51844

(3 Hours) [Total Marks: 100

Note: (*i*) All questions are compulsory.

(*ii*) Figures to the right indicate marks for respective parts.

Q.1	Choose correct alternative in each of the following:					(20)
	i.	Which of the following is an exact first order differential equation?				
		(a)	$(x^2+1)dx - xydy = 0$	(b)	$xdy + (3x-2y)dx = 0$	
		(c)	$2xydx + (x^2+2)dy = 0$	(d)	$x^2ydy - ydx = 0$	
		Ans	i.(c)			
	ii.	The particular solution of linear differential equation $x\frac{dy}{dx} = 4-2y$ satisfying initial condition $y(1) = 0$ is				
		(a)	$y = 4 - \frac{1}{x^2}$	(b)	$y = 2 - \frac{2}{x^2}$	
		(c)	$y = 1 - \frac{1}{x^2}$	(d)	None of the above.	
		Ans	ii.(b)			
	iii.	Family of orthogonal trajectories for the family of curves $xy = c$ is				
		(a)	$x^2 - y^2 = k$	(b)	$y^2 = kx$	
		(c)	$x^2 = ky$	(d)	$x^2 + y^2 = k$	
		Ans	iii(a)			
	iv.	A necessary and sufficient condition for a first order differential equation $M(x,y)dx + N(x,y)dy = 0$ to be exact is				
		(a)	$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$	(b)	$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$	
		(c)	$\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x} = 0$	(d)	$\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x} = 1$	
		Ans	iv.(a)			
	v.	The general solution of $4y'' + 12y' + 9y = 0$ is				
		(a)	$e^{-\frac{3}{2}x}\ (c_1 + c_2x)$	(b)	$c_1 + c_2e^{-\frac{3}{2}x}$	
		(c)	$c_1\cos x + c_2\sin(-\frac{3}{2}x)$	(d)	None of these	
		Ans	v.(a)			
	vi.	The auxiliary equation of the homogeneous differential equation $9y'' + 4y = 0$ is				
		(a)	$9m^2 + 4m = 0$	(b)	$9m^2 + 4 = 0$	
		(c)	$9m + 4 = 0$	(d)	None of these	
		Ans	vi.(b)			
	vii.	$y'' + P(x)y' + Q(x)y = R(x)$ is second order homogeneous differential equation if				
		(a)	$P(x) = 0$	(b)	$Q(x) = 0$	
		(c)	$R(x) = 0$	(d)	$R(x) =$ non zero constant	
		Ans	vii.(c)			
	viii.	If $y_1(x) = \cos\pi x$ and $y_2(x) = \sin\pi x$, value of Wronskian $W(y_1, y_2)$ is				
		(a)	π	(b)	2π	
		(c)	0	(d)	None of these.	
		Ans	viii.(a)			

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	ix.		The Wronskian of two solutions $\begin{cases} x = e^{t} \\ y = 2e^{t} \end{cases}$ and $\begin{cases} x = 3e^{2t} \\ y = 4e^{2t} \end{cases}$, of a homogeneous linear system of differential equations, is equal to	
		(a)	$-2e^{3t}$ (b) $3e^{2t}$	
		(c)	0 (d) None of these	
		Ans	ix. (a)	
	x.		Amongst the following, the pair of linearly independent solutions is ____	
		(a)	$\begin{cases} x = e^{t} \\ y = e^{t} \end{cases}$ and $\begin{cases} x = 2e^{3t} \\ y = e^{3t} \end{cases}$ (b) $\begin{cases} x = -e^{3t} \\ y = -e^{2t} \end{cases}$ and $\begin{cases} x = e^{3t} \\ y = e^{2t} \end{cases}$	
		(c)	$\begin{cases} x = e^{t} \\ y = -e^{3t} \end{cases}$ and $\begin{cases} x = -e^{t} \\ y = e^{3t} \end{cases}$ (d) None of these	
		Ans	x. (a)	
Q.2	a)		Attempt any **ONE** question from the following:	(08)
		i.	Show that the substitution $v = y^{1-n}$, reduces the Bernoulli's differential equation $\frac{dy}{dx} + Py = Qy^{n}$, (where $n \neq 0$, 1 and P, Q are integrable functions of x) to a linear first order O.D.E. in the variables x and v. Hence solve $\frac{dy}{dx} + y = xy^{3}$.	
			Sol: Dividing the given equation $\frac{dy}{dx} + Py = Qy^{n}$by y^{n}, we get, $y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$.	1
			Now putting $v = y^{1-n}$, we have $\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$ $\therefore$ The above O.D.E. transforms to	1
			$$\frac{1}{1-n}\frac{dv}{dx} + P(x)v = Q(x).$$ Finally letting $$P_1 = (1-n)P \quad \& Q_1 = (1-n)Q$$ yields the equation $\frac{dv}{dx} + P_1(x)v = Q_1(x)$, which is a linear O.D.E. and can be solved as usual.	2
			To solve the given differential equation put $v = y^{-2}$ giving $\frac{dv}{dx} - 2v = -2x$,	1
			which is linear and has solution, $v = e^{2x}(-\int 2xe^{-2x}dx + c)$ $$v = e^{2x}\left(xe^{-2x} + \frac{1}{2}e^{-2x} + c\right)$$	1
			$\therefore \frac{1}{y^{2}} - x - \frac{1}{2} = ce^{2x}$.	2
		ii.	Solve the given differential equation by reducing it to a homogenous differential equation: $\frac{dy}{dx} = \frac{x+y+4}{x-y-6}$	
			Sol: $\frac{dy}{dx} = \frac{x+y+4}{x-y-6}$ Put $x = X + h$and $y = Y + k$, so that $dx = dX$ and $dy = dY$, which implies $\frac{dy}{dx} = \frac{dY}{dX}$. Transforming to new variables the differential equation changes to	

