2e^{3t}[3cos3t + sin3t] \\ y(t) = c_1[5e^{3t}cos3t] + c_2[5^{3t}sin3t] \end{cases}$ is the general solution.	1
		iii.	Solve the following linear system:	

			$\begin{cases}\dfrac{dx}{dt} = 4x + 2y \\ \dfrac{dy}{dt} = 2x + 4y\end{cases}$	
			Solution: We know that the auxiliary equation of the system $\begin{cases}\dfrac{dx}{dt} = a_1x + b_1y \\ \dfrac{dy}{dt} = a_2x + b_2y\end{cases}$ is given by $m^2 - (a_1 + b_2)m + (a_1b_2 - a_2b_1) = 0$. Therefore, the auxiliary equation of the given system is $m^2 - (4 + 4)m + (16 - 4) = 0$. That is $m^2 - 8m + 12 = 0$. Thus, the roots of the auxiliary equation are 2 and 6. Consider the equations: $\begin{cases}2A + 2B = 0 \\ 2A + 2B = 0\end{cases}$ by considering $m = 2$. A nontrivial solution of this system is $A = 1$ and $B = -1$. Thus, we have $\begin{cases}x = e^{2t} \\ y = -e^{2t}\end{cases}$ as a solution. Similarly, by considering $m = 6$, we get $\begin{cases}-2A + 2B = 0 \\ 2A - 2B = 0\end{cases}$. This gives $A = 1$ and $B = 1$ as a nontrivial solution. Therefore, we have $\begin{cases}x = e^{6t} \\ y = e^{6t}\end{cases}$ as another solution. Check that these solutions are linearly independent. Therefore, $\begin{cases}x = c_1e^{2t} + c_2e^{6t} \\ y = -c_1e^{2t} + c_2e^{6t}\end{cases}$ is the general solution of the system.	2 1 1 2
		iv.	Obtain the general solution of a system of homogeneous linear first order O.D.E. with constant coefficients in two variables, when its auxiliary equation has two real and distinct roots	
			Solution: Let m_1 and m_2 be distinct roots of auxiliary equation $m^2 - (a_1 + b_2)m + a_1b_2 - a_2b_1 = 0$ for $\begin{cases}x'(t) = a_1x(t) + b_1y(t) \\ y'(t) = a_2x(t) + b_2y(t)\end{cases}$ - - - - - (*) Taking, $\begin{cases}x(t) = Ae^{mt} \\ y(t) = Be^{mt}\end{cases}$ We get, $mAe^{mt} = a_1Ae^{mt} + b_1Be^{mt}$ & $mBe^{mt} = a_2Ae^{mt} + b_2Be^{mt}$ $\therefore (m - a_1)A - b_1B = 0$ & $a_2A + (b_2 - m)B = 0$ These are solvable with non-zero values to A & B provided	2 1

(14)

