52722

(3 Hours)

[Total Marks : 100]

Q.1	Choose correct alternative in each of the following:		
	i.	(b)	
	ii.	(d)	
	iii.	(a)	
	iv	(d)	
	v	(a)	
	vi	(c)	
	vii	(c)	
	viii.	(b)	
	ix	(c)	
	x	(b)	
Q.2	a)	Attempt any ONE question from the following:	
	i		
	Ans		
	ii		
	Ans		
	b)	Attempt any TWO questions from the following:	
	i.		

Set 2

Q-1 i) b. 1, 2, c, 1, 2, c, - - -

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2 ii) d. $\frac{1}{7}$ iii) a. $\frac{2}{3}$ iv) a. closed.

Q-2.a. T^m: Every bdd sequence in $\mathbb{R}$ has a cgt subsequence.

P: Let (x_n) be a bdd seq in $\mathbb{R}$

$x : \mathbb{N} \to \mathbb{R}$, $x(n) = x_n \in \mathbb{R}$

Range: Image of seq. $= x(\mathbb{N}) = \{x_n / n \in \mathbb{N}\} \subseteq \mathbb{R}$

i) If $\{x_n / n \in \mathbb{N}\}$ is a finite set then $\exists$ a real number α s.t $x_n = \alpha$ for infinitely many $n \in \mathbb{N}$

Say for all $k \in S \subseteq \mathbb{N}$, $x_k = \alpha$, $k \in S$

Using well ordering principle $S \subseteq \mathbb{N}$ has least element n_1 & we can construct $n_1 < n_2 < \cdots < n_k < \cdots$ increasing seq of natural no.s in S s.t $x_{n_k} = \alpha$

Thus (x_{n_k}) is a subseq. of (x_n) which is constant $\therefore$ cgt subseq.

$\therefore$ Given bdd seq (x_n) has cgt. subseq. (x_{n_k})

$\therefore$ B.W Th^m is proved in this case.

case 2): Range is not finite set.

i.e $x(\mathbb{N}) = \{x_n / n \in \mathbb{N}\}$ is a infinite set

As (x_n) is a bdd seq in $\mathbb{R}$, $\exists M > 0$ s.t $|x_n| \leq M$, $\forall n \in \mathbb{N}$

$-M \leq x_n \leq M$, $\forall n \in \mathbb{N}$

$\therefore x_n \in [-M, M]$, $\forall n \in \mathbb{N}$

Let $J_0 = [-M, M]$

Bisect J_0 into two subintervals $[-M, 0]$ & $[0, M]$

Q.2.a. ii

Let $f:[a,b] \to \mathbb{R}$ be continuous. &
Suppos f has opposite signs at end pts.
Say $f(a) < 0$ & $f(b) > 0$, then $\exists\, c \in (a,b)$
s.t $f(c) = 0$

Pf: Let $J_0 = [a,b]$, $f:[a,b] \to \mathbb{R}$ cont. on $[a,b]$

using, $f(a) < 0$ & $f(b) > 0$

Let $c_1 = \frac{a+b}{2}$, midpt of J_0

$\therefore f(c_1)$ is '0' or +ve or -ve.

4 If $f(c_1) = 0$, nothing to prove.

5 If not, then bisect J_0

$[a, c_1]$ & $[c_1, b]$ are two subintervals of J_0. choose one of these so that f has opposite signs at end pt.

for eg. if $f(c_1) < 0$ we choose $[c_1, b]$

if $f(c_1) > 0$ we choose $[a, c_1]$

The chosen intervals is called

$J_1 = [a_1, b_1]$

$\therefore$ note f has opp. signs at end pts of J_1.

Now bisect J_1 choose one of subintervals $J_2 = [a_2, b_2]$ so that f takes value with opp. signs at end pts & proceed sim.