			$\frac{dY}{dX}=\frac{X+Y+(h+k+4)}{X-Y+(h-k-6)}$. Choosing h and k so that $h+k+4=0$ and $h-k-6=0$ we get $h=1$ and $k=-5$. The differential equation thus reduces to the first order homogenous equation $\frac{dY}{dX}=\frac{X+Y}{X-Y}$. Now putting $Y=vX$, the equation transforms to $v+X\frac{dv}{dx}=\frac{1+v}{1-v}$. Separating the variables, $\frac{dX}{X}=\frac{1-v}{1+v^2}dv$. Integrating we get, $logX=\tan^{-1}v-\frac{1}{2}\log(1+v^2)+c$ or that $\log X\sqrt{1+v^2}=\tan^{-1}v+c$. $\therefore \log\sqrt{X^2+Y^2}=\tan^{-1}\frac{Y}{X}+c$ or that $\log\sqrt{(x-1)^2+(y+5)^2}=\tan^{-1}\left(\frac{y+5}{x-1}\right)+c$.	4 4
	b)	Attempt any **TWO** questions from the following:		(12)
		i.	By Substituting $y=vx$, solve the differential equation: $$x\frac{dy}{dx}=y+\sqrt{x^2+y^2}$$	
			Sol: $y=vx$ $$\therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$$ $\therefore$ given differential equation becomes $$x\left(v+x\frac{dv}{dx}\right)=vx+\sqrt{x^2+v^2x^2}$$ $$\therefore x^2\frac{dv}{dx}=x\sqrt{1+v^2}$$ $$\therefore \frac{dv}{\sqrt{1+v^2}}=\frac{dx}{x}$$ $$\therefore \int\frac{dv}{\sqrt{1+v^2}}=\int\frac{dx}{x}$$ $$\therefore \log(v+\sqrt{1+v^2})=\log x+\log c$$ $$\therefore \log(v+\sqrt{1+v^2})=\log cx$$ $$\therefore v+\sqrt{1+v^2}=cx$$ Substituting value of $v=y/x$ we get, $y+\sqrt{x^2+y^2}=cx^2$.	1 1 1 1 2
		ii.	Find an integrating factor of the given differential equation and solve: $(2y+3xy^2)dx+(x+2x^2y)dy=0$.	
			Sol: The given differential equation is $y(2+3xy)dx+x(1+2xy)dy=0$, which is of the form $f_1(xy)ydx+f_2(xy)xdy=0$ $\therefore I.F.=\frac{1}{xy(1+2xy)-xy(2+3xy)}=\frac{-1}{xy(1+xy)}$ or $\frac{1}{xy(1+xy)}$. Multiplying the given differential equation by $I.F.$, we get the exact differential equation $\frac{2+3xy}{x(1+xy)}dx+\frac{1+2xy}{y(1+xy)}dy=0$.	1 2 1

(3)

			Its solution is, $\log c = \int \frac{2+3xy}{x(1+xy)} \partial x + \int \frac{2}{y} dy$ $= \log(1+xy) + 2\log x + 2\log y$ or that $x^2y^2(1+xy) = c$.	2
		iii.	Solve the differential equation: $$\frac{dy}{dx} + \frac{4xy}{1+x^2} = \frac{1}{(1+x^2)^3}$$	
			Sol: This is a linear differential equation $\frac{dy}{dx} + P\,y = Q$ with $$P = \frac{4x}{1+x^2}, \quad Q = \frac{1}{(1+x^2)^3}$$ $\therefore$ its solution is given by, $ye^{\int \frac{4x}{1+x^2} dx} = \int \frac{1}{(1+x^2)^3} e^{\int \frac{4x}{1+x^2} dx} dx + c$ i.e. $y\, e^{2\log(1+x^2)} = \int \frac{1}{(1+x^2)^3} e^{2\log(1+x^2)}\ dx + c$ i.e. $y\,(1+x^2)^2 = \int \frac{1}{(1+x^2)^3}(1+x^2)^2\ dx + c$ i.e. $y(1+x^2) = \int \frac{1}{1+x^2}\ dx + c$ i.e. $y(1+x^2) = \tan^1 x + c$	1 1 4
		iv.	A bacteria culture grows at a rate that is proportional to the number present. If it is found that the number doubles in 3 hours, how many may be expected at the end of 24 hours?	
			Sol: Let x denote the number of bacteria present in the culture at time t in hours. The number of bacteria in a certain culture grows at a rate that is proportional to the number present. Hence, $\frac{dx}{dt} \propto x$. $\therefore \frac{dx}{dt} = kx$ where (k> 0) is a constant $\therefore \frac{dx}{x} = kt\ dt$ Integrating both sides, we get, $\log x = kt + c$, where c is the constant of integration. If x_0 denotes the initial value of x, then by the above equation we get $c = \log x_0$ $\therefore \log x = kt + \log x_0$ or that $x = x_o e^{kt}$ ... (1) Now, given that, when $t = 3\ hrs, x = 2x_0$ $\therefore$ by (1), $2 = e^{3kt}$ or that $k = \frac{1}{3}\log 2$ Substituting the value of k in equation (1) gives, $$x = x_o e^{\frac{t}{3}\log 2} = x_0 2^{\frac{t}{3}}$$ $\therefore$ After 24 hrs. the number of bacteria will be $x = x_0 2^{\frac{24}{3}} = 2^8 x_0$ i.e. 2^8times the initial number.	3 3
Q.3	a)	Attempt any **ONE** question from the following:		(08)

