Examination : SYBSc Semester IV **Subject : Mathematics**

Exam Date : 26th April, 2019. **Q.P.Code : 66040**

(3 Hours) [Total Marks: 100]

Note: (i) All questions are compulsory.

(ii)Figures to the right indicate marks for respective parts.

Q.1	Choose correct alternative in each of the following (20)			
$i.$	A group G is said to be Abelian if			
	(a)	$\forall x, y \in G, xy = yx$	(b)	$\forall x, y \in G, xy \neq yx$
	(c)	For some $x, y \in G, xy = yx$	(d)	None of the above
	Ans	(a)		
$ii.$	The set $\mathbb{Q}$ forms a group under the binary operation			
	(a)	'$+$'	(b)	'$-$'
	(c)	'.'	(d)	None of the above
	Ans	(a)		
$iii.$	Let D_n denote the dihedral group. Then $\|D_n\| =$			
	(a)	n	(b)	$2n$
	(c)	$n!$	(d)	None the above
	Ans	(b)		
$iv.$	Let H be a subgroup of a group G. then			
	(a)	$\forall x, y \in H, xy^{-1} \in H$	(b)	$\forall x, y \in H, xy^{-1} \in G$
	(c)	$\forall x, y \in H, xy^{-1} \notin H$	(d)	None of the above
	Ans	(a)		
$v.$	The order of 0 in the cyclic group of integers $\mathbb{Z}$ under addition is			
	(a)	0	(b)	Infinite order
	(c)	1	(d)	None of the above
	Ans	(c)		

$vi.$	Let G=$\mathbb{C}^*$ be the multiplicative group of non-zero complex numbers and $i \in$ G then o(i) is			
	(a)	1	(b)	2
	(c)	3	(d)	4
	Ans	(d)		
$vii.$	Which of the following is false?			
	(a)	Any infinite cyclic group has exactly two generators	(b)	A subgroup of a cyclic group need not be cyclic.
	(c)	There is only one subgroup of order d where d is a divisor of n for a cyclic group of order n	(d)	The multiplicative group of n^{th} roots of unity is cyclic.
	Ans	(b)		
$viii.$	Let $G = (\mathbb{C}^*, \cdot)$ and $H = \{z \in \mathbb{C}^* : \|z\| = 1\}$ then the cosets of H in G are			
	(a)	$\{z \in \mathbb{C}^* : \|z\| = k\}\ \forall k \in \mathbb{R}^+$	(b)	$\{z \in \mathbb{C}^* : \|z \cdot w\| = 1\}\ \forall w \in \mathbb{C}^*$
	(c)	$\{z \in \mathbb{C}^* : \|z + w\| = 1\}\ \forall w \in \mathbb{C}^*$	(d)	None of these
	Ans	(a)		
$ix.$	The number of group homomorphism from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{30}$ is			
	(a)	6	(b)	7
	(c)	8	(d)	None of these
	Ans	(a) 6		
$x.$	Which of the following groups are isomorphic			
	(a)	$(\mathbb{Z}_{4,}+)$ and V_4 (Klien-4 group)	(b)	$(\mathbb{Z}_{4,}+)$ and μ_4 (4^{th} root of unity)
	(c)	V_4 and μ_4	(d)	None of these
	Ans	(b) $(\mathbb{Z}_{4,}+)$ and μ_4 (4^{th} root of unity)		

Q2.	Attempt any **ONE** question from the following:	(08)
$a)$	i.	Show that $\mathbb{Z}_n$, the set of residue class of modulo n form a group under the binary operation '+'.
	Ans	First we prove that addition in $\mathbb{Z}_n$ is well defined Suppose $\bar{a} = \bar{c}$ and $\bar{b} = \bar{d}$ $\Rightarrow a \equiv c \pmod{n}$ and $b \equiv d \pmod{n}$ $\Rightarrow a + b \equiv c + d \pmod{n}$ $\Rightarrow \overline{a+b} = \overline{c+d}$ $\Rightarrow +$ is well defined in $\mathbb{Z}_n$. Consider any $\bar{a}, \bar{b}, \in \mathbb{Z}_n$ $\bar{a} + \bar{b} = \overline{a+b} \in \mathbb{Z}_n$ after modulo n Also, $\bar{a} + \left(\bar{b} + \bar{c}\right) = \bar{a} + \left(\overline{b+c}\right) = \overline{a+b+c}$ $= \left(\overline{a+b}\right) + \bar{c} = \left(\bar{a} + \bar{b}\right) + \bar{c}$ by properties of $\mathbb{Z}_n$ $\Rightarrow +$ is associative Now, $\exists\, \bar{0} \in \mathbb{Z}_n$ such that for any $\bar{a} \in \mathbb{Z}_n$ $\bar{a} + \bar{0} = \overline{a+0} = \bar{a} = \bar{0} + \bar{a}$ $\Rightarrow \bar{0}$ is additive identity in $\mathbb{Z}_n$ Also, $\forall\, \bar{a} \in \mathbb{Z}_n, \exists \bar{b} = \overline{n-a} \in \mathbb{Z}_n$ such that $\bar{a} + \bar{b} = \bar{a} + \left(\overline{n-a}\right)$ $= \overline{a+n-a}$ $= \bar{n} = \bar{0} = \bar{b} + \bar{a}$ $\Rightarrow \bar{b}$ is additive inverse of $\bar{a}$ $\therefore (\mathbb{Z}_n, +)$ is a group.
	ii.	Prove that if for $a \in G, O(a) = m$ then $O\left(a^k\right) = \dfrac{m}{g.c.d.(m,k)}$.

