1

52718

(3 Hours) [Total Marks : 100]

Q.1	Choose correct alternative in each of the following:		
	i.	(b) Interval [0,9].	
	ii.	(c) $\mathbb{R}\setminus\mathbb{Q}$.	
	iii.	(c) $(n+1)$.	
	iv	(a) [1,6].	
	v	(a)	
	vi	(c)	
	vii	(c)	
	viii.	(c)	
	ix	(d)	
	x	(c)	
Q.2	a)	Attempt any ONE question from the following:	
	i	Using Nested Interval Theorem prove that the set $\mathbb{R}$ of real numbers is uncountable.	
	Ans	If possible assume $\mathbb{R}$ is countable. Then we can list the elements of $\mathbb{R}$ as $I = \{x_1, x_2, \dots x_n, \dots\}$ where all x_i's are distinct. First a closed interval I_1 of I is selected such that $x_1 \notin I_1$(this can be done by dividing the interval I). Next a closed interval I_2 of I_1 is selected such that $x_2 \notin I_2$ and so on. Therefore we get $I_1 \supseteq I_2 \supseteq \cdots$ (nested). By Nested Interval Theorem, $\bigcap_{n=1}^{\infty} I_n \neq \phi$.	

		Let $c \in \bigcap_{n=1}^{\infty} I_n$. Then $\neq x_n$, $n \in \mathbb{N}$. Therefore I contains a point c which is other than x_n , $n \in \mathbb{N}$. Thus the enumeration of I is incomplete. Hence I is uncountable. That is, the set $\mathbb{R}$ is uncountable.	
	ii	State and prove Nested Interval Theorem in $\mathbb{R}$.	
	Ans	*Statement:* If $I_n = [a_n, b_n]$, $n \in \mathbb{N}$ is a nested sequence of closed bounded intervals such that $\lim_{n\to\infty}(b_n - a_n) = 0$, then there exists unique number r in $\bigcap_{n=1}^{\infty} I_n$. *Proof:* Part I $a_n \leq b$, $\forall\, n \in \mathbb{N}$. Therefore the set $S = \{a_n : n \in \mathbb{N}\}$ is bounded above set. Let $r = \sup\{a_n : n \in \mathbb{N}\}$. Then $a_n \leq r$, $\forall\, n \in \mathbb{N}$. Claim : For any fixed $n \in \mathbb{N}$, b_n is an upper bound of S. For $n \leq k$, $I_n \supseteq I_k$. Therefore $a_k \leq b_k \leq b_n$. For $n > k$, $I_k \supseteq I_n$. Therefore $a_k \leq a_n \leq b_n$. Hence the claim. Therefore $r \leq b_n \forall\, n \in \mathbb{N}$. That is, $a_n \leq r \leq b_n$ $\forall\, n \in \mathbb{N}$. Therefore $r \in \bigcap_{n=1}^{\infty} I_n$. Part II (Uniqueness) Suppose $\exists\, s \neq r$ such that $s, r \in \bigcap_{n=1}^{\infty} I_n$. Then $-s \leq b_n - a_n$ $\forall\, n \in \mathbb{N}$. Since $\lim_{n\to\infty}(b_n - a_n) = 0$, it follows that $s = r$.	
	b)	Attempt any TWO questions from the following:	
	i.	Using Nested Interval Theorem prove that if $f: [a;\ b] \to \mathbb{R}$ is a continuous function with $f(a)\,f(b) < 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$.	