Thus we obtain seq $\{J_n\}$ of closed subintervals s.t

1) $J_n = [a_n, b_n]$, then $f(a_n) < 0$ & $f(b_n) > 0$

II) $J_n \supseteq J_{n+1}$, $\forall n \in \mathbb{N}$

III) $\ell(J_n) = (b-a)/2^n = b_n - a_n$

By nested interval th^m $\bigcap_{n \in \mathbb{N}} J_n \neq \phi$

& $\ell(J_n) \to 0$ as $n \to \infty$

$\therefore \bigcap_{n \in \mathbb{N}} J_n = \{c\}$ = singlton set

Claim: $f(c) = 0$

$c \in J_n$ $\forall n \in \mathbb{N}$.

$\Rightarrow c \in [a_n, b_n]$, $\forall n \in \mathbb{N}$

$|c - a_n| \le \ell(J_n) = \frac{b-a}{2^n} \quad \forall\, n \in \mathbb{N}$ (5)

$\Rightarrow$ ~~$c \in [a_n, b_n]\ \forall$~~

$$0 \le \lim_{n\to\infty} |c - a_n| \le 0$$

$\therefore \lim_{n\to\infty} a_n = c$ by sandwich th^m.

$\lim |c - b_n| \le \frac{b-a}{2^n}, \ \forall\, n \in \mathbb{N}$

$\therefore \lim_{n\to\infty} b_n = c$

As $c \in [a, b] \Rightarrow f$ is cont at c.

$\therefore$ By cont of f at c;

$(a_n) \to c \Rightarrow f(a_n) \to f(c)$

But $f(a_n) \le 0$, $f(b_n) \ge 0$; $\forall\, n \in \mathbb{N}$

$\Rightarrow f(c) \le 0$, & $f(c) \ge 0$

$\therefore f(c) = 0$

$\therefore \exists\, c \in (a, b)$ s.t $f(c) = 0$

In particular if $f : [a, b] \to \mathbb{R}$ cont. on $[a, b]$

& if $r \in \mathbb{R}$ s.t $f(a) < r < f(b)$ then $\exists\, x_0 \in (a, b)$

s.t $f(x_0) = r$

Define $g(x) = f(x) - r$, g is cont. as f is

Apply previous prop. as $g(a) < 0$ & $g(b) > 0$

$\exists\, c \in (a, b)$ s.t $g(c) = 0$

$\Rightarrow f(c) = r$

Q2 bii)

$F_n \subseteq \mathbb{R}$, $n \in \mathbb{N}$

$F_n = [n, \infty)$ is a closed set.

$[F_n = (-\infty, n)]$ opt. compli of ↑.

$F_n \supseteq F_{n+1}$, $\forall n \in \mathbb{N}$

$l(F_n) \not\to 0$ as $n \to \infty$

Nested interval th^m can't be applicable

Let $x \in \mathbb{R}$

s.t $x \in \cap F_n$

$x \in [n, \infty)$, $\forall n \in \mathbb{N}$.

$\implies n \leq x$, $\forall n \in \mathbb{N}$

$\implies x$ is upper of $\mathbb{N}$

A contradiction $\forall x \in \mathbb{R}$

s.t $x \in \cap F_n$

$\cancel{F_1 \subseteq F_2 \subseteq F_3 \subseteq} \cdots$

$\cancel{F_n \not\subseteq F_{n+1}}$ $\cap F_n = \phi$ in this case $n \in \mathbb{N}$

$\cancel{\bigcap_{n=1}^{\infty} F_n = (-1, 1)}$

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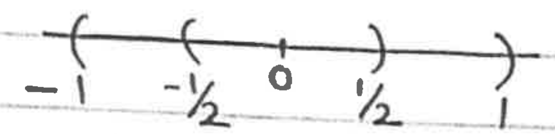
$F_n = \left[-\frac{1}{n}, \frac{1}{n} \right]$

F_n's are closed interval

$F_1 = [-1, 1] = [-1, 1]$, $F_2 = [-1/2, 1/2]$

$\therefore$ F_n's are closed & $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$



$\therefore$ F_n's are nested.

$l(F_n) \to 0$, $2/n \to (1/n - (-1/n)) = 1/n + 1/n$

$l(F_n) \to 0$ as $n \to \infty$

Nested interval th^m can be satisfied.

