Answer Key

Exam: S.Y.B.ScSem IV (Choice Based) April 2019 **Subject: Mathematics Paper III**

Exam date: 02/05/2019 **Q.P.Code: 65872**

(3 Hours) [Total Marks: 100]

Note: (i) All questions are compulsory.

(ii) Figures to the right indicate marks for respective parts.

Q.1 Choose correct alternative in each of the following (20)

$i.$ The order and degree of differential equation $\sqrt{\left(\frac{d^3y}{dx^3}\right)^2 + x + 4\left(\frac{dy}{dx}\right)^3} = 3x^3$ is

(a) 1 and 3 (b) 2 and 1

(c) 2 and 2 (d) 3 and 2

Ans (d)

$ii.$ Which of the following equations is exact?

(a) $xdx + ydy + \frac{xdy - ydx}{x^2 + y^2} = 0$ (b) $x^2y^2dx + (y^2 + x)dy = 0$

(c) $(3x + y)^2dx + x^2dy = 0$ (d) None of these.

Ans **(a)**

$iii.$ Which of the following substitution converts non-homogeneous differential equation $(x + y)dy + (5x - 3y + 8)dx = 0$ into a homogeneous differential equation?

(a) $x = X - 1, y = Y + 1$ (b) $x = X + 1, y = Y + 1$

(c) $x = X + 1, y = Y - 1$ (d) $x = X - 1, y = Y - 1$

Ans **(a)**

$iv.$ Which of the following functions are not linearly independent on $[1, 2]$?

(a) $y = \sin x$ and $y = \cos x$ (b) $y = x^2$ and $y = x^3$

(c) $y = \log x^2$ and $y = \log x^3$ (d) None of these

Ans **(c)**

$v.$ One of the solutions of the differential equation $y'' - y = 0$ is

(a) $y = k$, where $k \in \mathbb{R}\backslash\{0\}$ (b) $y = \sin x$

(c) $y = e^x$ (d) None of these

Ans **(c)**

$vi.$ The Wronskian of $y_1 = \sin x$ and $y_2 = \cos x$ is

(a) 1 (b) 0

(c) 2 (d) None of these

Ans **(a)**

$vii.$ The roots of the auxiliary equation of the differential equation $y'' - 4y' + 3y = 0$ are

(a) 1 and 4 (b) 3 and 4

(c) 1 and 3 (d) None of these

Ans **(c)**

$viii.$ The auxiliary equation of the following linear system of homogeneous differential equations

$\begin{cases} \frac{dx}{dt} = a_1x + b_1y \\ \frac{dy}{dt} = a_2x + b_2y \end{cases}$ is

(a) $m^2 - (a_1 + b_2)m + a_1b_2 - a_2b_1 = 0$

(b) $m^2 - (a_2 + b_1)m + a_1b_2 - a_2b_1 = 0$

(c) $m^2 + (a_1 + b_2)m + a_1b_2 - a_2b_1 = 0$

(d) $m^2 - (a_2 + b_1)m + a_2b_1 - a_1b_2 = 0$

Ans **(a)**

$ix.$ One of the solutions of the homogeneous linear system of differential equations $\begin{cases} \frac{dx}{dt} = y - x \\ \frac{dy}{dt} = 3x + y \end{cases}$ is

(a) $\begin{cases} x = 3e^{2t} \\ y = e^{2t} \end{cases}$

(b) $\begin{cases} x = e^{2t} \\ y = 3e^{2t} \end{cases}$

(c) $\begin{cases} x = e^{t} \\ y = 3e^{t} \end{cases}$

(d) None of these

Ans **(b)**

$x.$ The auxiliary equation of the linear system of homogeneous differential equations $\begin{cases} \frac{dx}{dt} = 3x + 2y \\ \frac{dy}{dt} = -5x + y \end{cases}$ has

(a) Real and distinct roots

(b) Roots which are complex conjugates

(c) Real and repeated roots

(d) Does not have any roots

Ans **(b)**

Q2. Attempt any **ONE** question from the following: (08)