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		i.	Let $y_1(x)$ and $y_2(x)$ be any two solutions of $y'' + P(x)y' + Q(x)y = 0$. Then show that their Wronskian $W(x) = W(y_1, y_2)$ is either identically zero or never zero.	
			Proof As, $y_1(x)$and$y_2(x)$ are solutions to $y'' + P(x)y' + Q(x)y = 0$ $\therefore\ y_1'' + P(x)y_1' + Q(x)y_1 = 0\ \dots\ (1)$ $y_2'' + P(x)y_2' + Q(x)y_2 = 0\ \dots\ (2)$ $W(x) = W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x)$ Since $y_1(x)$ and $y_2(x)$ are solutions, therefore W(x)*is* differentiable Differentiating $W(x)$ w.r.t x $W'(x) = y_1(x)y_1''(x) - y_2(x)y_1''(x)$ $y_2(x) \times (1) - y_1(x) \times (2)$, we get $(y_1''(x)y_2(x) - y_2''(x)y_1(x)) + P(x)(y_1'(x)y_2(x) - y_2'(x)y_1(x)) = 0$ $W'(x) + P(x)W(x) = 0$ $\therefore \frac{dW}{dx} = -P(x)\,W$ $\therefore \int_{x_0}^{x} \frac{dW}{W} = \int_{x_0}^{x} -P(x)\ dx$ $\log W(x) - \log W(x_0) = -\int_{x_0}^{x} P(x)\ dx$ $W(x) = W(x_0)e^{-\int_{x_0}^{x} P(x)\,dx}$ If $W(x_0) = 0$ *then* $W(x) = 0\ \ \forall x \in [a, b]$. Hence, W is identically zero. If $W(x_0) \neq 0$ *then* $W(x) \neq 0\ \ \forall x \in [a, b]$. Hence, W is never zero.	2 1 1 1 2 1
		ii.	Let $y'' + py' + q = 0$ be a homogeneous differential equation with constant coefficients. And let $m^2 + am + b = 0$ be a corresponding auxiliary equation with roots m_1 and m_2. Then, Discuss the general solution of the differential equation when $(a) m_1$ and m_2 are complex roots. $(b) m_1$ and m_2 are real and unequal roots.	
			Solution: (a) $m_1$and m_2 are the roots of the auxiliary equation of the differential equation $y'' + py' + q = 0$, where p and q are constant. Auxiliary equation: $m^2 + pm + q = 0$ with roots m_1 and m_2.	

			$(a) m_1$ and m_2 are complex roots. Let $m_1 = \alpha + i\,\beta$ and $m_2 = \alpha - i\,\beta$ Two solutions to differential equation are $y_1 = e^{(\alpha+i\,\beta)x}$ and $y_2 = e^{(\alpha-i\,\beta)x}$ $\therefore W(y_1, y_2) = (\alpha - i\beta)e^{(\alpha+i\beta)x+(\alpha-i\beta)x} - (\alpha + i\beta)e^{(\alpha+i\beta)x+(\alpha-i\beta)x}$ $= -2\beta e^{2\alpha x} \neq 0$	2
			$\therefore y_1$ and y_2 are linearly independent, $\therefore$ general solution is $y = c'_1 e^{(\alpha+i\beta)x} + c'_2 e^{(\alpha-i\beta)x}$ $\therefore y = e^{\alpha x}(c'_1 e^{i\beta x} + c'_2 e^{-i\beta x})$ By putting $e^{i\beta x} = \cos\beta x + i\sin\beta\, x$ & $e^{-i\beta x} = \cos\beta x - i\sin\beta\, x$ We get, General solution is $\boldsymbol{y = e^{\alpha x}(c_1 \cos\beta\, x + c_2 \sin\beta x)}$	2
			(b) when $m_1 \neq m_2$(Real and unequal roots) Two solutions to differential equation are $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$	1
			$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2\, e^{m_2 x} \end{vmatrix} = (m_2 - m_1)e^{(m_1+m_2)x}$ $\neq 0$ as $m_1 \neq m_2$	2
			$\therefore y_1$ and y_2 are linearly independent. $\therefore$ general solution to given differential equation is $\boldsymbol{y = c_1 e^{m_1 x} + c_2 e^{m_1 x}}$	1
	b)	Attempt any **TWO** questions from the following:		(12)
		i.	Show that $y(x) = c_1 x + c_2 x^2$ is solution for the equation $x^2 y'' - 2xy' + 2y = 0$, hence find particular solution,if $y(1) = 3, y'(1) = 5$.	
			Solution: $y(x) = c_1 x + c_2 x^2, y'(x) = c_1 + 2c_2 x, y''(x) = 2c_2$	2
			$\therefore x^2 y'' - 2xy' + 2y$ $= x^2(2c_2) - 2x(c_1 + 2c_2 x) + 2(c_1 x + c_2 x^2)$ $= 0$ $\therefore y(x) = c_1 x + c_2 x^2$ is solution for the equation $x^2 y'' - 2xy' + 2y = 0$	1