	Ans	Given $f(a)f(b) < 0$. Therefore either $f(a) < 0$ or $f(b) < 0$. Suppose $f(a) < 0 < f(b)$. Let $I_1 = [a_1, b_1]$, $(a = a_1, b = b_1)$. Let $p_1 = \frac{a_1+b_1}{2}$. If $f(p_1) = 0$, then $c = p_1$ and we are done. If $f(p_1) \neq 0$, then either $f(p_1) > 0$ or $f(p_1) < 0$. i) If $f(p_1) < 0$, then let $a_2 = p_1$, $b_2 = b_1$. ii) If $f(p_1) > 0$, then let $a_2 = a_1$, $b_2 = p_1$. Let $I_2 = [a_2, b_2]$. Then $\ell(I_2) = \frac{b_1-a_1}{2}$. Let $p_2 = \frac{a_2+b_2}{2}$. If $(p_2) = 0$, then $c = p_2$ and we are done. Else proceed similarly. Therefore if $f(p_n) = 0$ for some $n \in \mathbb{N}$, then $p_n = c$ and we are done. Else we obtain a nested sequence of closed bounded intervals $I_n = [a_n, b_n]$ such that $\forall\, n \in \mathbb{N}$ $f(a_n) < 0$ and $f(b_n) > 0$ and $\ell(I_n) = \frac{b_n-a_n}{2^{n-1}} \to 0$ as $n \to \infty$. Therefore by nested interval theorem, $\bigcap_{n=1}^{\infty} I_n = \{c\}$ for $c \in (a, b)$. Hence there exists $c \in (a, b)$ such that $f(c) = 0$.	
	ii.	If $I_n = \left(0, \frac{1}{n}\right)$ for all $n \in \mathbb{N}$ then prove that $\bigcap_{n=1}^{\infty} I_n = \phi$.	
	Ans	Let $I_n = \left(0, \frac{1}{n}\right)$ $\forall\, n \in \mathbb{N}$. Since $\frac{1}{1} > \frac{1}{2} > \frac{1}{2} > \cdots > \frac{1}{n} > \cdots$ we get $I_1 \supset I_2 \supset I_3 \supset \cdots I_n \supset \cdots$ Also $\lim_{n\to\infty} \left(\frac{1}{n} - 0\right) = 0$. Note that $0 \notin I_n$ $\forall\, n \in \mathbb{N}$. So $0 \notin \bigcap_{n=1}^{\infty} I_n$. To prove that $\bigcap_{n=1}^{\infty} I_n = \phi$. If there is $\xi \in \bigcap_{n=1}^{\infty} I_n$ such that $0 < \xi < \frac{1}{n}$ $\forall\, n \in \mathbb{N}$, by Archimedean property of $\mathbb{R}$, $\xi = 0$, which is contradiction. Hence $\bigcap_{n=1}^{\infty} I_n = \phi$.	
	iii.	State Intermediate value theorem and using it show that if $f: [0, 1] \to \mathbb{R}$ is continuous function which takes only rational values then f must be a constant function.	

	Ans	Intermediate Value Theorem : Let $f: [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) < 0 < f(b)$. Then there exists $c \in (a, b)$ such that $f(c) = 0$. Let $f: [0,1] \to \mathbb{R}$ be a continuous rational valued function. To prove that f is constant. On the contrary, let us assume that f is not constant. Then $\exists\ x, y \in [0,1]$ such that $f(x) = q_1 \in \mathbb{Q}$ and $f(y) = q_2 \in \mathbb{R}\backslash\mathbb{Q}$ with $q_1 \neq q_2$. Since $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists$ an irrational number $t \in \mathbb{R}\backslash\mathbb{Q}$ such that $q_1 < t < q_2$. That is, $f(x) < t < f(y)$. Since f is continuous on $[0,1]$, By Intermediate Value Theorem, $z \in [0,1]$ such that $f(z) = t \in \mathbb{R}\backslash\mathbb{Q}$, which is contradiction. Hence f is constant function.	
	iv.	If the function $f : (0, 1) \to \mathbb{R}$ is continuous and injective on $(0, 1)$, then using Intermediate value theorem prove that f is strictly monotonic on $(0, 1)$.	
	Ans	Suppose f is not strictly monotonic. Therefore $\exists\ x_1 < x_2 < x_3$ such that $x_1, x_2, x_3 \in (0,1)$. Case 1) i) $f(x_1) > f(x_3)$. Therefore $f(x_2) > f(x_1) > f(x_3)$. By Intermediate value theorem $\exists c \in (x_2, x_3)$ such that $(c) = f(x_1)$, contradicts the injectivity of f. ii) $f(x_1) < f(x_3)$. Therefore $f(x_1) < f(x_3) < f(x_2)$. By Intermediate value theorem $\exists c \in (x_1, x_2)$ such that $(c) = f(x_3)$, contradicts the injectivity of f. Case 2) Similar.	
Q.3	a)	Attempt any ONE question from the following:	
	i	Let $f: [a, b] \to \mathbb{R}$ be a bounded function. Prove that f is R-integrable on $[a, b]$ iff	