$a)$ i. Show that the general solution of the linear differential equation $\frac{dy}{dx} + P(x)y = Q(x)$ is $y = e^{-\int P\,dx}\left[\int Q \cdot e^{\int P\,dx}dx + c\right]$. Hence find solution to $(x^2+1)\frac{dy}{dx} + 2xy = 4x^2$

Ans

$$\frac{d}{dx}\left(ye^{\int Pdx}\right) = y\frac{d}{dx}\left(e^{\int Pdx}\right) + e^{\int Pdx}\frac{dy}{dx}$$

$$= ye^{\int Pdx}\frac{d}{dx}\left(\int Pdx\right) + e^{\int Pdx}\frac{dy}{dx}$$

$$= ye^{\int Pdx}P + e^{\int Pdx}\frac{dy}{dx}$$

$$= e^{\int Pdx}\left(\frac{dy}{dx} + Py\right)$$

$$= e^{\int Pdx}Q(x) \text{ as } \frac{dy}{dx} + P(x)y = Q(x)$$

$$\Rightarrow ye^{\int Pdx} = \int Q \cdot e^{\int P\,dx}dx + c$$

$$\Rightarrow y = e^{\int Pdx}\left[\int Q \cdot e^{\int P\,dx}dx + c\right]$$

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$$(x^2+1)\frac{dy}{dx}+2xy=4x^2 \Rightarrow \frac{dy}{dx}+\frac{2xy}{(1+x^2)}=\frac{4x^2}{(1+x^2)}$$

$$\int P\,dx=\log(1+x^2) \Rightarrow e^{\int Pdx}=(1+x^2)$$

The solution is $y(1+x^2)=\int\frac{4x^2}{(1+x^2)}(1+x^2)dx+C$

$$\Rightarrow y(1+x^2)=\frac{4x^3}{3}+C$$ 3

ii. Show that the necessary and sufficient condition for the differential equation $M(x,y)dx+N(x,y)dy=0$ to be exact is $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ where the functions M,N possess continuous first order partial derivatives in domain under consideration

Ans **The condition is necessary**:

Suppose the equation $Mdx+Ndy=0$ is exact, then there exists function $f(x,y)$ such that $Mdx+Ndy=df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy \Rightarrow M=\frac{\partial f}{\partial x}, N=\frac{\partial f}{\partial y}$

$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial^2 f}{\partial y\partial x}, \frac{\partial N}{\partial x}=\frac{\partial^2 f}{\partial x\partial y}$ But first order partial derivatives are continuous 4

$$\Rightarrow \frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y} \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

The condition is sufficient:

Suppose$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Let $v=\int Mdx$ treating y as constant, then $\frac{\partial v}{\partial x}=M$

$$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial^2 v}{\partial y\partial x}=\frac{\partial^2 v}{\partial x\partial y}=\frac{\partial N}{\partial x} \Rightarrow \frac{\partial}{\partial x}\left(N-\frac{\partial v}{\partial y}\right)=0$$

$$\Rightarrow N-\frac{\partial v}{\partial y}=\text{a function of y say } g(y) \Rightarrow N=\frac{\partial v}{\partial y}+g(y)$$

$$Mdx+Ndy=\frac{\partial v}{\partial x}dx+\left(\frac{\partial v}{\partial y}+g(y)\right)dy=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+g(y)dy$$

$=dv+g(y)dy=d[v+g(y)] \Rightarrow Mdx+Ndy$ is exact

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Q.2 Attempt any **TWO** questions from the following: (12)