		$\begin{bmatrix} m-a_1 & -b_1 \\ a_2 & (b_2-m) \end{bmatrix}$ is singular i.e. provided $\begin{vmatrix} m-a_1 & -b_1 \\ a_2 & (b_2-m) \end{vmatrix} = 0$ *i.e.* $m^2-(a_1+b_2)m+a_1b_2-a_2b_1=0$ Essentially now only one equation is sufficient as the other equation is a multiple of the first. $\therefore (m-a_1)A-b_1B=0$ gives $\frac{B}{A}=\frac{m-a_1}{b_1}$ whenever $b_1\neq 0$ Take $A=b_1$ and $B=m-a_1$	1
		Then $\begin{cases} x(t)=b_1e^{mt} \\ y(t)=(m-a_1)e^{mt} \end{cases}$ *solves* $*$ Taking $m=m_1$ and $m=m_2$ we get two solutions with Wronskian $W(t)$ $W(t)=\begin{vmatrix} b_1e^{m_1t} & (m_1-a_1)e^{m_1t} \\ b_1e^{m_2t} & (m_2-a_1)e^{m_2t} \end{vmatrix} = b_1(m_2-m_1)e^{(m_1+m_2)t}\neq 0$ *as* $b_1\neq 0$ & $m_1\neq m_2$ Gives two linearly independent solutions and a linear combination of these with arbitrary constants c_1 & c_2 gives the general solution $\begin{cases} x(t)=c_1b_1e^{m_1t}+c_2b_1e^{m_2t} \\ y(t)=c_1(m_1-a_1)e^{m_1t}+c_2(m_2-a_1)e^{m_2t} \end{cases}$	1
		Also $(a_2)A+(b_2-m)B=0$ gives $\frac{A}{B}=\frac{m-b_2}{a_2}$ whenever $a_2\neq 0$ Take $A=m-b_2$ and $B=a_2$ $\begin{cases} x(t)=(m-b_2)e^{mt} \\ y(t)=a_2e^{mt} \end{cases}$ solves $*$ Gives similarly a general solution if $a_2\neq 0$ But if $b_1=0$ & $a_2=0$ then $m^2-(a_1+b_2)m+a_1b_2-a_2b_1=0$ gives $m_1=a_1$ & $m_2=b_2$ $\begin{cases} x(t)=c_1e^{a_1t} \\ y(t)=c_2e^{b_2t} \end{cases}$ is the general solution to $*$	1
Q.5		Attempt any **FOUR** questions from the following:	(20)
	a)	Verify that the following differential equation is exact and solve: $\left(y\left(1+\frac{1}{x}\right)+\cos y\right)dx+(x+\log x-x\sin y)dy=0.$	
		Here, $\frac{\partial M}{\partial y}=1+\frac{1}{x}-\sin y=\frac{\partial N}{\partial x}$ $\therefore$ exact	2
		Solution is, $c=\int M\partial x+\int N(\text{terms free from } x)dy$ $=\int\left(y\left(1+\frac{1}{x}\right)+\cos y\right)\partial x=y(x+\log x)+x\cos y.$	3
	b)	Solve: $\sin^2 y\,dx+\cos^2 x\,dy=0.$	
		Separating the variables, we get,	2 3

		$$\frac{dx}{\cos^2 x} + \frac{dy}{\sin^2 y} = 0$$ $\sec^2 x\, dx + \csc^2 y dy = 0.$ Integrating we get, $tanx - coty = c.$	
	c)	Solve the differential equation $y'' + y = x$ by the method of variation of parameters.	
	Ans	Associated auxiliary equation of homogeneous system is $x^2 + 1 = 0$ hence $y_1(x) = Cos\ x\ and\ y_2(x) = Sinx$ are the linearly independent solutions of homogeneous system. $y_P(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$ Is particular solution of non-homogeneous system. Now $W(y_1, y_2)$=1 $$v_1(x) = \int -xSinx\ dx = xCosx - Sinx$$ $v_2(x) = \int xCosx\ dx = xSinx + Cosx$ ∴ General solution is y(x), y(x) = $Cosx + c_2Sinx + (xCosx - Sinx)Cosx + (xSinx + Cosx)Sinx$	2 1 1 1 1
	d)	Show that $y = c_1x + c_2x^{-2}$ is a solution of $x^2y'' + 2xy' - 2y = 0$ on any interval not containing the origin	
	Ans	$$x^2y'' + 2xy' - 2y = 0\ \therefore y'' + \frac{2x}{x^2}y' - \frac{2}{x^2}y = 0$$ $\therefore P + Qx = 0$, Where $P = \frac{2x}{x^2}, Q = \frac{-2}{x^2} \therefore y_1 = x\ is\ a\ solution \therefore y_1' = 1$ And $y_1'' = 0$ Clearly, $x^2y_1'' + 2xy_1' - 2y_1 = 0$ therefore y_1 is a solution to given differential equation. Similarly, for, $$y_2 = x^{-2}$$ $$\therefore y_2' = -2x^{-3}$$ And $y_2'' = 6x^{-4}$ Clearly, $x^2y_2'' + 2xy_2' - 2y_2 = 0$ therefore y_1 is a solution to given differential equation. Here y_1 and y_2 are linearly independent, also $x \neq 0$ $\therefore y = c_1x^{-1} + c_2x^5$is a solution of $x^2y'' - 3xy' - 5y = 0$ on any interval, not containing the origin.	1 1 1 1 1 1
	e)	Show that both $x_1 = 2e^{5t}, y_1 = e^{5t}$ and $x_2 = e^{-t}, y_2 = -e^{-t}$are solutions of the system $\begin{cases} \frac{dx}{dt} = 3x + 4y \\ \frac{dy}{dt} = 2x + y \end{cases}$ Also show that these two solutions are linearly independent	
		Solution: Consider. $\begin{cases} x = 2e^{5t} \\ y = e^{5t} \end{cases}$ Therefore, $\frac{dx}{dt} = 10e^{5t}\quad \frac{dy}{dt} = 5e^{5t}$ and $\quad 3x + 4y = 6e^{5t} + 4e^{5t} = \frac{dx}{dt} = 10e^{5t}$	2