$F_n \supseteq F_{n+1}$

$\bigcap_{n=1}^{\infty} F_n$ is a singlton set

Note $0 \in F_n$ $\forall\, n \in \mathbb{N}$;

$\bigcap_{n \in \mathbb{N}}^{\infty} F_n = \{0\}$

Set 4 - 40

Q 2. (b) (iii) Given for any $x \in I$, $\exists\ y \in I$ s.t

$|f(y)| \le \frac{1}{2} |f(x_1)|$ --- *

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We construct a sequence as follows:

Choose any $x_1 \in I$

By * $\exists\ y = x_2 \in I$ s.t. $|f(x_2)| \le \frac{1}{2} |f(x_1)|$

Similarly for $x = x_2$, $\exists\ y = x_3$ s.t

$|f(x_3)| \le \frac{1}{2} |f(x_2)|$

$\therefore |f(x_3)| \le \frac{1}{2^2} |f(x_1)|$

Proceeding Similarly we ~~get~~ get x_n

s.t $|f(x_n)| \le \frac{1}{2} |f(x_{n-1})| \le \cdots \le \frac{1}{2^n} |f(x_1)|$

$\therefore \lim_{n\to\infty} |f(x_n)| \le \lim_{n\to\infty} \frac{1}{2^n} |f(x_1)|$

$\le |f(x_1)| \times 0 = 0$

$\because |f(x_n)| \ge 0$,

$\therefore \lim_{n\to\infty} |f(x_n)| = 0$ ——— (*)$_1$

$\because x_n \in I, \forall n$ $\therefore (x_n)$ is a bounded sequence.

$\therefore$ By Bolzano-Weierstrass th^m for sequences

$\exists$ a convergent subsequence say $(x_{n_k}) \to c$

$\because f$ is conti. on I, by sequential conti.

$(f(x_{n_k})) \to f(c)$.

By (*)$_1$, $\lim_{k\to\infty} f(x_{n_k}) = 0$ $\therefore f(c) = 0$

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Q2 b(iv) $x_n = (2 + (-1)^n)$, $\forall\ n \in \mathbb{N}$

Find two cgt subseq.

$x_1 = (2 + (-1)^1) = 1$

$x_2 = (2 + (-1)^2) = 3$

$x_3 = (2 + (-1)^3) = 1$

$x_4 = (2 + (-1)^4) = 3$

$\vdots$

$\therefore\ x_{2n-1} = 1 \quad \forall\ n \in \mathbb{N}$ which is cgt subsequence of (x_n)

$x_{2n} = 3 \quad \forall\ n \in \mathbb{N}$ ~~which~~ is also cgt subseq. of (x_n)

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[illegible]

[illegible] $0.1\dot{9}$

$$= \frac{1}{10} + \frac{9}{10}\left\{\frac{1}{10} + \frac{1}{10^2} + \cdots + \frac{1}{10^n} + \cdots\right\}$$

$$= \frac{1}{10} + \frac{9}{10} \lim_{n\to\infty}\left(\frac{1}{10} + \frac{1}{10^2} + \cdots + \frac{1}{10^n}\right)$$

$$= \frac{1}{10} + \frac{9}{10} \lim_{n\to\infty} \frac{1}{10}\,\frac{\left(1 - 1/10^n\right)}{\left(1 - 1/10\right)}$$

$$= \frac{1}{10} + \frac{9}{100} \times \frac{10}{9} \lim_{n\to\infty}\left(1 - \frac{1}{10^n}\right)$$

$$= \frac{1}{10} + \frac{1}{10} = \frac{2}{10} = 0.2$$

Q-5 b. $f, g : [a,b] \longrightarrow \mathbb{R}$ be continuous fu^n on $[a,b]$

$f(a) > g(a)$ & $f(b) < g(b)$

Define $h : [a,b] \to \mathbb{R}$ as

$h(x) = f(x) - g(x)$

f, g ~~can't~~ · cont. on $[a,b]$

$\therefore$ by alg. of cont. fu^n.

h is cont. on $[a,b]$

$h(a) = f(a) - g(a) > 0$

$h(b) = f(b) - g(b) < 0$

$\therefore h$ has opp. signs at end pts.