$b)$ i. Solve the differential equation$(x^2y-2xy^2)dx-(x^3-3x^2y)dy=0$

Ans Checking the DE is homogeneous 1

$$\frac{\partial M}{\partial y}=x^2-4xy \text{ and } \frac{\partial M}{\partial y}=-3x^2+6xy \text{ and hence isNon exact}$$ 1

$$IF=\frac{1}{Mx+Ny}=\frac{1}{x^2y^2}$$ 1

Multiplying throughout by $\frac{1}{x^2y^2}$

$$\left(\frac{1}{y}-\frac{2}{x}\right)dx+\left(-\frac{x}{y^2}+\frac{3}{y}\right)dy=0 \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=-\frac{1}{y^2}$$ 1

$$\int M dx = \int \left(\frac{1}{y} - \frac{2}{x}\right) dx = \frac{x}{y} - 2\log x$$

$$\int N dy (\text{terms not containing x}) = \int \frac{3}{y} dy = 3\log y$$

The general solution is $y^3 e^{\frac{x}{y}} = cx^2$ 2

Equivalently using substitution $y = vx$ the differential equation can be solved

ii. Using Bernoulli's method find the general solution to $x\frac{dy}{dx} + y + 2x^6y^4 = 0$.

Ans

$$x\frac{dy}{dx} + y = -2x^6y^4 \Rightarrow \frac{1}{y^4}\frac{dy}{dx} + \frac{1}{x}\frac{1}{y^3} = -2x^5$$

Substituting $\frac{1}{y^3} = v, -\frac{3}{y^4}\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow -\frac{1}{3}\frac{dv}{dx} + \frac{v}{x} = -2x^5$ 2

$$\int P\, dx = \log\left(\frac{1}{x^3}\right) \Rightarrow e^{\int Pdx} = \frac{1}{x^3}$$ 1

$$\Rightarrow v\frac{1}{x^3} = \int \frac{1}{x^3} 6x^5 + c \Rightarrow \frac{1}{y^3} = 2x^2 + cx^3$$ 3

iii. Find the family of orthogonal trajectories of the family of parabolas $y^2 = 4ax$.

Ans

$y^2 = 4ax \Rightarrow \frac{y^2}{x} = 4a$

Differentiating both sides with respect to x

$$\frac{2y}{x}\frac{dy}{dx} - \frac{y^2}{x^2} = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{2x}$$ 2

The slope of orthogonal trajectories$=-\frac{1}{\frac{dy}{dx}}$ 1

The differential equation of orthogonal trajectories is obtained by solving

$$-\frac{1}{\frac{dy}{dx}} = \frac{y}{2x} \Rightarrow \frac{dy}{dx} = -\frac{2x}{y} \Rightarrow \frac{y^2}{2} + x^2 = c$$ 3

iv. Solve $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$

Ans

$$\frac{\partial M}{\partial y} = 2xy + 6x^2y^2 \ and\ \frac{\partial N}{\partial x} = 2xy - 3x^2y^2 - \text{Non Exact}$$ 1

The equation is of the form $yf_1(xy)dx + xf_2(xy)dy = 0$ and the integrating factor is $\frac{1}{Mx - Ny}$ if $Mx - Ny \neq 0$ 1

$$IF = \frac{1}{3x^3y^3}$$ 1

Multiplying throughout by $\frac{1}{3x^3y^3}$

$$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0 \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -\frac{3}{x^2y^2}$$ 1

$$\int M dx = \int \left(\frac{1}{3x^2y} + \frac{2}{3x}\right) dx = \frac{-1}{3xy} - \frac{2}{3}\log x$$

$$\int N dy (\text{terms not containing x}) = \int -\frac{3}{y} dy = -3\log y$$ 2

The solution is $cy = x^2 e^{-\frac{1}{xy}}$

Q3. Attempt any **ONE** question from the following: (08)

$a)$ i. Let y_1 be a non-zero solution of the homogeneous differential equation $y'' + P(x)y' + Q(x) = 0$. Assuming $y_2 = vy_1$, show that another solution (which is linearly independent of y_1) of the same differential equation can be found out, where $v = \int \frac{1}{y^2} e^{-\int P dx} dx$.

If $y_1 = x$ is one solution of $x^2 y'' + xy' - y = 0$, then find another solution of the same, which is linearly independent of y_1.

Ans Consider the homogeneous differential equation $y'' + P(x)y' + Q(x)y = 0$. Assume that y_1 is a solution of the same. We want to obtain another solution y_2, which is linearly independent of y_1. So, assume $y_2 = vy_1$.