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			$y(1) = 3 \quad \Rightarrow \quad c_1 + c_2 = 3$ $y'(1) = 5 \quad \Rightarrow \quad c_1 + 2c_2 = 5$ $\therefore c_1 = 1, c_2 = 2$ $\therefore$ particular solution is $\boldsymbol{y(x) = x + 2x^2}$	2 1
		ii.	Using method of variation of parameter solve the differential equation $y'' - y = \frac{2}{1+e^x}$	
			Solution: Auxiliary equation to corresponding homogeneous d.e: $m^2 - 1 = 0$ Roots are $m_1 = 1, m_2 = -1$ General solution to $y'' - y = 0$ is $y = c_1 e^x + c_2 e^{-x}$ $y_1(x) = e^x, y_2(x) = e^{-x} \;\therefore W(x) = W(y_1, y_2) = -2$ The particular solution of $y'' - y = \frac{2}{1+e^x}$ is $y_p = v_1 y_1 + v_2 y_2$ Where $v_1 = -\int \frac{y_2 R}{W} = -e^{-x} + \log(e^{-x} + 1)$ and $v_2 = \int \frac{y_1 R}{W} = -\log(1 + e^x)$ $\therefore$ The particular solution of $y'' - y = \frac{2}{1+e^x}$ is $y_p = (-e^{-x} + \log(e^{-x} + 1))e^x + (-\log(1 + e^x))e^{-x}$ $\therefore$ General solution of $y'' - y = \frac{2}{1+e^x}$ is $\boldsymbol{y_g} = c_1 e^x + c_2 e^{-x} + (-e^{-x} + \log(e^{-x} + 1))e^x + (-\log(1 + e^x))e^{-x}$	1 1 1 2 1
		iii.	Solve the non-homogenous differential equation $y'' + y = 2x^3 - x + 3$, by method of undetermined coefficients.	
			Solution: Auxiliary equation to corresponding homogeneous d.e: $m^2 + 1 = 0$ Roots are $m_1 = i, m_2 = -i$ General solution to $y'' + y = 0$ is $y = c_1 \cos x + c_2 \sin x$ For particular integral, let trial function be $y_p = Ax^3 + Bx^2 + Cx + D, y_p' = 3Ax^2 + 2Bx + C, y_p'' = 6Ax + 2B$ Substituting in $y'' + y = 2x^3 - x + 3$, we get $A = 2, B = 0, C = -13, D = 3$ $\therefore y_p = 2x^3 - 13x + 3$ General solution to $y'' + y = 2x^3 - x + 3$ is $\boldsymbol{y} = c_1 \cos x + c_2 \sin x + 2x^3 - 13x + 3$	2 1 1 2

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		iv.	Find another linearly independent solution for the differential equation $(1-x^2)y'' - 2xy' + 2y = 0$, if $y_1 = x$ is one of the solution.	
			Solution: $y_1 = x$ is one of the solution of $(1-x^2)y'' - 2xy' + 2y = 0$ i.e $y_1 = x$ is one of the solution of $y'' - \left(\frac{2x}{1-x^2}\right)y' + \left(\frac{2}{1-x^2}\right)y = 0$ ∴ other linearly independent solution is $y_2(x) = y_1(x)\int \frac{1}{y_1^2} e^{-\int P(x)\,dx}\,dx$ $= x\int \frac{1}{x^2} e^{\int\left(\frac{2x}{1-x^2}\right)dx}\,dx \;=\; x\int \frac{1}{x^2} e^{-\log(1-x^2)}\,dx$ $= x\int \frac{1}{x^2(1-x^2)}\,dx \;=\; x\int \left(\frac{1}{x^2} + \frac{1}{1-x^2}\right)dx$ $= x\left(-\frac{1}{x} + log\left\|\frac{1-x}{1+x}\right\|\right)$ $\therefore y_2(x) = x\left(-\frac{1}{x} + log\left\|\frac{1-x}{1+x}\right\|\right)$ is the required other solution.	2 1 3
Q.4	*a)*	Attempt any **ONE** question from the following:		(08)
		i.	State the theorem for existence and uniqueness of first order homogeneous linear system of ODE in two variables. Let $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ be two solutions of the homogeneous system $\begin{cases} \frac{dx}{dt} = a_1(t)x + b_1(t)y \\ \frac{dy}{dt} = a_2(t)x + b_2(t)y \end{cases}$, where a_1, a_2, b_1, b_2 are continuous functions on $[a,b]$. Show that the linear independence and continuity of the solutions on $[a,b]$ implies $\begin{cases} x = c_1x_1(t) + c_2x_2(t) \\ y = c_1y_1(t) + c_2y_2(t) \end{cases}$ to be the general solution of the homogeneous system on $[a,b]$.	
			Solution: The linear system of homogeneous first order ordinary differential equations given as $\begin{cases} \frac{dx}{dt} = a_1(t)x(t) + b_1(t)y(t) \\ \frac{dy}{dt} = a_2(t)x(t) + b_2(t)y(t) \end{cases}$ ----------- * where $a_1(t)$, $a_2(t)$, $b_1(t)$, $b_2(t)$ are continuous functions defined on interval $[a, b]$ has a unique solution $((x(t), y(t))$ satisfying initial conditions $x(t_0) = x_0$ & $y(t_0) = y_0$ for some $t_0 \in [a, b]$.	2

Let $(x_1(t), y_1(t))$ & $(x_2(t), y_2(t))$ where $x_1(t), y_1(t), x_2(t), y_2(t)$ are continuously differentiable on [a,b], be two linearly independent solutions of $*$, i.e.,$W(t) \neq 0, \forall t \in [a,b]$.