$\therefore$ By IMVP of cont. fu^n $\exists$ $c \in (a,b)$

s.t $h(c) = 0$

$\therefore f(c) - g(c) = 0$

$\therefore f(c) = g(c)$, $c \in (a,b)$

Q.3	a)	Attempt any ONE question from the following:	
	i	Let $f: [a, b] \to \mathbb{R}$ be a bounded function. Prove that f is R-integrable on $[a, b]$ iff for any $\in > 0$, there exists a partition $P_\in$ of $[a, b]$ such that $U(f, P_\in) - L(f, P_\in) < \in$.	

Ans

Proof: ($\Rightarrow$) Given f is R integrable on $[a, b]$.

T.P.T: $\forall\, \epsilon > 0, \exists$ apartition $P_\epsilon$of $[a, b]$ such that, $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$.

Let, $\epsilon > 0$ be any real number, as f is R integrable,

$$\therefore U(f) = L(f)$$

Where, $U(f) = \inf\{U(f, P): P$ is any partion of $[a, b]\}$

And $L(f) = \sup\{L(f, P): P$ is any partion of $[a, b]\}$

$\therefore$ for given $\epsilon > 0, \exists$ a partition P_1 of $[a, b]$ such that,

$U(f) \leq U(f, P_1) < U(f) + \frac{\epsilon}{2}$ (1)

Also, for given $\epsilon > 0, \exists$ a partition P_2 of $[a, b]$ such that,

$L(f) - \frac{\epsilon}{2} < L(f, P_2) \leq L(f)$ or $-L(f) \leq -L(f, P_2) < -L(f) + \frac{\epsilon}{2}$ (2)

from (1) and (2)

$$U(f) - L(f) \leq U(f, P_1) - L(f, P_2) < U(f) - L(f) + \epsilon$$

$\therefore 0 \leq U(f, P_1) - L(f, P_2) < \epsilon$ (3) $(\because U(f) = L(f))$ ... 3 marks

Now taking $P_\epsilon = P_1 \cup P_2$,

$$\therefore U(f, P_\epsilon) \leq U(f, P_1) \;\&\; L(f, P_\epsilon) \geq L(f, P_2) (\because P_1 \subseteq P_\epsilon \;\&\; P_2 \subseteq P_\epsilon)$$

$\therefore U(f, P_\epsilon) - L(f, P_\epsilon) \leq U(f, P_1) - L(f, P_2) < \epsilon$ by (3)

$\therefore U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$ ...... 3 marks

($\Leftarrow$) Given: $\forall\, \epsilon > 0, \exists$ apartition $P_\epsilon$of $[a, b]$ such that, $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$.

T.P.T: f is R integrable on $[a, b]$.

i.e. T.P.T: $L(f) = U(f)$

$\because\; \forall\, \epsilon > 0, \exists$ apartition P_ϵ of $[a, b]$ such that, $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$

We know that, $U(f) \leq U(f, P_\epsilon) \;\&\; L(f) \geq L(f, P_\epsilon)$

$$\therefore 0 \leq U(f) - L(f) \leq U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon (\because U(f) \geq L(f))$$

$\therefore 0 \leq U(f) - L(f) < \epsilon$ ...2 marks

$\therefore U(f) = L(f)$

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ii If f is an R-integrable function on $[a, b]$ then prove that $\left|\int_a^b f\right| \leq \int_a^b |f|$

Ans Proof:

Given: f is R integrable on $[a, b]$.

Claim: $|f|$ is R integrable on $[a, b]$.

Let $P = \{x_0 = a, x_1, x_2, \ldots \ldots x_n = b\}$ be any partition of $[a, b]$

Let $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ and $M_i' = \sup\{|f|(x) : x \in [x_{i-1}, x_i]\}$

$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$ and $m_i' = \inf\{|f|(x) : x \in [x_{i-1}, x_i]\}$, $i = 1,2, \ldots . n$.