Hence, $y_2' = vy_1' + v' y_1$ and $y_2'' = vy_1'' + 2v' y_1' + v'' y_1$ 2

Substitute y_2, y_2', y_2'' in the differential equation and rearrange.

Thus, $v(y'' + Py' + Q) + v'' y_1 + v'(2y_1' + Py_1) = 0$.

That is, $v'' y_1 + v'(2y_1' + Py_1) = 0$.

So, $\frac{v''}{v'} = -2\frac{y_1'}{y_1} - P$ 4

This gives, $v = \int \frac{1}{y_1^2} e^{-\int P dx} dx$

Given, $y_1 = x$ is one solution of $x^2 y'' + xy' - y = 0$. This can be written as $y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0$.

To get another solution, use $v = \int \frac{1}{y^2} e^{-\int P dx} dx$ 2

This gives, $v = \int \frac{1}{x^2} e^{-\int \frac{1}{x} dx} dx = \int \frac{1}{x^2} e^{-\log x} dx = \int \frac{1}{x^3} dx = \frac{-1}{2x^2}$

Hence, $y_2 = -\frac{1}{2x^2} x = -\frac{1}{2x}$

ii. Explain the method of variation of parameters to find the general solution of a non-homogeneous differential equation, when the general solution of the corresponding homogeneous differential equation is known.

Ans The method of variation of parameters is used to find a particular solution of a non-homogeneous differential equation, when the general solution of the corresponding homogeneous differential equation is known.

So assume that $y = c_1y_1 + c_2y_2$ is the general solution of the homogeneous differential equation $y_2^{''} + Py_1 + Qy = 0$ and suppose we want to find out a particular solution of $y_2^{''} + Py_1 + Qy = R$. 2

Let $y = v_1y_1 + v_2y_2$ be a particular solution of $y_2^{''} + Py_1 + Qy = R$.

Consider $y^{'} = (v_1y_1^{'} + v_2y_2^{'}) + (v_1^{'}y_1 + v_2^{'}y_2)$

Assume $(v_1^{'}y_1 + v_2^{'}y_2) = 0$

Hence, $y^{''} = v_1y_1^{''} + v_1^{'}y_1^{'} + v_2y_2^{''} + v_2^{'}y_2^{'}$

Substituting and rearranging we get $v_1^{'}y_1^{'} + v_2^{'}y_2^{'} = R$ 4

Consider $v_1^{'}y_1 + v_2^{'}y_2 = 0$ and $v_1^{'}y_1^{'} + v_2^{'}y_2^{'} = R$

Solving these simultaneously we get $v_1^{'} = -\frac{y_2R}{W}$ and $v_2^{'} = \frac{y_1R}{W}$

Note that $W \neq 0$, since y_1 and y_2 are linearly independent.

2

Finally, we have $v_1 = \int -\frac{y_2R}{W}dx$ and $v_2 = \int \frac{y_1R}{W}dx$

Thus, $y = y_1\int -\frac{y_2R}{W}dx + y_2\int \frac{y_1R}{W}dx$ is a particular solution.

Q3. Attempt any **TWO** questions from the following: (12)

$b)$ i. Show that any linear combination of two solutions of the homogeneous equation $y^{''} + P(x)y^{'} + Q(x)y = 0$ is also a solution of the same.

Hence or otherwise, show that $e\sin x + \pi\cos x$ is a solution of $y^{''} + y = 0$.

Ans Let y_1 and y_2 be solutions of $y^{''} + P(x)y^{'} + Q(x)y = 0$.