		for any $\in > 0$, there exists a partition $P_\in$ of $[a, b]$ such that $U(f, P_\in) - L(f, P_\in) < \in$.	
	Ans	Proof: ($\Rightarrow$) Given f is R integrable on $[a, b]$. T.P.T: $\forall \epsilon > 0, \exists$ apartition $P_\epsilon$of $[a, b]$ such that, $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$. Let, $\epsilon > 0$ be any real number, as f is R integrable $\therefore U(f) = L(f)$ Where, $U(f) = \inf\{U(f, P): P$ is any partion of $[a, b]\}$ And $L(f) = \sup\{L(f, P): P$ is any partion of $[a, b]\}$ $\therefore$ for given $\epsilon > 0, \exists$ a partition P_1 of $[a, b]$ such that, $U(f) \leq U(f, P_1) < U(f) + \frac{\epsilon}{2}$ (1) Also, for given $\epsilon > 0, \exists$ a partition P_2 of $[a, b]$ such that, $L(f) - \frac{\epsilon}{2} < L(f, P_2) \leq L(f)$ or $-L(f) \leq -L(f, P_2) < -L(f) + \frac{\epsilon}{2}$ (2) from (1) and (2) $$U(f) - L(f) \leq U(f, P_1) - L(f, P_2) < U(f) - L(f) + \epsilon$$ $\therefore 0 \leq U(f, P_1) - L(f, P_2) < \epsilon$ (3) ($\because U(f) = L(f)$)… 3 marks Now taking $P_\epsilon = P_1 \cup P_2$, $\therefore U(f, P_\epsilon) \leq U(f, P_1)$ & $L(f, P_\epsilon) \geq L(f, P_2)$($\because P_1 \subseteq P_\epsilon$ & $P_2 \subseteq P_\epsilon$) $\therefore U(f, P_\epsilon) - L(f, P_\epsilon) \leq U(f, P_1) - L(f, P_2) < \epsilon$ by (3) $\therefore U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$ …… 3 marks ($\Leftarrow$) Given: $\forall \epsilon > 0, \exists$ apartition $P_\epsilon$of $[a, b]$ such that, $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$. T.P.T: f is R integrable on $[a, b]$. i.e. T.P.T: $L(f) = U(f)$ $\because \forall \epsilon > 0, \exists$ apartition $P_\epsilon$of $[a, b]$ such that, $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$ We know that, $U(f) \leq U(f, P_\epsilon)$& $L(f) \geq L(f, P_\epsilon)$	

		$\therefore 0 \le U(f) - L(f) \le U(f,P_\epsilon) - L(f,P_\epsilon) < \epsilon (\because U(f) \ge L(f))$ $\therefore 0 \le U(f) - L(f) < \epsilon \quad \therefore U(f) = L(f)$	2M
	ii	Let $f: [a,b] \to \mathbb{R}$ be a bounded function if f is R-integrable on $[a,c]$ and $[c,b]$ then prove that f is R-integrable on $[a,b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$	
	Ans	Given f is integrable on $[a,c]$and $[c,b]$. for any $\in > 0$ $\exists$ a partition P_1 of $[a,c]$ such that $U(f,P_1) - L(f,P_1) < \in$ for any $\in > 0$ $\exists$ a partition P_2 of $[c,b]$ such that $U(f,P_2) - L(f,P_2) < \in$ since $U(f,P) = U(f,P_1) + U(f,P_2)$ *and* $L(f,P) = L(f,P_1) + L(f,P_2)$ Take $P = P_1 \cup P_2 \Longrightarrow U(f,P) \le U(f,P_1)$ *and* $L(f,P) \ge L(f,P_1)$ 2M $\therefore U(f,P) - L(f,P) \le U(f,P_1) - L(f,P_1) < \in$ $\therefore f$ is R $-$ integrable on $[a,b]$ for any $\in > 0$ $\exists$ partitions P_1 and P_2 of $[a,c]$ and $[c,b]$ then one can find a partition of $[a,b]$ such that $U(f,P) - L(f,P) < \in \Longrightarrow f$ is R $-$ integrable on $[a,b]$. 2M Claim : $\int_a^b f = \int_a^c f + \int_c^b f$ LHS = $\int_a^b f = \int_c^{\bar{b}} f \le U(f,P) < \in + L(f,P) < \in + L(f,P_1) + L(f,P_2) < \in + \int_a^c f + \int_c^b f$ therefore $\int_a^b f \le \int_a^c f + \int_c^b f$ 2M Similarly one can show $\int_a^b f \ge \int_a^c f + \int_c^b f$ 1M Hence $\int_a^b f = \int_a^c f + \int_c^b f$ 1M	
	b)	Attempt any TWO questions from the following:	
	i.	Let f be a bounded function on [a, b]. Let P and $P'$are two partitions of [a, b] with $P \subseteq P'$. Show that $L(f,P') \ge L(f,P)$	
	Ans	Let $P = \{x_0, x_1, \cdots\cdots\cdots, x_n\}$ be a partition of [a,b]. Given P is subset of P'	

Let $y_1, y_2, \cdots\cdots, y_m$ are extra points which are in P' but not in P.