Then ,

$$\begin{cases} x_1'(t) = a_1(t)x_1(t) + b_1(t)y_1(t) \\ y_1'(t) = a_2(t)x_1(t) + b_2(t)y_1(t) \end{cases} \text{and}$$

$$\begin{cases} x_2'(t) = a_1(t)x_2(t) + b_1(t)y_2(t) \\ y_2'(t) = a_2(t)x_2(t) + b_2(t)y_2(t) \end{cases}$$

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Now $(c_1x_1)'(t) = c_1x_1'(t)$, $(c_1y_1)'(t) = c_1y_1'(t)$,

$(c_2x_2)'(t) = c_2x_2'(t)$, $(c_2y_2)'(t) = c_2y_2'(t)$

$$a_1(t)(c_1x_1)(t) + b_1(t)(c_1y_1)(t) = c_1(a_1(t)x_1(t) + b_1(t)y_1(t))$$

$$a_2(t)(c_1x_1)(t) + b_2(t)(c_1y_1)(t) = c_1(a_1(t)x_2(t) + b_1(t)y_2(t))$$

Similarly for c_2x_2 & c_2y_2

Thus verifying that $(c_1x_1 + c_2x_2, c_1y_1 + c_2y_2)$ satisfies $*$

Now let $((x(t), y(t))$ satisfying initial conditions $x(t_0) = x_0$ & 1
$y(t_0) = y_0$ for some $t_0 \in [a,b]$ be a particular solution to $*$ then consider the equations

$$c_1x_1(t_0) + c_2x_2(t_0) = x_0$$

$$c_1y_1(t_0) + c_2y_2(t_0) = y_0$$

$$\therefore \begin{bmatrix} x_1(t_0) & x_2(t_0) \\ y_1(t_0) & y_2(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

The above matrix equation has a unique solution provided 2

$$det \begin{bmatrix} x_1(t_0) & x_2(t_0) \\ y_1(t_0) & y_2(t_0) \end{bmatrix} \neq 0$$

But as $W(t) \neq 0, \forall t \in [a,b]$ hence $W(t_0) \neq 0$

$\therefore$ For above values of c_1, c_2, $(c_1x_1 + c_2x_2, c_1y_1 + c_2y_2)$ Satisfies $*$ but due to uniqueness of solution satisfying initial conditions hence

$c_1x_1 + c_2x_2 = x$ & $c_1y_1 + c_2y_2 = y$.

Thus $(c_1x_1 + c_2x_2, c_1y_1 + c_2y_2)$ is general solution to $*$ for arbitrary constants c_1 & c_2. 2

ii. State the theorem for existence and uniqueness of a solution to a homogeneous linear system of first order ODE in two variables. What is the auxiliary equation of such a system with constant coefficients? State the two linearly independent solutions, in case the roots of the

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auxiliary equation are (i) Real and distinct, (ii) Distinct complex and (iii) Real and equal.

Solution:

The linear system of homogeneous first order ordinary differential equations given as

$$\begin{cases} \frac{dx}{dt} = a_1(t)x + b_1(t)y \\ \frac{dy}{dt} = a_2(t)x + b_2(t)y \end{cases}$$

where $a_1(t)$, $a_2(t)$, $b_1(t)$, $b_2(t)$ are continuous functions defined on interval $[a, b]$ has a unique solution $((x(t), y(t))$ satisfying initial conditions $x(t_0) = x_0$ & $y(t_0) = y_0$ for some $t_0 \in [a, b]$. 1

The auxiliary equation of the homogeneous linear system in two variables with constant coefficients

$$\begin{cases} \frac{dx}{dt} = a_1 x + b_1 y \\ \frac{dy}{dt} = a_2 x + b_2 y \end{cases} \quad \ldots (1)$$

is given by $m^2 - (a_1 + b_2)m + a_1 b_2 - a_2 b_1 = 0$.

Let $\begin{cases} x = Ae^{mt} \\ y = Be^{mt} \end{cases}$ be a solution of (1). Then we get the following linear algebraic system: 1

$$(a_1 - m)A + b_1 B = 0 \quad \ldots (2)$$

$$a_2 A + (b_2 - m)B = 0$$

Roots of the auxiliary equation are:

(i) Real and distinct:

If m_1 and m_2 are the distinct real roots, then the general solution of system (1) is given by

$$\begin{cases} x = c_1 A_1 e^{m_1 t} + c_2 A_2 e^{m_2 t} \\ y = c_1 B_1 e^{m_1 t} + c_2 B_2 e^{m_2 t} \end{cases}$$

where A_1, B_1 is a non-trivial solution of system (2) when $m = m_1$ and 2
A_2, B_2 is a non-trivial solution of system (2) when $m = m_2$.