Therefore,

$$y_1^{''} + P(x)y_1^{'} + Q(x)y_1 = 0$$

And 2

$$y_2^{''} + P(x)y_2^{'} + Q(x)y_2 = 0$$

Consider, $(c_1y_1 + c_2y_2)^{''} + P(x)(c_1y_1 + c_2y_2)^{'} + Q(x)(c_1y_1 + c_2y_2)$

$= (c_1y_1)^{''} + P(x)(c_1y_1)^{'} + Q(x)(c_1y_1) +$

$$(c_2y_2)^{''} + P(x)(c_2y_2)^{'} + Q(x)(c_2y_2)$$

$= c_1[(y_1)^{''} + P(x)(y_1)^{'} + Q(x)(y_1)] +$

$$c_2[(y_2)^{''} + P(x)(y_2)^{'} + Q(x)(y_2)]$$

$=c_1[0]+c_2[0]=0$ 2

Thus, any linear combination of two solutions of $y'' + P(x)y' + Q(x)y = 0$ is also a solution of the same.

Consider, $y'' + y = 0$. 2

Note that if $y = \sin x$ then $y'' = -\sin x = -y$.
Similarly, if $y = \cos x$, then, $y'' = -\cos x = -y$.
Thus, both $\sin x$ and $\cos x$ are solutions of $y'' + y = 0$. Therefore, any linear combination of these two is also a solution of the same differential equation.
Thus, $e\sin x + \pi\cos x$ is a solution of $y'' + y = 0$.

ii. Verify that $y_1 = x^2$ is one solution of $x^2y'' + xy' - 4y = 0$, and find the general solution of the same.

Ans Let $y = x^2$. Then $y' = 2x$ and $y'' = 2$. 2
Substitute these in the LHS of $x^2y'' + xy' - 4y = 0$.
Consider the LHS
$= x^2y'' + xy' - 4y$
$=x^2(2) + x(2x) - 4(x^2)$
$=2x^2 + 2x^2 - 4x^2 = 0$ 2

Hence, $y_1 = x^2$ is one solution of $x^2y'' + xy' - 4y = 0$.

To find the second solution, use $y_2 = vy_1$, where $v = \int \frac{1}{y^2} e^{-\int Pdx}\,dx$. Write, the given equation as $y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 0$

Hence, $v = \int \frac{1}{x^4} e^{-\int \frac{1}{x}dx}\,dx = \int \frac{1}{x^4} e^{-\log x}\,dx = \int \frac{1}{x^5}\,dx = \frac{1}{-4x^4}$ 2
Thus, $y_2 = \frac{1}{-4x^4}x^2 = \frac{1}{-4x^2}$

Therefore, $y = c_1x^2 + \frac{c_2}{-4x^2}$ is the general solution of the given differential equation.

iii. By using the method of undetermined coefficients, find the general solution of the differential equation, $y'' - y' - 6y = 20e^{-2x}$.

Ans The corresponding homogeneous equation is $y'' - y' - 6y = 0$
The corresponding auxiliary equation is $m^2 - m - 6 = 0$
This has -2 and 3 as roots. 2
The roots are real and distinct and therefore, $e^{-2x}$and e^{3x} both are solutions of the homogeneous equation. Moreover, these solutions are linearly independent.

Therefore, the general solution of the homogeneous equation is
$y = c_1 e^{-2x} + c_2 e^{3x}$.

Since, the RHS of the given equation is an exponential function, $20e^{-2x}$, we should start with Ae^{ax} and A should be found out.. Note that, the power is -2 (that is, $a = -2$), which is a root of the auxiliary equation. 2

Therefore, $A = \frac{1}{2a+p}$, where $a = -2$ and $p = -1$
Thus, $A = \frac{1}{-4-1} = -\frac{1}{5}$
Hence, $y = -\frac{1}{5}e^{-2x}$ is a particular solution of the given differential equation. 2
Finally, the general solution of the given differential equation is

$$y = c_1 e^{-2x} + c_2 e^{3x} - \frac{1}{5}e^{-2x}$$

iv. By using the method of variation of parameters, find the general solution of the differential equation: $y'' + 4y = \tan 2x$.