To show that,

$$M_i' - m_i' \leq M_i - m_i, \quad i = 1,2, \ldots . n.$$

Let, $x, y \in [x_{i-1}, x_i]$

$$m_i \leq f(x) \leq M_i$$
$$m_i \leq f(y) \leq M_i$$
$$\therefore m_i - M_i \leq f(x) - f(y) \leq M_i - m_i$$
$$\therefore\ -m_i - M_i \leq f(x) - f(y) \leq M_i - m_i$$

Consider,

$$\begin{aligned} |f(x)| &= |f(x) - f(y) + f(y)| \\ &\leq |f(x) - f(y)| + |f(y)| \\ &\leq M_i - m_i + |f(y)| \end{aligned}$$

Here, $y \in [x_{i-1}, x_i]$

$\therefore |f(x)| \leq M_i - m_i + |f(y)|, \ \forall x \in [x_{i-1}, x_i]$

$\therefore M_i - m_i + |f(y)|$is an upper bound of $\{f(x) : x \in [x_{i-1}, x_i]\}$.

$\therefore M_i' \leq M_i - m_i + |f(y)|$, ($\because M_i'$ is least of upper bound)

$$\therefore M_i' - M_i + m_i \leq |f(y)|, \forall\, y \in [x_{i-1}, x_i]$$

$\therefore M_i' - M_i - m_i$is lower bound of $\{f(x) : x \in [x_{i-1}, x_i]\}$.

$\therefore M_i' - M_i + m_i \leq m_i'$($\because m_i'$ is greatest lower bound)

$\therefore\ M_i' - m_i' \leq M_i - m_i, \ \ i = 1,2, \ldots . n.$ ......3 marks

Multiplying above relation by $(x_i - x_{i-1})$ and adding above n relations we have,

$$U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P) \qquad (*)$$

As, f is R integrableon $[a, b]$.

Hence, for given $\epsilon > 0, \exists$ partition P_ϵ of $[a, b]$ such that,

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$$

$\therefore$ by $(*)$

$$U(|f|, P_\epsilon) - L(|f|, P_\epsilon) < \epsilon$$

$\therefore$ $|f|$ is R integrable on $[a, b]$. ……… 3 marks

We know that,

$$-|f(x)| \leq f(x) \leq |f(x)| \;\; \forall\, x \in [a, b]$$

$$\therefore \; -\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|$$

$\therefore \left|\int_a^b f\right| \leq \int_a^b |f|$ …………..2 marks

Hence, proved.

b) Attempt any TWO questions from the following:

i. Let $f: [a, b] \to \mathbb{R}$ be a monotonic increasing function. Show that f is Riemann integrable on $[a, b]$.

Ans Claim: if f is increasing function on [a,b] then f is R integrable.

Let $P = \{x_0, x_1, \cdots\cdots\cdots, x_n\}$ be a partition of [a,b]

As f is increasing on $[x_{i-1}, x_i]$ such that $M_i = f(x_i)$ and $m_i = f(x_{i-1})$ where

$M_i = sup\{f(x)/x \in [x_{i-1}, x_i]\}$ & $m_i = inf\{f(x)/x \in [x_{i-1}, x_i]\}$ ……..2Marks

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} (Mi - mi)(x_i - x_{i-1}) \leq \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\|P\|$$

$$= (f(b) - f(a))\,\|P\|$$

Select P such that $\|P\| < \frac{\epsilon}{f(b) - f(a) + 1}$

…………. 2 marks

Hence $U(P, f) - L(P, f) < \frac{\epsilon}{f(b) - f(a) + 1}(f(b) - f(a)) < \epsilon$ ……….2marks

ii Let P = { 2, 2.2, 2.4, 2.6, 2.8, 3 } be a partition of [2, 3] and $f : [2, 3] \longrightarrow \mathbb{R}$ is a function such that $f(x) = 2x + 4$ then verify that $L(P, f) \leq U(P, f)$

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	Ans	$f(x_k) = 2x_k^2 + 4 \qquad x_k - x_{k-1} = 0.2$	

Interval	m_k	M_k	$m_k(x_k - x_{k-1})$	$M_k(x_k - x_{k-1})$	
[2, 2.2]	8	8.4	1.6	1.68	
[2.2, 2.4]	8.4	8.8	1.68	1.76	
[2.4, 2.6]	8.8	9.2	1.76	1.84	3M
[2.6, 2.8]	9.2	9.6	1.84	1.92	
[2.8, 3]	9.6	10	1.92	2	

$L(P,f) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = 8.8 \qquad U(P,f) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = 9.2$ 2M

Therefore $L(P,f) \leq U(P,f)$ 1M

iii. Let $f : [-1, 1] \longrightarrow \mathbb{R}$ defined by $f(x) = x \quad if\ x \in [-1,1]\cap\mathbb{Q}$

$= 0 \quad if\ x \in [-1,1] \setminus \mathbb{Q}$

Check whether f is Riemann integrable.