	Ans	Consider $(a^k)^{m_1} = (a^{k_1 d})^{m_1} = (a^{k_1})^{dm_1}$ $= (a^{k_1})^m$ $= (a^m)^{k_1}$ $= e^{k_1}$ (as $o(a) = m$) $= e$ $\therefore$ $(a^k)^{m_1} = e$ and $o(a^k) = n$ $\Rightarrow n \mid m_1$ ... (*) Now, $o(a^k) = n \Rightarrow (a^k)^n = e$ $\Rightarrow a^{kn} = e$ But $o(a) = m$ $\Rightarrow m \mid kn$ $\Rightarrow m_1 d \mid k_1 dn$ by (1) $\Rightarrow m_1 \mid n$ as $(m_1, k_1) = 1$ (**) Thus, by (*), (**) $m_1 = n$ $n = m_1 = \frac{m}{d}$ (by (1)) $= \frac{m}{(m,k)}$ Thus, $n = \frac{m}{(m,k)}$ (i.e.) $o(a^k) = \frac{m}{(m,k)}$ (as $o(a^k) = n$)	
Q.2	Attempt any **TWO** questions from the following:		(12)
$b)$	i.	Let G be a group. Prove that p) Identity element of G is unique. q) The inverse of every element in G is unique.	
	Ans	p) Let there are two identities e *and* e' $xe = ex = e \ \ \forall x \in G$ $xe' = e'x = e' \ \ \forall x \in G$ $\therefore e'e = ee' = e$ and $ee' = e'e = e'$ ∴e=e' q) Let $x \in G$ has two inverses y *and* y' $y = y + e = y + (x + y') = (y + x) + y' = e + y' = y'$	3 3
	ii.	If $a^2 = e$ for every a in a group G then show that G is abelian group.	
	Ans	$(ab)(ba) = ab^2a = aea = a^2 = e$ But $(ab)(ab) = e$ By uniqueness of inverse $ab = ba$	2 2 2
	iii.	Let $G = GL_2(\mathbb{R})$. Let $H = \{A \in G \mid \det A = 2^n$, for some $n \in \mathbb{Z}\}$.Prove that H is a subgroup of G.	
	Ans	Let $A, B \in H$ $\therefore$ $\lvert A \rvert = 2^n$ and $\lvert B \rvert = 2^m$, for some $n, m \in \mathbb{Z}$	2