Ans The corresponding homogeneous equation is $y'' + 4y = 0$
The corresponding auxiliary equation is $m^2 + 4 = 0$
This has $+2i$ and $-2i$ as roots.
The roots are non-real and distinct and therefore, we can write two solutions of the homogeneous differential equation as $e^{ax} \cos bx$ and $e^{ax} \sin bx$, where $a = 0$ and $b = 2$. 2
Therefore, the general solution of the homogeneous differential equation is
$c_1 \cos 2x + c_2 \sin 2x$

We have to use the method of variation of parameters to find the general solution of the given equation.
The Wronskian of $\cos 2x$ and $\sin 2x$ is $\begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$.

A particular solution is given by

$$y = y_1 \int -\frac{y_2 R(x)}{W} dx + y_2 \int \frac{y_1 R(x)}{W} dx$$

$$y = \cos 2x \int -\frac{\sin 2x \tan 2x}{2} dx + \sin 2x \int \frac{\cos 2x \tan 2x}{2} dx$$ 3

$$y = -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

1

Thus, the general solution of the given differential equation is

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

Q4. Attempt any **ONE** question from the following: (08)

$a)$ i. Define Wronskian $W(t)$ of the two solutions

$\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ of the homogeneous system $\begin{cases} \frac{dx}{dt} = a_1(t)x + b_1(t)y \\ \frac{dy}{dt} = a_2(t)x + b_2(t)y \end{cases}$

where $a_1(t), a_2(t), b_1(t), b_2(t)$ are continuous functions on $[a, b]$. Show that their Wronskian is either identically zero or nowhere zero on $[a, b]$.

Ans Definition of Wronskian…….. **(2M)**

$w(t) = x_1(t)y_2(t) - x_2(t)y_1(t).$ ……**(1M)**

$\therefore \frac{dw}{dt} = \frac{dx_1}{dt}y_2 + \frac{dy_2}{dt}x_2 - \frac{dx_2}{dt}y_1 - x_2\frac{dy_1}{dt}$……. **(1M)**

On substituting values of the derivatives and solving, we get

$\frac{dw}{dt} = (a_1(t) + b_2(t))w(t)$ which is in the variable separable form…… **(2M)**

On solving we get,

$w(t) = ce^{\int a_1(t)+b_2(t)dt}$……. **(2M)**

ii. Find the general solution of system $\begin{cases} \frac{dx}{dt} = a_1x + b_1y \\ \frac{dy}{dt} = a_2x + b_2y \end{cases}$ where $a_1, a_2, b_1 and\ b_2$ are constants when the roots of auxiliary equation are real and distinct.

Ans Auxiliary equation for the system of equation $\begin{cases} \frac{dx}{dt} = a_1(t)x + b_1(t)y \\ \frac{dy}{dt} = a_2(t)x + b_2(t)y \end{cases}$ is

$\begin{vmatrix} a_1 - m & b_1 \\ a_2 & b_2 - m \end{vmatrix} = 0$ …….. **(1M)**

Suppose m_1 and m_2 are distinct and real root of the above equation.

Let $\begin{cases} x_1(t) = A_1e^{m_1t} \\ y_1(t) = B_1e^{m_1t} \end{cases}$ and $\begin{cases} x_2(t) = A_2e^{m_2(t)} \\ y_2(t) = B_2e^{m_2t} \end{cases}$.

i)To show that Since $(x_1(t), y_1(t))and\ (x_2(t), y_2(t))$ are solutions of the above system, A_1, B_1, A_2, B_2 are non trivial …….(2M)

ii) To show that $A_1B_2 = A_2B_1$ by showing the following is not possible.

- Both $B_1 \neq 0$ and $B_2 \neq 0$
- Atleast one of B_1, B_2 equals to zero. …….**(5M)**

Q4. Attempt any **TWO** questions from the following: (12)

$b)$ i. Show that $x = e^{4t}, y = e^{4t}$ and $x = e^{-2t}, y = -e^{-2t}$ are linearly independent solutions to the homogeneous system $\begin{matrix} \frac{dx}{dt} = x + 3y \\ \frac{dy}{dt} = 3x + y \end{matrix}$. Hence find the general solution.