Let $P_1 = P \cup \{y_1\}$.let $y_1 \in [x_{j-1}, x_j]$ .....2 marks

$$L(P,f) - L(P_1, f) = (m_j - m_j')(y_1 - x_{j-1}) + (m_j - m_j'')(x_j - y_1) \le 0$$

Where $m_j = inf\{f(x)/x \in [x_{j-1}, x_j]\}$

$m_j' = inf\{f(x)/x \in [x_{j-1}, y_1]\}$ and $m_j'' = inf\{f(x)/x \in [y_1, x_j]\}$

As $m_j' \ge m_j$ and $m_j'' \ge m_j$....... 3 marks

Therefore $L(P_1, f) \ge L(P, f)$

Similarly , $L(P_2, f) \ge L(P1, f)$

$L(P_m, f) \ge L(P_{m-1}, f) \ge L(P_{m-2}, f) \dots\dots.. \ge L(P, f)$

but $P_m = P'$

....1mark

$L(P', f) \ge L(P, f)$

ii

Show that the function $f : [0, 2] \to \mathbb{R}$ is Riemann integrable, where

$f(x) = -2 \quad for\ 0 \le x < 1$

$\quad\ = 4 \quad for\ 1 \le x \le 2$

Ans

For $\in > 0$ By Archimedian property, $\exists\, n \in \mathbb{N}$ such that $n > {}^{12}/_{\in} \Rightarrow {}^{12}/_{n} < \in$ 1M

Let $P = \{0, 1 - {}^{1}/_{n}, 1 + {}^{1}/_{n}, 2\}$ be a partition of [0, 2].

Let m_k and M_k be infimum and supremum of f respectively on sub-intervals of [0, 2]

$$U(P,f) - L(P,f) = \sum_{k=1}^{3} M_k(x_k - x_{k-1}) - \sum_{k=1}^{3} m_k(x_k - x_{k-1}) = \sum_{k=1}^{3}(M_k - m_k)(x_k - x_{k-1})$$ 2M

$$= (-2-(-2)\left(1-\frac{1}{n}\right) + (4-(-2))\left(\frac{2}{n}\right) + (4-4)\left(1-\frac{2}{n}\right)$$ 1M

$$= 0 + 6\times\frac{2}{n} + 0 = \frac{12}{n} < \in$$ 1M

$\therefore$ f is R-integrable on [0, 2]. 1M

8

	iii.	Show that any constant function is Riemann integrable.	
	Ans	As f is constant function $\Rightarrow f(x) = c \quad \forall x \in [a,b]$ $P = \{a = x_0, x_1, \cdots\cdots\cdots, x_n = b\}$ be partition of $[a,b] \Rightarrow m_k = M_k = c \ \forall x \in [a,b]$ $L(P,f) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n} c(x_k - x_{k-1}) = c(b-a) \Rightarrow L(f) = c(b-a)$ $U(P,f) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} c(x_k - x_{k-1}) = c(b-a) \Rightarrow U(f) = c(b-a)$ $\therefore L(f) = U(f) \Rightarrow f$ is R-integrable.	1M 2M 2M 1M
	iv.	Using Riemann Criterion, show that the function : $[0, 1] \longrightarrow \mathbb{R}$ defined by $f(x) = 2x - 1$ is Riemann integrable.	
	Ans	By Archimedian property, $\exists\ n \in \mathbb{N}$ such that $n > 2/\epsilon \Rightarrow 2/n < \epsilon$ Let $P = \{0, 1/n, 2/n, \cdots\cdots\cdots, 1\}$ be a partition of [0, 1]. $x_k - x_{k-1} = 1/n$ and $x_k = k/n$ Since f is increasing, hence $M_k = 2x_k - 1$ and $m_k = 2x_{k-1} - 1$ $U(P,f) - L(P,f) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) - \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n} 2(x_k - x_{k-1})(x_k - x_{k-1})$ $= \sum_{k=1}^{n} 2\frac{1}{n}\frac{1}{n} < 2\ \frac{1}{n^2}\ n < \frac{2}{n} < \epsilon$ $\therefore\ f$ is R-integrable.	1M 1M 1M 2M 1M
Q.4	a)	Attempt any ONE question from the following:	
	i	Let f(x) and g(x) be two R-integrable functions on $[a, \infty]$ such that $f(x) \leq g(x), \forall x \in [a, \infty]$ if i) $\int_a^\infty g(x)dx$ is convergent then show that $\int_a^\infty f(x)dx$ is convergent	