(ii) Complex :

If m_1 and m_2 are distinct complex roots, then the general solution of system (1) is given by

$$\begin{cases} x = e^{at}[c_1(A_1 \cos bt - A_2 \sin bt) + c_2(A_1 \sin bt + A_2 \cos bt)] \\ y = e^{at}[c_1(B_1 \cos bt - B_2 \sin bt) + c_2(B_1 \sin bt + B_2 \cos bt)] \end{cases}$$

(10)

			where $m = a \pm ib$ and $A = A_1 + iA_2, B = B_1 + iB_2$ is a non-trivial solution of system (2) when $m = a + ib$. (iii) Real and equal: If $m = m_1 = m_2$, then the general solution of system (1) is given by $$\begin{cases} x = c_1Ae^{mt} + c_2(A_1 + tA_2)e^{mt} \\ y = c_1Be^{mt} + c_2(B_1 + tB_2)e^{mt} \end{cases}$$ where A, B is a non-trivial solution of system (2) when we put value of m and A_1, A_2, B_1 and B_2 are found by substituting $\begin{cases} x = (A_1 + tA_2)e^{mt} \\ y = (B_1 + tB_2)e^{mt} \end{cases}$ in the system (1).	2 2
	b)	Attempt any **TWO** questions from the following:		(12)
		i.	Find the general solution of the system: $\begin{cases} \frac{dx}{dt} = 4x - 2y \\ \frac{dy}{dt} = 5x + 2y \end{cases}$	
			Solution: Auxiliary equation is $m^2 - (4+2)m + 8 + 10 = 0$ Roots are $m = 3 \pm 3i$ Taking $\begin{cases} x(t) = Ae^{mt} \\ y(t) = Be^{mt} \end{cases}$ & $m = 3 + 3i \therefore x'(t) = 4x(t) - 2y(t) \therefore$ $mAe^{mt} = 4Ae^{mt} - 2Be^{mt}$ $\therefore (4 - m)A - 2B = 0 \; as \; e^{mt} \neq 0 \therefore using \; m = 3 + 3i \; we \; get \; (1 - 3i)A - 2B = 0$ Take $A = 1 + 3i \; then \; (1 - 3i)(1 + 3i) = 2B \therefore B = 5$ $\therefore \begin{cases} x(t) = (1 + 3i)e^{(3+3i)t} \\ y(t) = 5e^{(3+3i)t} \end{cases}$ $\therefore \begin{cases} x(t) = e^{3t}[cos3t - 3sin3t + i(3cos3t + sin3t)] \\ y(t) = e^{3t}[5cos3t + i(5sin3t)] \end{cases}$ Consider $\begin{cases} x_1(t) = e^{3t}[cos3t - 3sin3t] \\ y_1(t) = e^{3t}[5cos3t] \end{cases}$ *and* $\begin{cases} x_2(t) = e^{3t}[3cos3t + sin3t] \\ y_2(t) = e^{3t}[5sin3t] \end{cases}$ Then check these solve $\begin{cases} \frac{dx}{dt} = 4x - 2y \\ \frac{dy}{dt} = 5x + 2y \end{cases}$ Also Wronskian of these solutions $W(t) =$ $\begin{vmatrix} e^{3t}(cos3t - sin3t) & 5e^{3t}cos3t \\ e^{3t}(3cos3t + sin3t) & 5e^{3t}sin3t \end{vmatrix} = -15e^{6t} \neq 0$ Hence these are linearly independent solutions	2 2 1

			$\therefore \begin{cases} x(t) = c_1e^{3t}[cos3t - 3sin3t] + c_2e^{3t}[3cos3t + sin3t] \\ y(t) = c_1[5e^{3t}cos3t] + c_2[5^{3t}sin3t] \end{cases}$ is the general solution.	1
		ii.	Find the general solution of the following linear system: $\begin{cases} \frac{dx}{dt} = -4x - y \\ \frac{dy}{dt} = x - 2y \end{cases}$	
			Solution: We know that the auxiliary equation of the system $\begin{cases} \frac{dx}{dt} = a_1x + b_1y \\ \frac{dy}{dt} = a_2x + b_2y \end{cases}$ is given by $m^2 - (a_1 + b_2)m + (a_1b_2 - a_2b_1) = 0$. Therefore, the auxiliary equation of the given system is $m^2 + 6m + (8+1) = 0$. That is, $m^2 + 6m + 9 = 0$. That is, $(m+3)(m+3) = 0$. Therefore, the roots of the auxiliary equation are –3 and –3. Start with $\begin{cases} x = Ae^{mt} \\ y = Be^{mt} \end{cases}$, where $m = -3$. This gives, $-3Ae^{-3t} = -4Ae^{-3t} - Be^{-3t}$. Thus, $A = -B$. Take $A = 1$ and $B = -1$. This gives $\begin{cases} x = e^{-3t} \\ y = -e^{-3t} \end{cases}$ as one of the solutions of the system. To obtain the second solution, start with $\begin{cases} x = (A_1 + A_2t)e^{-3t} \\ y = (B_1 + B_2t)e^{-3t} \end{cases}$ Substituting this in the given system and simplifying we get $A_1 + A_2 + B_1 = 0$ $A_2 + B_2 = 0$ $-A_1 - B_1 + B_2 = 0$ $-A_2 - B_2 = 0$ From second and fourth equations, we have $A_2 + B_2 = 0$. Take $A_2 = 1$ and $B_2 = -1$. Substituting these in other equations, we get $A_1 + B_1 = -1$. Take $A_1 = -1$ and $B_1 = 0$. Inserting all these four values, we get another solution as $\begin{cases} x = (-1+t)e^{-3t} \\ y = -te^{-3t} \end{cases}$ Thus, the general solution of the system is given by, $\begin{cases} x = c_1e^{-3t} + (-1+t)c_2e^{-3t} \\ y = -c_1e^{-3t} - c_2te^{-3t} \end{cases}$	1 1 2 2
		iii.	Find the general solution of the following linear system:	