Ans

- To show that $x = e^{4t}, y = e^{4t}$ and $x = e^{-2t}, y = -e^{-2t}$ are solutions of $\frac{dx}{dt} = x + 3y$, $\frac{dy}{dt} = 3x + y$ ……….**(3M)**
- $w(t) = -2e^{2t} \neq 0$**…..(2M)**
- The general solution is linear combination of $x = e^{4t}, y = e^{4t}$ and $x = e^{-2t}, y = -e^{-2t}$**…..(1M)**

ii. Find the general solution of the following system $\frac{dx}{dt} = 4x - 3y$, $\frac{dy}{dt} = 8x - 6y$.

Ans

- Auxiliary equation $\begin{vmatrix} 4-m & -3 \\ 8 & -6-m \end{vmatrix} = m^2 + 2m = 0$
- Roots are $m = 0, -2$. **….(1M)**
- For m=0, the solution $(x_1, y_1) = (A_1 e^{0t}, B_1 e^{0t}) = (A_1, B_1)$= (3, 4)**….(2M)**
- For m= -2, the solution $(x_2, y_2) = (A_2 e^{-2t}, B_2 e^{-2t}) = (e^{-2t}, 2e^{-2t})$**…..(2M)**
- The general solution is linear combination of $(x_1, y_1) and\ (x_2, y_2)$**…(1M)**

iii. Find the general solution of the system $\begin{cases} \frac{dx}{dt} = x - 4y \\ \frac{dy}{dt} = x + y \end{cases}$

Ans

- Auxiliary equation $\begin{vmatrix} 1-m & -4 \\ 1 & 1-m \end{vmatrix} = m^2 - 2m + 5 = 0$
- Roots are $m = 1 \pm 2i$. **….(2M)**
- Take $m = 1 + 2i$, then we get the system
- $\begin{bmatrix} -2i & -4 \\ 1 & -2i \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

  $\therefore A = 2iB$.

  Let $B = 1$ then $A = 2i$.

  Then $A_1 = 0, A_2 = 2, B_1 = 1\ and\ B_2 = 0$**…..(2M)**

  The general solution is linear combination of $(x_1, y_1) and\ (x_2, y_2)$**…(2M)**

iv. Let $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ be two solutions of the homogeneous system $\begin{cases} \frac{dx}{dt} = a_1(t)x + b_1(t)y \\ \frac{dy}{dt} = a_2(t)x + b_2(t)y \end{cases}$ where $a_1(t), a_2(t), b_1(t), b_2(t)$are continuous functions on $[a, b]$. Then show that $\begin{cases} x(t) = c_1 x_1(t) + c_2 x_2(t) \\ y(t) = c_1 y_1(t) + c_2 y_2(t) \end{cases}$ is also a solution of the above system of equation for $t \in [a,b]$.

Ans Since (x_1, y_1) is a solution of the above system, we get

$\frac{dx_1}{dt} = a_1(t)x_1 + b_1(t)y_1$

$\frac{dy_1}{dt} = a_2(t)x_1 + b_2(t)y_1$

and (x_2, y_2) is a solution of the above system,

$\frac{dx_2}{dt} = a_1(t)x_2 + b_1(t)y_2$

$\frac{dy_2}{dt} = a_2(t)x_2 + b_2(t)y_2$ **…….(1M)**

Let $x(t) = c_1x_1(t) + c_2x_2(t)$

$y(t) = c_1y_1(t) + c_2y_2(t)$**….. (1M)**

Then show that $\frac{dx}{dt} = a_1(t)x + b_1(t)y$, $\frac{dy}{dt} = a_2(t)x + b_2(t)y$ **…….(4M)**

Q5. Attempt any **FOUR** questions from the following: (20)

$a)$ Solve $(5xy + 4y^2 + 1)dx + (x^2 + 2xy)dy = 0$

Ans $M(x,y) = 5xy + 4y^2 + 1$, $N(x,y) = x^2 + 2xy$,

$\frac{\partial M}{\partial y} = 5x + 8y, \frac{\partial N}{\partial x} = 2x + 2y$. Hence DE is Non Exact 1