		$2x+y=4e^{5t}+e^{5t}=\frac{dy}{dt}=5e^{5t}$ Thus, $\begin{cases} x=2e^{5t} \\ y=e^{5t} \end{cases}$ satisfies the given system. Similarly, $\begin{cases} x=e^{-t} \\ y=-e^{-t} \end{cases}$ consider. Therefore, $\frac{dx}{dt}=-e^{-t}$ and $\frac{dy}{dt}=e^{-t}$ $3x+4y=3e^{-t}+4(-e^{-t})=\frac{dx}{dt}=-e^{-t}$ $2x+y=2e^{-t}+(-e^{-t})=\frac{dy}{dt}=e^{-t}$ Thus, $\begin{cases} x=e^{-t} \\ y=-e^{-t} \end{cases}$ too satisfies the given system. Consider the Wronskian of the two solutions: $W=\begin{bmatrix} 2e^{5t} & e^{-t} \\ e^{5t} & -e^{-t} \end{bmatrix}=-3e^{4t}$ Since, exponential function never vanishes, $W \neq 0$. Thus, these solutions are linearly independent	2 1
	$f)$	Show that $(-2e^{t}\sin 2t, e^{t}\cos 2t)$ and $(2e^{t}\cos 2t, e^{t}\sin 2t)$ are linearly independent solutions of $\begin{cases} \frac{dx}{dt}=x-4y \\ \frac{dy}{dt}=x+y \end{cases}$	
		Solution: Consider $\begin{cases} x=-2e^{t}\sin 2t \\ y=e^{t}\cos 2t \end{cases}$ .Therefore, $\frac{dx}{dt}=-2\left(2e^{t}\cos 2t+e^{t}\sin 2t\right)$ $\quad$ $\frac{dy}{dt}=-2e^{t}\sin 2t+e^{t}\cos 2t$ $=-2e^{t}\sin 2t-4e^{t}\cos 2t$ and $=x+y$ $=x-4y$ Thus, $\begin{cases} x=-2e^{t}\sin 2t \\ y=e^{t}\cos 2t \end{cases}$ satisfies the given system. Similarly, consider $\begin{cases} x=2e^{t}\cos 2t \\ y=e^{t}\sin 2t \end{cases}$ $\frac{dx}{dt}=2\left(-2e^{t}\sin 2t+e^{t}\cos 2t\right)$ $\quad$ $\frac{dy}{dt}=2e^{t}\cos 2t+e^{t}\sin 2t$ $=2e^{t}\cos 2t-4e^{t}\sin 2t$ and $=x+y$ $=x-4y$	2 1

(13)

		Thus, $\begin{cases} x = 2e^t \cos 2t \\ y = e^t \sin 2t \end{cases}$ satisfies the given system. Now consider the Wronskian of the two solutions: $W = \begin{vmatrix} -2e^t \sin 2t & 2e^t \cos 2t \\ e^t \cos 2t & e^t \sin 2t \end{vmatrix}$ $= -2e^{2t} \sin^2 2t - 2e^{2t} \cos^2 2t$ $= -2e^{2t} \left(\sin^2 2t + \cos^2 2t\right)$ $= -2e^{2t}$ Since, exponential function never vanishes, we have $W \neq 0$. Thus, the solutions are linearly independent.	1 1

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