Ans Divide the interval $[-1, 1]$ into $2n$ equal parts each of length $\frac{1-(-1)}{2n} = \frac{1}{n}$

Let $P = \left\{-1, -1+\frac{1}{n}, -1+\frac{2}{n}, \cdots\cdots, -1+\frac{n-1}{n}, 0, \frac{1}{n}, \frac{2}{n}, \cdots\cdots, \frac{n-1}{n}, 1\right\}$

Let m_k and M_k , $k = 1\ to\ n$ be infimum and supremum of f respectively on $\left[-1, -1+\frac{1}{n}\right]$, $\left[-1+\frac{1}{n}, -1+\frac{2}{n}\right], \cdots\cdots, \left[-1+\frac{n-1}{n}, 0\right]$.

$\Rightarrow m_1 = -1,\ m_k = -1 + \frac{k-1}{n},\ k = 2\ to\ n \quad$ and $\quad M_k = 0,\ k = 1\ to\ n$ 1M

Let $m'_k\ and\ M'_k$, $k = 1\ to\ n$ be infimum and supremum of f respectively on $\left[0, \frac{1}{n}\right]$, $\left[\frac{1}{n}, \frac{2}{n}\right], \cdots\cdots, \left[\frac{n-1}{n}, 1\right] \Rightarrow m'_k = 0,\ M'_k = {}^{k}/_{n},\ i = 1\ to\ n$ 1M

$$L(P,f) = \sum_{k=1}^{n}(m_k + m'_k)(x_k - x_{k-1}) = \frac{1}{n}\left[-1 + \left(-1+\frac{1}{n}\right) + \cdots + \left(-1+\frac{n-1}{n}\right) + 0 + \cdots 0\right]$$

$$= \frac{-1}{n^2}\left[n + (n-1) + \cdots\cdots + 2 + 1\right] = -\frac{1}{2}\left(1+\frac{1}{n}\right) \Rightarrow L(f) = -\frac{1}{2}$$ 2M

$$U(P,f) = \sum_{k=1}^{n}(M_k + M'_k)(x_k - x_{k-1}) = \frac{1}{n}\left[0 + \cdots + 0 + \frac{1}{n} + \frac{2}{n} + \cdots + \frac{n}{n}\right]$$

		$= \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{1}{2}\left(1+\frac{1}{n}\right) \Rightarrow U(f) = -\frac{1}{2}$ $\therefore L(f) \neq U(f) \Rightarrow f$ is not R-integrable.	1M 1M
	iv.	Show that the function $f : [0, 3] \longrightarrow \mathbb{R}$ is Riemann integrable, where $f(x) = 1/2 \quad for\ 0 \le x < 1$ $\quad = 2 \quad for\ 1 \le x < 2$ $\quad = 5 \quad for\ 2 \le x \le 3$	
	Ans	For any $\in > 0$ By Archimedian property, $\exists\, n \in \mathbb{N}$ such that $n > 9/\in \Rightarrow 9/n < \in$ 1M Let $P = \{0, 1 - 1/n, 1 + 1/n, 2 - 1/n, 2 + 1/n, 3\}$ be a partition of [0, 3]. Let m_k and M_k be infimum and supremum of f respectively on sub-intervals of [0,3] $U(P, f) - L(P, f) = \sum_{k=1}^{5}(M_k - m_k)(x_k - x_{k-1})$ $= \left(\frac{1}{2}-\frac{1}{2}\right)\left(1-\frac{1}{n}\right) + \left(2-\frac{1}{2}\right)\left(\frac{2}{n}\right) + (2-2)\left(1-\frac{2}{n}\right) + (5-2)\left(\frac{2}{n}\right) + (5-5)\left(1-\frac{1}{n}\right)$ $= 0 + \frac{3}{2}\times\frac{2}{n} + 0 + 3\times\frac{2}{n} + 0 = \frac{9}{n} < \in$ $\therefore f$ is R-integrable on [0, 3].	 1M 1M 2M 1M
Q.4	a)	Attempt any ONE question from the following:	
	i	(1)Let $f : [a,b] \to \Re$ be continuous and g(x) be R-integrable on $[a,b]$ such that If $f'(x) = 0, \forall x[a,b]$ then show that f is constant function on $[a,b]$. (2) $g(x) \ge 0, \forall x$. Then show that $\exists c \in [a,b]$ such that $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$	
	Ans	(1) since f is continuous on $[a,b] \Rightarrow$ f is bounded on $[a,b]$ $m \le f(x) \le M \Rightarrow m\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le M\int_a^b g(x)dx$ $\Rightarrow \int_a^b f(x)g(x)dx = \mu\int_a^b g(x)dx \qquad (1)$	