		ii) $\int_a^\infty f(x)dx$ is divergent then show that $\int_a^\infty g(x)dx$ is divergent	
	Ans	i) since $\int_a^\infty g(x)dx$ is convergent $\Rightarrow \exists c \in (a,\infty)$ such that $\lim_{\in\to\infty}\int_a^\in g(x)dx = c$ $f(x) \le g(x), \forall x \in [a,\infty] \Rightarrow \lim_{\in\to\infty}\int_a^\in f(x)dx \le \lim_{\in\to\infty}\int_a^\in g(x)dx = c$ Hence $\int_a^\infty f(x)dx$ is convergent ii)since $\int_a^\infty f(x)dx$ is divergent $\Rightarrow \lim_{\in\to\infty}\int_a^\in f(x)dx = \infty$ since $f(x) \le g(x), \forall x \in [a,\infty] \Rightarrow \lim_{\in\to\infty}\int_a^\in f(x)dx \le \lim_{\in\to\infty}\int_a^\in g(x)dx$ $\Rightarrow \infty \le \lim_{\in\to\infty}\int_a^\in g(x)dx \Rightarrow \int_a^\infty g(x)dx$ is divergent.	
	ii	Prove that (a) $\beta(m,n) = \beta(n,m)$ (b) $\beta(m,n+1) = \frac{n}{m+n}\beta(m,n)$.	
	Ans	Put $x = 1-t$ in $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx = \int_1^0 (1-t)^{m-1}t^{n-1}(-dt) = \int_0^1 t^{m-1}(1-t)^{n-1}dt = \beta(n,m)$. $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ $= \frac{1}{m}x^m(1-x)^n\|_0^1 + \frac{n}{m}\int_0^1 x^m(1-x)^{n-1}dx =$ $-\frac{n}{m}\int_0^1 x^m(1-x)^{n-1}(1-x-1)dx = -\frac{n}{m}[\beta(m,n+1) - \beta(m,n)]$ Hence the result.	
	b)	Attempt any TWO questions from the following:	
	i.	Let $f(x) = \cos(2x-\pi), x \in [3\pi/4, \pi]$.Find the value of C that satisfy mean value	

		theorem of integration of f.
	Ans	Mean value theorem for integral is $\int_a^b f(x)dx = f(c)(b-a)$ Putting the values we get, $\left[\frac{\sin(2x-\pi)}{2}\right]_{3\pi/4}^{\pi} = \frac{-\pi}{4}\cos 2c$ $\frac{-1}{2}\sin \pi/2 = \frac{-\pi}{4}\cos 2c \Rightarrow \cos 2c = \frac{2}{\pi} \Rightarrow c = \frac{1}{2}\cos^{-1}(2/\pi)$
	ii.	Solve improper integral $\int_1^3 \frac{1}{(x-1)^2}dx$.
	Ans	$\int_1^3 \frac{1}{(x-1)^2}dx = \lim_{x\to 1}\int_x^3 \frac{1}{(t-1)^2}dt = \lim_{x\to 1}\left[\frac{-1}{(t-1)}\right]_x^3 = \infty$
	iii.	Define Gamma function and hence prove that $\Gamma(n+1) = n!$.
	Ans	(2 marks) Definition. $\Gamma(n+1) = \int_0^\infty x^n e^{-x}dx = -e^{-x}x^n\vert_0^\infty + \int_0^\infty nx^{n-1}e^{-x}dx = n\Gamma(n)$. (4 marks)
	iv.	Sketch the region and change the order of integration of $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} f(x,y)dy\,dx$.
	Ans	Sketching 3 marks. $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} f(x,y)dy\,dx = \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)dx\,dy$. (3 marks)
Q.5		Attempt any FOUR questions from the following:
	a)	Show that *0.23* and *0.2299 ... 9 ...* represents the same rational number.
	Ans	Let $x_1 = 0.23$ $= \frac{2}{10} + \frac{3}{100} = \frac{20+3}{100} = \frac{23}{100}$.