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			$\begin{cases} \frac{dx}{dt} = x + y \\ \frac{dy}{dt} = 4x - 2y \end{cases}$	
			Solution: We know that the auxiliary equation of the system $\begin{cases} \frac{dx}{dt} = a_1x + b_1y \\ \frac{dy}{dt} = a_2x + b_2y \end{cases}$ is given by $m^2 - (a_1 + b_2)m + (a_1b_2 - a_2b_1) = 0$. The auxillary equation is $m^2 + m - 6 = 0$ or $(m+3)(m-2) = 0$ Distinct roots $m = -3$ *and* $m = 2$ m=-3 $4A + B = 0$ $A = 1\ B = -4$ $\begin{cases} x = e^{-3t} \\ y = -4e^{-3t} \end{cases}$ $m = 2$ $-A + B = 0$ $4A - 4B = 0$ $A = 1$ $B = 1$ $\begin{cases} x = e^{2t} \\ y = e^{2t} \end{cases}$ The general solution is $\begin{cases} x = c_1e^{-3t} + c_2e^{2t} \\ y = -4c_1e^{-3t} + c_2e^{2t} \end{cases}$	2 1 1 2
		iv.	Show that both $x_1 = 2e^{5t}, y_1 = e^{5t}$ and $x_2 = e^{-t}, y_2 = -e^{-t}$ are solutions of the system $\begin{cases} \frac{dx}{dt} = 3x + 4y \\ \frac{dy}{dt} = 2x + y \end{cases}$. Also show that these two solutions are linearly independent	
			Solution: Consider. $\begin{cases} x = 2e^{5t} \\ y = e^{5t} \end{cases}$ Therefore, $\frac{dx}{dt} = 10e^{5t}\ \frac{dy}{dt} = 5e^{5t}$ and $3x + 4y = 6e^{5t} + 4e^{5t} = \frac{dx}{dt} = 10e^{5t}$ $2x + y = 4e^{5t} + e^{5t} = \frac{dy}{dt} = 5e^{5t}$	

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		Thus, $\begin{cases} x = 2e^{5t} \\ y = e^{5t} \end{cases}$ satisfies the given system.	2
		Similarly, $\begin{cases} x = e^{-t} \\ y = -e^{-t} \end{cases}$ consider. Therefore, $\frac{dx}{dt} = -e^{-t}$ and $\frac{dy}{dt} = e^{-t}$ $3x + 4y = 3e^{-t} + 4(-e^{-t}) = \frac{dx}{dt} = -e^{-t}$ $2x + y = 2e^{-t} + (-e^{-t}) = \frac{dy}{dt} = e^{-t}$ Thus, $\begin{cases} x = e^{-t} \\ y = -e^{-t} \end{cases}$ too satisfies the given system.	2
		Consider the Wronskian of the two solutions: $W = \begin{bmatrix} 2e^{5t} & e^{-t} \\ e^{5t} & -e^{-t} \end{bmatrix} = -3e^{4t}$ Since, exponential function never vanishes, $W \neq 0$. Thus, these solutions are linearly independent.	2
Q.5		Attempt any **FOUR** questions from the following:	(20)
	a)	Verify that the following differential equation is exact and solve: $\left(4x^3y^3 + \frac{1}{x}\right) dx + y^2\left(3x^4 - \frac{1}{y}\right) dy = 0$	
		Sol: $\left(4x^3y^3 + \frac{1}{x}\right) dx + \left(3x^4 - \frac{1}{y}\right) dy = 0$ Here, $\frac{\partial M}{\partial y} = 12x^3y^2 = \frac{\partial N}{\partial x}$ $\therefore$ the given differential equation is exact.	2
		Its solution is, $\int M\partial x + \int N(\text{terms free from } x)dy = c$ i.e. $\int \left(4x^3\, y^3 + \frac{1}{x}\right) \partial x + \int (-y)dy = c$	1
		i.e. $x^4y^3 + \log x - \frac{y^2}{2} = c.$	2
	b)	Solve: $\frac{dy}{dx} = (9x - y + 1)^2.$	
		Sol: $\frac{dy}{dx} = (9x - y + 1)^2$ Put $u = 9x - y + 1,$	1
		so that $\frac{du}{dx} = 9 - \frac{dy}{dx}$ or that $\frac{dy}{dx} = 9 - \frac{du}{dx}$ $\therefore$ the given differential equation becomes, $9 - \frac{du}{dx} = u^2.$	1
		Separating the variables, $0 = dx + \frac{du}{u^2 - 3^2}.$ Integrating,	1