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		Being continuous f takes every value between m and M $\therefore \exists c \in [a,b]$ such that $f(c)=\mu$ .from eq (1) we get the result . since f is differentiable hence continuous on $[a,b]$ Therefore LMVT holds $\forall x \in [a,x], x \le b$ $\therefore \exists c \in [a,b]$ such that $f'(x)=\dfrac{f(x)-f(a)}{x-a} \Rightarrow f(x)=f(a), \forall x \Rightarrow f$ is constant function. (2)	
	ii	Show that $\int_0^{\infty} x^{n-1}e^{-x}dx$ is convergent iff n>0. Also show $\Gamma(1)=1$.	
	Ans	Let $f(x)=x^{n-1}e^{-x}$. 0 is the point of infinite discontinuity of f if $n<1$. Let $\int_0^{\infty} f(x)dx=\int_0^1 f(x)dx+\int_1^{\infty} f(x)dx$. (2 marks) Converges at 0: $g(x)=\dfrac{1}{x^{1-n}}\cdot \lim\limits_{x\to 0}\dfrac{f(x)}{g(x)}=1.$ Since $\int_0^1 g(x)dx$ converges iff$1-n<1$ $i.e. n>0$. (2 marks) Converges at 1: For given n and large values of x, we have $e^x>x^{n+1} \Rightarrow e^{-x}<x^{-n-1} \Rightarrow$ $x^{n-1}e^{-x} \quad < x^{n-1}x^{-n-1}=\frac{1}{x^2}$. Since $\int_1^{\infty}\frac{1}{x^2}dx$ converges, $\int_1^{\infty} x^{n-1}e^{-x}dx$ $\forall n$. (2 marks) $\Gamma(1)=\displaystyle\int_0^{\infty} e^{-x}=[-e^{-x}]_0^{\infty}=1.$	
	b)	Attempt any TWO questions from the following:	
	i.	By using comparison test ,check the convergence of integral $\displaystyle\int_1^{\infty}\frac{1}{1+e^x}dx$.	
	Ans	$1+e^x \ge e^x \Rightarrow \displaystyle\int_1^{\infty}\frac{1}{1+e^x}dx \le \int_1^{\infty}\frac{1}{e^x}dx$ Since $\displaystyle\int_1^{\infty}\frac{1}{e^x}dx=\frac{1}{e} \Rightarrow$ convergent.	
	ii.	Let $x=\displaystyle\int_0^{y}\frac{dt}{\sqrt{1+t^2}}, y \ge 0$ .Show that $\dfrac{d^2y}{dx^2}=y$.	
	Ans	differentiate with respect to x, we get $1=\dfrac{1}{\sqrt{1+y^2}}\dfrac{dy}{dx}$	