		$\|AB^{-1}\| = \|A\|\|B\|^{-1} = \frac{2^n}{2^m} = 2^{n-m}$ $\therefore AB^{-1} \in H$ by 1-step test .	2 2
	iv.	Let α=(1 2 5)(6 13 5)and $\beta = (1\ 3\ 4)(2\ 6\ 5)(2\ 3\ 4)$ .Write α and β as a product of disjoint cycles. Further, verify the following. p) $O(\alpha) = O(\alpha^{-1})$ q) $O(\alpha\beta) = O(\beta\alpha)$ r) $O(\alpha\beta\alpha^{-1}) = O(\beta)$	
	Ans	$\alpha = (1\ 3\ )(2\ 5\ 6\ ), \beta = (1\ 3\ )(2\ 4\ 6\ \ 5)$ $\alpha^{-1} = (1\ 3)(2\ 6\ 5),\ \ O(\alpha) = O(\alpha^{-1}) = 6$ $\alpha\beta = (2\ 4\ ), \beta\alpha = (4\ 6), O(\alpha\beta) = O(\beta\alpha) = 2$ $\alpha\beta\alpha^{-1} = (1\ 3)(2\ 6\ 5\ 4), \quad O(\alpha\beta\alpha^{-1}) = O(\beta) = 8$	 2 2 2
Q3.	Attempt any **ONE** question from the following:		(08)
a)	i.	Prove that subgroup of a cyclic group is cyclic	
	Ans	Let G =(a) be a cyclic group and H be a subgroup of G. If H ={e} then H is cyclic. On the other hand if H≠ {e} choose x ∈H, $x \neq$ e.∴ $x =$ a^n for some n≠0 Since x *and* x^{-1} are in H hence some positive power of a belongs to H. Choose the least positive power say m. $(a^m) \subseteq H$ ------(1) Now if b∈H then b=a^k and we can write k=qm+r ; 0≤r<m ∴ a^r∈H hence r=0 .i.e. H⊆(a^m) ---------(2) Hence H=(a^m) --------------- [from (1) & (2)]	 2 2 2 2
	ii.	Prove that if G be a finite cyclic group of order n then G has ϕ(n) *generators*.	
	Ans	Let G = (a) be a finite cyclic group of order n and let b ∈ G, where b=a^m and suppose b is a generator of G. ∴ a=b^k∴ a= $(a^m)^k$. ∴ $a^{mk-1} = e$. But n is o(G) hence n divides $mk - 1$. ∴ $mk - 1 = nt$. i.e. $mk - nt = 1$ ∴ m and n are relatively prime. Conversely let m and n be relatively prime. ∴ There exists x and y such that $mx + ny = 1$ $\therefore a^1 = a^{mx+ny} = a^{mx}.a^{ny} = a^{mx}$	 2 2 1 1

		$\therefore a \in (a^m) \therefore G \subseteq (a^m)$, .i.e. G=$(a^m)$ Hence there are ϕ(n) generators for G.	1 1
Q3.	Attempt any **TWO** questions from the following: (12)		
b)	i.	Show that the group of positive rational numbers under multiplication is not cyclic.	
	Ans	Suppose a & b are relatively prime positive integers and that $(\frac{a}{b}) = \mathbb{Q}^+$ Then there is some positive integer k so that $\left(\frac{a}{b}\right)^k$=2 k$\neq 0,1,-1$ If $k > 1$ then a^k=2b^k so that 2 divides a. Also as $k > 1$ hence 4 divides a and as a consequence 2 divides b Which contradicts that a & b are relatively prime. A similar contradiction occurs if $k < -1$ Hence$(\frac{a}{b}) = \mathbb{Q}^+$ *is not possible .i.e.* $\mathbb{Q}^+$is not cyclic.	2 1 1 1 1 1
	ii.	List all the elements of $\mathbb{Z}_{40}$ that have order 10.	
	Ans	4 3x4 7x4 9x4	6
	iii.	Show that an infinite cyclic group has exactly two generators.	
	Ans	Let G = (a) be an infinite cyclic group and let b∈G be another generator of G, so that G = (b). Since b ∈ G , b=a^m and a=b^n $\therefore\ a = a^{mn}$.i.e.$a^{mn-1} = a^0 = e.$ Since all powers of a are distinct in an infinite cyclic group, we have $mn - 1 = 0. \therefore\ m = \pm 1$, showing that b= a^{-1} is the only other generator of G.	 3 3
	iv.	Show that if G is a group with more than p – 1 elements of order p, where p is a prime then G cannot be cyclic.	
	Ans	There are two cases for G : 1. G is infinite cyclic 2. G is finite cyclic. G cannot be infinite cyclic, since an infinite cyclic group has no two elements of prime order [all powers of the generator element	 2