		Let $x_2 = 0.22999 \dots 9 \dots$ $= \frac{2}{10} + \frac{2}{100} + \frac{9}{1000}\left[1 + \frac{1}{10} + \cdots + \frac{1}{10^n} + \cdots\right]$ $= \frac{22}{100} + \frac{9}{1000}\left[\lim_{n\to\infty}\left(1 + \frac{1}{10} + \cdots + \frac{1}{10^n}\right)\right]$ $= \frac{22}{100} + \frac{9}{1000}\left[\lim_{n\to\infty} 1\frac{1-\frac{1}{10^n}}{1-\frac{1}{10}}\right]$ $= \frac{22}{100} + \frac{9}{1000}\frac{10}{9}\left[\lim_{n\to\infty}\left(1 - \frac{1}{10^n}\right)\right]$ $= \frac{22}{100} + \frac{1}{100} = \frac{23}{100} = x_1.$	
	b)	Show that the equation $\cos x = x$ has a solution in $[0, \frac{\pi}{2}]$.	
	Ans	Let $f(x) = \cos x - x$. $f(0) = \cos 0 - 0 = 1 > 0$. $f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} - \frac{\pi}{2} = -\frac{\pi}{2} < 0$. Also f is continuous on $[0, \frac{\pi}{2}]$. Therefore by Intermediate value theorem, $\exists\, c \in (0, \frac{\pi}{2})$ such that $f(c) = 0$. That is, $\cos c - c = 0$. Therefore $\cos c = c$, $c \in (0, \frac{\pi}{2})$.	
	c)	Prove that the function $: [-1, 1] \longrightarrow \mathbb{R}$ defined by $f(x) = \lvert x \rvert$ is Riemann integrable and evaluate $\int_{-1}^{1} f(x)dx$.	
	Ans	$f(x) = \lvert x \rvert = x \quad if\ x \geq 0$ $= -x \quad if\ x < 0$	1M

		Divide the interval [−1, 1] into $2n$ equal parts each of length $\frac{1-(-1)}{2n}=\frac{1}{n}$ as $P=\{-1=x_0,\ x_1,\cdots\cdots, x_n=0, x_{n+1},\cdots\cdots, x_{2n}=1\} \Rightarrow x_k-x_{k-1}={}^{1}/_{n}$ $x_k=-1+{}^{k}/_{n}$, $for\ k=0\ to\ n$ and $x_k={}^{(k-n)}/_{n}$, $for\ k=n+1\ to\ 2n$ $\int_{-1}^{1} f(x)dx=\lim_{n\to\infty}\sum_{k=1}^{2n} f(x_k)(x_k-x_{k-1})=\lim_{n\to\infty}\left[\sum_{k=1}^{n}(-x_k)+\sum_{k=n+1}^{2n}x_k\right]\frac{1}{n}$ $=\lim_{n\to\infty}\frac{1}{n}\left[\sum_{k=1}^{n}\left(1-\frac{k}{n}\right)+\sum_{k=n+1}^{2n}\left(\frac{k-n}{n}\right)\right]=\lim_{n\to\infty}\frac{1}{n}\left[n-\frac{1}{n}\frac{n(n+1)}{2}+\frac{1}{n}\sum_{k=n+1}^{2n}(k-n)\right]$ $=\lim_{n\to\infty}\frac{1}{n}\left[n-\frac{n+1}{n}+\frac{1}{n}(1+2+\cdots\cdots+n)\right]=1$	1M 1M 2M
	d)	Using properties of Riemann integration check whether the function $f:[0,1]\longrightarrow\mathbb{R}$ defined by $f(x)=0 \quad if\ x=0$ $=\frac{1}{n} \quad if\ \frac{1}{n+1}<x\le\frac{1}{n}$ is Riemann integrable.	
	Ans	$f(x)$ can be defined as, $f(x)=1 \quad if\ {}^{1}/_{2}<x\le 1$ $={}^{1}/_{2} \quad if\ {}^{1}/_{3}<x\le{}^{1}/_{2}$ $={}^{1}/_{3} \quad if\ {}^{1}/_{4}<x\le{}^{1}/_{3}$ ⋮ ⋮ $={}^{1}/_{n-1} \quad if\ {}^{1}/_{n}<x\le{}^{1}/_{n-1}$ $=0 \quad if\ x=0$ $\because \lvert f(x)\rvert\le 1\ \forall x\in[0,1]\Rightarrow f$ is bounded. Further f is monotonic increasing $\Rightarrow f$ is R-integrable.	2M 1M 2M
	e)		
	Ans		

	f)	Find the volume of the solid bounded by $z = 1 + x^2 + y^2$, the planes $x = 3, y = 2$ and the co ordinate planes.	
	Ans	$\int_0^2 \int_0^3 (1 + x^2 + y^2)\, dxdy = 32$	
		*********************************	