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x^2 + 2xy}(3x + 6y) = \frac{3}{x}$$

$IF = e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$ 1

Multiplying throughout by x^3

$$x^3(5xy + 4y^2 + 1)dx + x^3(x^2 + 2xy)dy = 0 \Longrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 5x^4 + 8yx^3$$

$$\int M dx = \int (5x^4y + 4x^3y^2 + x^3)\,dx = x^5y + x^4y^2 + \frac{x^4}{4}$$

$$\int N dy(\text{terms not containing x}) = \int 0\,dy = 0$$

The solution is $x^5y + x^4y^2 + \frac{x^4}{4} = c$ 3

$b)$ Find the solution to the differential equation $\frac{dx}{dy} + 2x\tan y = \sin y$ given that $x = 0$ when $y = \frac{\pi}{3}$.

Ans $\int P dy = \int 2\tan(y)dy = \int \frac{2\sin(y)}{\cos(y)}dy = -2\log(\cos(y)) \Longrightarrow e^{\int P dy} = \frac{1}{\cos^2(y)}$ 1

The solution is

$$\frac{x}{\cos^2(y)} = \int \frac{1}{\cos^2(y)}\sin(y)dy + c = \frac{1}{\cos(y)} + c \Longrightarrow x = \cos(y) + c\cos^2(y)$$ 2

When $y = \frac{\pi}{3}, x = 0 \Longrightarrow 0 = \frac{1}{2} + c\frac{1}{4} \Longrightarrow c = -2 \Longrightarrow x = \cos(y) - 2\cos^2(y)$ 2

$c)$ Show that the functions $x^3$and $x^2|x|$ are linearly independent on $[-1, 1]$.

Ans Two functions $y_1$and y_2 are said to be linearly dependent on a given interval $[a, b]$, provided there exists a constant k, such that either $y_2 = ky_1$ or $y_1 = ky_2$ on $[a, b]$. If no such k exists, then $y_1$and y_2 are linearly independent. (2)

Thus, if there exists a constant k such that $x^3 = kx^2|x|$ or $x^2|x| = kx^3$, for each $x \in [-1,1]$, then the functions would be linearly dependent. We need to show that there doesn't exist any such k. (1)

Note that, $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

Thus, if $x \geq 0$, then $x^3 = x^2|x|$ and if $x < 0$, then $x^3 = -x^2|x|$. (2)

Thus, no such k exists and therefore, the functions are linearly independent on $[-1,1]$

$d)$ Solve: $y'' - 11y' + 28y = 0$.

Ans The given equation is $y'' - 11y' + 28y = 0$.
The corresponding auxiliary equation is $m^2 - 11m + 28 = 0$ 2
That is, $(m - 4)(m - 7) = 0$
Therefore, the roots of the auxiliary equation are 4 and 7.
The roots are real and distinct. 2
Therefore, e^{4x} and e^{7x} both are solutions of the given differential equation. From the theory of "homogeneous differential equations with constant coefficients", the solutions are linearly independent and hence, the general solution of the given equation $c_1e^{4x} + c_2e^{7x}$ 1

$e)$ Define the system of homogeneous linear differential equations of order 1. State the condition for two solutions (x_1, y_1) and (x_2, y_2) to be linearly independent. Also write the general solution.

Ans Definition of system of homogeneous 1st order linear differential equation **……(1M).**
Condition for linear independence by using definition or Wronskian**…..(2M)**
General solution given two linearly independent solutions
(x_1, y_1) *and* (x_2, y_2). **…..(2M)**

$f)$ Show that $x = -\frac{1}{4}e^{-3t}, y = e^{-3t}$ and $x = e^{2t}, y = e^{2t}$ are solutions of the homogeneous system

$$\begin{cases} \frac{dx}{dt} = x + y \\ \frac{dy}{dt} = 4x - 2y \end{cases}$$

Ans Verifying each ordered pair is the solution **…..(2M).**
If entire answer is right give full marks.

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