		in an infinite cyclic group are distinct]. Now if G is finite cyclic then G can have only one subgroup for each divisor of its order. A subgroup of order p has exactly p – 1 elements of order p. Another element of order p will give rise to another subgroup of order p. This results in more than one subgroup of order p which is not possible for finite cyclic groups. Combining both the cases G is not cyclic.	1 2 1
Q4.	Attempt any **ONE** question from the following:		(08)
$a)$	i.	Let H be a subgroup of a group G and $a, b \in G$ then show that (p) $a \in aH$ (q) $aH = bH$ or $aH \cap bH = \emptyset$ (r) $\|aH\| = \|bH\|$	
	Ans	(p) Since $e \in H \Longrightarrow ae \in aH \Longrightarrow a \in aH$ (q) case (i) If $aH \cap bH = \emptyset$ then done Case (ii) If $aH \cap bH \neq \emptyset$ Let $x \in aH \cap bH$ then for $h_1, h_2 \in H$ $\Longrightarrow x = ah_1$ and $x = bh_2 \Longrightarrow a = xh_1^{-1}$ Let $y \in aH \Longrightarrow y = ah = xh_1^{-1}h = bh_2h_1^{-1}h \in bH \Longrightarrow aH \subseteq bH$ Similarly one can show $bH \subseteq aH \Longrightarrow aH = bH$ (r) Define a map $f: aH \to bH$ by $f(ah) = bh$, $h \in H$ Show f is bijective map that gives $\|aH\| = \|bH\|$	1M 4M 3M
	ii.	Let $f: G \longrightarrow G'$ is onto group homomorphism. Prove that (p) If H is subgroup of G then $f(H) = \{f(h)/h \in H\}$ is subgroup of G'. (q) If H' is subgroup of G' then $f^{-1}(H') = \{a \in G / f(a) \in H'\}$ is subgroup of G and $ker f \subseteq f^{-1}(H')$.	
	Ans	(p) Since $H \subseteq G$ and $e \in H \Longrightarrow f(H) \subseteq G'$ and $f(e) = e' \in f(H)$ <u>Claim</u> : $xy^{-1} \in f(H)$ where $x, y \in f(H)$ For $a, b \in H$ such that $x = f(a), y = f(b)$ Now $xy^{-1} = f(a)(f(b))^{-1} = f(ab^{-1}) \in f(H)$ as $ab^{-1} \in H$	 3M

		$\therefore f(H)$ is subgroup of G'. (q) Since $H' \subseteq G'$ and $f(e) = e' \in H' \Longrightarrow f^{-1}(H') \subseteq G$ and $e \in f^{-1}(H') \Longrightarrow f^{-1}(H')$ is non−empty subset of G. Claim : $ab^{-1} \in f^{-1}(H')$ where $a, b \in f^{-1}(H')$ As $a, b \in f^{-1}(H')$ gives $f(a) = x \in H', f(b) = y \in H' \Longrightarrow xy^{-1} \in H'$ Now $f(ab^{-1}) = f(a)(f(b))^{-1} = xy^{-1} \in H' \Longrightarrow ab^{-1} \in f^{-1}(H')$ $f^{-1}(H')$ is subgroup of G. Let $a \in kerf \Longrightarrow f(a) = e' \in H' \Longrightarrow a \in f^{-1}(H') \Longrightarrow kerf \subseteq f^{-1}(H')$	4M 1M
Q4.	Attempt any **TWO** questions from the following:		(12)
b)	i.	Let H and K be two subgroups of G. If $o(H) = p$, a prime integer, then show that either $H \cap K = \{e\}$ or $H \subseteq K$.	
	Ans	Since H and K be two subgroups of $G \Longrightarrow H \cap K$ is also subgroup of G. Further $H \cap K \subseteq H \Longrightarrow H \cap K$ is also subgroup of H. By Lagrange's theorem , $o(H \cap K) \mid o(H) \Longrightarrow o(H \cap K) \mid p$ $o(H \cap K) = 1 \; or \; p$ If $o(H \cap K) = 1 \Longrightarrow H \cap K = \{e\}$ If $o(H \cap K) = p = o(H)$, also $H \cap K \subseteq H$ gives $H \cap K = H$ Hence $H \subseteq K$.	6M
	ii.	Let G be a group of order pq where p and q are distinct prime integers. Show that every subgroup $H \neq G$ is a cyclic subgroup of G.	
	Ans	Since H is subgroups of G By Lagrange's theorem , $o(H) \mid o(G) \Longrightarrow o(H) \mid pq$ As p and q are distinct primes and $H \neq G \Longrightarrow o(H) = 1$ or p	3M