		$\frac{dy}{dx}=\sqrt{1+y^2} \Rightarrow \frac{d^2y}{dx^2}=\frac{y}{\sqrt{1+y^2}}\frac{dy}{dx}$ Put the value of $\frac{dy}{dx}$ in above equation ,we get the result.	
	iii.	Express $\int_0^1 x^m(1-x^2)^n\,dx$ in terms of beta function.	
	Ans	$z=x^2, dx=\frac{dz}{2\sqrt{z}}.\ I=\int_0^1 z^{\frac{m}{2}}(1-z)^n\frac{dz}{2\sqrt{z}}=\frac{1}{2}\int_0^1 z^{\frac{m-1}{2}}(1-z)^n\,dz=\frac{1}{2}\beta\left(\frac{m+1}{2},n+1\right).$	
	iv.	Use polar co ordinates to find the volume of the solid above *xy*- plane bounded by the paraboloid $z=12-3x^2-3y^2$.	
	Ans	Volume=$\int_0^{2\pi}\int_0^1(1-r^2)\,r\,drd\theta=\frac{\pi}{2}$.	
Q.5		Attempt any FOUR questions from the following:	
	a)		
	Ans		
	b)		
	Ans		
	c)	Prove that the function $f:[1,3]\longrightarrow\mathbb{R}$ defined by $f(x)=x-2$ is Riemann integrable and evaluate $\int_1^3 f(x)dx$.	
	Ans	Since f is continous on $[1,3]\Rightarrow f$ is R-integrable on $[1,3]$. Let $P=\{1=x_0,x_1,\cdots\cdots\cdots,x_n=3\}$ is partition of $[1,3]$ into n equal parts. $x_k-x_{k-1}=2/n$ and $x_k=1+2k/n$ $\int_1^3 f(x)dx=\lim_{n\to\infty}\sum_{k=1}^{n}f(x_k)(x_k-x_{k-1})=\lim_{n\to\infty}\sum_{k=1}^{n}(x_k-2)\frac{2}{n}$ $=\lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n}\left[2\frac{k}{n}-1\right]=\lim_{n\to\infty}\frac{2}{n}\left[\frac{2}{n}\frac{n(n+1)}{2}-n\right]=0$	1M 1M 1M 2M
	d)	Using Riemann Criterion, show that the function $f:[0,1]\longrightarrow\mathbb{R}$ defined by $f(x)=x+2$ is Riemann integrable.	

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	Ans	For any $\in > 0$ Claim : $U(P,f) - L(P,f) < \in$ By Archimedian property, $\exists\, n \in \mathbb{N}$ such that $n > 1/\in \Rightarrow 1/n < \in$ Let $P = \{0, 1/n, 2/n, \cdots\cdots\cdots, 1\}$ be a partition of [0, 1]. $x_k - x_{k-1} = 1/n$ and $x_k = k/n$ Since f is increasing, hence $M_k = x_k + 2$ and $m_k = x_{k-1} + 2$ $U(P,f) - L(P,f) = \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) = \sum_{k=1}^{n}(x_k - x_{k-1})(x_k - x_{k-1})$ $= \sum_{k=1}^{n}\frac{1}{n}\frac{1}{n} < \frac{1}{n^2}\times n < \in$ $\therefore$ f is R-integrable.	1M 1M 1M 1M 1M
	e)	Let $f:[0,1]\to\Re$ such that $f(x)=\begin{cases}0, 0\le x\le 1/2\\ 1, 1/2 \le\ne x\le 1\end{cases}$ and $F:[0,1]\to\Re$ such that $F(x)=\int_0^x f(x)dx$ .Discuss the continuity of f and differentiability of F at $x = 1/2$. Examine whether $F'(1/2) = f(1/2)$.	
	Ans	$\therefore \lim_{x\to 1/2^+} f(x) = 1$ and $\therefore \lim_{x\to 1/2^-} f(x) = 0$ implies f is not continuous. $F'(1/2) = \lim_{h\to 0}\frac{F(1/2+h) - F(1/2)}{h} = \lim_{h\to 0}\frac{1}{h}\int_{1/2}^{1/2+h} f(x)dx = \lim_{h\to 0}\frac{1}{h}\int_{1/2}^{1/2+h} 1dx = 1$ $\therefore F'(1/2) \ne f(1/2)$	
	f)	Reverse the order of integration and evaluate $\int_0^1\int_{y^2}^{y} xdxdy$.	
	Ans	$\int_0^1\int_{y^2}^{y} xdxdy = \int_0^1\int_x^{\sqrt{x}} xdydx = \frac{1}{15}$.	
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