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		$x+\frac{1}{2(3)}\log\frac{u-3}{u+2}=c$ $x+\frac{1}{6}\log\left(\frac{9x-y+1-3}{9x-y+1+3}\right)=c$ $x+\frac{1}{6}\log\left(\frac{9x-y-2}{9x-y+4}\right)=c$	2
	c)	Find the solution of the differential equation $y''-2y'+5y=0$, with $y(\pi)=0$ and $y'(\pi)=2$.	
		Solution: Auxiliary equation to differential equation is $m^2-2m+5=0$ Roots are $m_1=1+2i,\ m_2=1-2i$ General solution to $y''+12y'+36y=0$ is $y(x)=e^x(c_1cos2x+c_2sin2x)$ $\therefore y'(x)=e^x(c_1cos2x+c_2sin2x)+e^x(-2c_1sin2x+2c_2cos2x)$ $y(\pi)=0 \quad\Rightarrow\quad c_1e^{\pi}=0 \quad\Rightarrow\quad c_1=0$ $y'(\pi)=2 \quad\Rightarrow\quad e^{\pi}(c_1)+e^{\pi}(2c_2)=2$ $\therefore\ c_1=-0\ and\ c_2=e^{-\pi}$ $\therefore$ General solution to $y''-2y'+5y=0$, is $y(x)=e^x(e^{-\pi}sin2x)=e^{(x-\pi)}sin2x$	1 1 1 1 1
	d)	Using Wronskian determinant method, determine whether the functions $y_1=sinx\|$ and $y_2=cosx,\quad x\in\mathbb{R}$ are linearly dependent or independent.	
		Solution: $W(x)=W(y_1,y_2)=\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x)\end{vmatrix}$ $=\begin{vmatrix} sinx & cosx \\ cosx & -sinx\end{vmatrix}$ $=-1$ $y_1=sinx$ and $y_2=cosx$ are linearly independent.	2 2 1
	e)	Solve the following linear system: $\begin{cases}\frac{dx}{dt}=x+2y\\ \frac{dy}{dt}=3x+2y\end{cases}$	
		Solution: We know that the auxiliary equation of the system $\begin{cases}\frac{dx}{dt}=a_1x+b_1y\\ \frac{dy}{dt}=a_2x+b_2y\end{cases}$ is given by $m^2-(a_1+b_2)m+(a_1b_2-a_2b_1)=0$. Therefore, the auxiliary equation of the given system is $m^2-3m+(2-6)=0$.	

		That is, $m^2-3m-4=0$. That is, $(m+1)(m-4)=0$. Thus, the roots of the auxiliary equation are 4 and -1. Start with $\begin{cases} x=Ae^{mt} \\ y=Be^{mt} \end{cases}$, where $m=4$. This gives, $4Ae^{4t}=Ae^{4t}+2Be^{4t}$. Thus, $3A=2B$. Take $A=2$ and $B=3$. Thus, one of the solutions of the given system is $\begin{cases} x=2e^{4t} \\ y=3e^{4t} \end{cases}$ Consider again, $\begin{cases} x=Ae^{mt} \\ y=Be^{mt} \end{cases}$, where $m=-1$. This gives, $-Ae^{-t}=Ae^{-t}+2Be^{-t}$. Thus, $-2A=2B$. Take $A=1$ and $B=-1$.Thus, $\begin{cases} x=e^{-t} \\ y=-e^{-t} \end{cases}$ is another solution of the system. Therefore, the general solution of the system is given by, $\begin{cases} x=2c_1e^{4t}+c_2e^{-t} \\ y=3c_1e^{4t}-c_2e^{-t} \end{cases}$	2 1 1 1
	$f)$	Show that both $x=e^{4t}, y=e^{4t}$ and $x=e^{-2t}, y=-e^{-2t}$ are solutions of the system: $\begin{cases} \frac{dx}{dt}=x+3y \\ \frac{dy}{dt}=3x+y \end{cases}$. Show that these solutions are linearly independent solutions on every closed and bounded interval.	
		Solution: Consider $\begin{cases} x=e^{4t} \\ y=e^{4t} \end{cases}$. Therefore, $\frac{dx}{dt}=4e^{4t}=e^{4t}+3e^{4t}=x+3y$ and $\frac{dy}{dt}=4e^{4t}=3e^{4t}+e^{4t}=3x+y$ Thus, $\begin{cases} x=e^{4t} \\ y=e^{4t} \end{cases}$ satisfies the given system. Similarly, consider $\begin{cases} x=e^{-2t} \\ y=-e^{-2t} \end{cases}$. Therefore, $\frac{dx}{dt}=-2e^{-2t}=e^{-2t}-3e^{-2t}=x+3y$ and $\frac{dy}{dt}=2e^{-2t}=3e^{-2t}-e^{-2t}=3x+y$ Thus, $\begin{cases} x=e^{-2t} \\ y=-e^{-2t} \end{cases}$ too satisfies the given system.	2 2

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		Consider the Wronskian of the two solutions: $W = \begin{vmatrix} e^{4t} & e^{-2t} \\ e^{4t} & -e^{-2t} \end{vmatrix}$ $= -e^{2t} - e^{2t}$ $= -2e^{2t}$ Since, exponential function never vanishes, $W \neq 0$. Thus, these solutions are linearly independent.	1

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