		or q If $o(H) = 1 \Longrightarrow H = \{e\} \Longrightarrow H$ is cyclic. If $o(H) = p$ and p is prime $\Longrightarrow H$ is cyclic. If $o(H) = q$ and q is prime $\Longrightarrow H$ is cyclic.	1M 1M 1M
	iii.	Let G be an abelian group of order n and $(m,n) = 1,\ m \in \mathbb{Z}$ then show that $f: G \longrightarrow G$ defined by $f(x) = x^m, \forall x \in G$ is an automorphism.	
	Ans	Since $(m,n) = 1 \Longrightarrow mp + nq = 1$ for $p, q \in \mathbb{Z}$ Also $o(G) = n \Longrightarrow x^n = e$, $y^n = e$ for any $x, y \in G$ Therefore $x = x^1 = x^{mp+nq} = x^{mp}x^{nq} = x^{mp}$, similarly $y = y^{mp}$ As G is abelian, $f(xy) = (xy)^m = x^m\, y^m = f(x)f(y)$ $\Longrightarrow f$ is homomorphism Now $f(x) = f(y) \Longrightarrow x^m = y^m \Longrightarrow x^{mp} = y^{mp} \Longrightarrow x = y$ $\Longrightarrow f$ is injective. Let $y \in G \Longrightarrow y^p \in G \Longrightarrow f(y^p) = y^{pm} = y \Longrightarrow f$ is surjective. Therefore $f: G \longrightarrow G$ is an automorphism.	6M
	iv.	Show that the map $f: GL_2(\mathbb{R}) \longrightarrow GL_2(\mathbb{R})$ defined by $f(A) = (A^t)^{-1}$ is an group automorphism.	
	Ans	Consider $f(AB) = [(AB)^t]^{-1} = (B^tA^t)^{-1} = (A^t)^{-1}(B^t)^{-1} = f(A)\, f(B)$ Let $f(A) = f(B) \Longrightarrow (A^t)^{-1} = (B^t)^{-1} \Longrightarrow A^t = B^t \Longrightarrow A = B$ Let $B \in GL_2(\mathbb{R})$ then $(B^{-1})^t \in GL_2(\mathbb{R})$ such that $f((B^{-1})^t) = B$ Therefore f is an automorphism.	6M
Q5.	Attempt any **FOUR** questions from the following: (20)		
$a)$	Define Center $Z(G)$ of a group G. Show that $Z(G)$ is a subgroup of G.		
Ans	$Z(G) = \{x \in G \mid xg = gx\ \forall g \in G\}$		1

Let $x, y \in Z(G)$ and $g \in G$

$xyg = xgy = gxy$

$\therefore xy \in Z(G)$ — 2

Let $x \in Z(G)$ and $g \in G$

$xg = gx$

$g = x^{-1}gx$

$gx^{-1} = x^{-1}g$

$\therefore x^{-1} \in Z(G)$ — 2

By 2-step test

$Z(G)$ is a subgroup of G.

b) Construct composition table of $U(10)$ under multiplication modulo 10. Also find the order of each of its elements.

Ans

	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

3

$O(1) = 1, O(3) = 4, O(7) = 4, O(9) = 2$ — 2

c) How many elements does the group U(10) have? List them. Also find order of each element. Is U(10) cyclic?

Ans

U(10)= {1,3,7,9} — 1

o(1)=1, o(3)=o(7)=4, o(9)=2. — 3

As there is an element of order 4 which is the order of U(10), hence U(10) is cyclic. — 1

d) Show that a cyclic group is abelian.

Ans

Let G=(a) .

Let $x, y \in$ G be any

$\therefore x = a^r$ and $y = a^s$ — 2

$\therefore x * y = a^r * a^s = a^{r+s} = a^{s+r}$ [Using + is commutative in integers] — 2

$= a^s * a^r = y * x$ — 1

$\therefore$ By definition G is abelian.

e) Give an example of a group G and a subgroup H of G such that $aH = bH$ but

	$Ha \neq Hb$ for some $a, b \in G$.	
Ans	$G = S_3$, $H = \{e, (12)\}$ then for $a = (13)$ and $b = (123)$ $aH = bH = \{(13), (123)\}$ but $Ha = \{(13), (132)\}$ and $Hb = \{(23), (123)\}$ $\Longrightarrow Ha \neq Hb$	5M
f)	Show that the map $f: (\mathbb{C}, +) \longrightarrow (\mathbb{C}, +)$ defined by $f(a + bi) = a - bi$ is an group isomorphism.	
Ans	Let $x = a + bi$ then map can be defined as $f(x) = \bar{x}$ Now $f(x + y) = \overline{x + y} = \bar{x} + \bar{y} = f(x) + f(y) \Longrightarrow f$ is homomorphism Suppose $f(x) = f(y) \Longrightarrow \bar{x} = \bar{y} \Longrightarrow \bar{\bar{x}} = \bar{\bar{y}} \Longrightarrow x = y \Longrightarrow f$ is injective Let $y \in (\mathbb{C}, +) \Longrightarrow \bar{y} \in (\mathbb{C}, +) \Longrightarrow f(\bar{y}) = \bar{\bar{y}} = y \Longrightarrow f$ is surjective. Therefore $f: (\mathbb{C}, +) \longrightarrow (\mathbb{C}, +)$ is an group isomorphism.	5M

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