S.Y. B.A./B. Sc. Semester IV
Mathematics Paper I (Old)
Exam Date 23/04/19
Q.P.Code 66034
Answer Key

(3 Hours) [Total Marks: 100]

Note: (i) All questions are compulsory.
(ii) Figures to the right indicate marks for respective parts.

Q.1	Choose correct alternative in each of the following (20)			
$i.$	Which of the following sequences does not have a convergent subsequence			
	(a)	$(n+2)$	(b)	1, 0,2 ,1 ,0, 2,…………….
	(c)	$\left(\frac{1}{n}\right)$	(d)	None of these
Ans	(a)	$(n+2)$		
$ii.$	If the decimal representation of a number x is non terminating, non periodic then the number x belongs to			
	(a)	$\mathbb{Q}$	(b)	$\mathbb{R}\setminus\mathbb{Q}$
	(c)	$\mathbb{N}$	(d)	None of these
Ans	(b)	$\mathbb{R}\setminus\mathbb{Q}$		
$iii.$	Which of the following sets is uncountable			
	(a)	$\mathbb{R}$	(b)	$\mathbb{N}$
	(c)	$\mathbb{Q}$	(d)	None of these
Ans	**(a)**	$\mathbb{R}$		
$iv.$	Let P and Q be any two Partitions of interval [a,b]. Then the statement that is always true is			
	(a)	$L(P,f) \leq U(Q,f)$	(b)	$U(P,f) \leq U(Q,f)$
	(c)	$U(P,f) \leq L(Q,f)$	(d)	None of these
Ans	(a)	$L(P,f) \leq U(Q,f)$		
v.	$f : [a,b] \to \mathbb{R}$ is R-integrable on $[a,b]$. Then			
	(a)	f^2 and $\|f\|$are R-integrable on $[a,b]$.	(b)	$f^2$is R-integrable on $[a,b]$ but $\|f\|$ may or may not be so.
	(c)	Both f^2 and $\|f\|$ are not R-integrable on $[a,b]$.	(d)	None of these

Ans	(a)	f^2 and $\lvert f \rvert$ are R-integrable on $[a, b]$.		
vi.	Let $f : [0,100] \to \mathbb{R}$ be defined as $f(x) = \lfloor x \rfloor$, (Where $\lfloor x \rfloor$ is floor function of x). Then			
	(a)	f is discontinuous hence not R-integrable.	(b)	f is R-integrable and $\int_0^{100} f(x)dx = 5000$
	(c)	f is R-integrable and $\int_0^{100} f(x)dx = 4950$	(d)	None of these
Ans	(c)	f is R-integrable and $\int_0^{100} f(x)dx = 4950$		
vii.	Gamma function $\Gamma(n)$ is defined as			
	(a)	$\int_0^1 e^{-x} x^{n-1}\, dx$, n>0	(b)	$\int_0^1 e^{x} x^{n-1}\, dx$, n>0
	(c)	$\int_0^\infty e^{-x} x^{n-1}\, dx$, n>0	(d)	$\int_0^\infty e^{x} x^{n-1}\, dx$, n>0
Ans	(c)	$\int_0^\infty e^{-x} x^{n-1}\, dx$, n>0		
viii.	Find $\int_0^{\frac{\pi}{2}} sin^6 x\, dx$			
	(a)	0	(b)	$\frac{5\pi}{12}$
	(c)	$\frac{5\pi}{32}$	(d)	$\frac{5\pi}{6}$
Ans	(c)	$\frac{5\pi}{32}$		
ix.	β(m,n) =			
	(a)	$\int_0^1 \frac{x^{m-1}}{(1+x)^{m-n}}\, dx$	(b)	$\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$
	(c)	$\int_0^1 \frac{x^{n-1}}{(1+x)^{m-n}}\, dx$	(d)	None of these
Ans	(b)	$\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$		
x.	$\int_0^1 \int_0^{1-y} f(x,y)dxdy$=			
	(a)	$\int_0^1 \int_0^x f(x,y)dydx$	(b)	$\int_0^1 \int_0^{1-x} f(x,y)dydx$
	(c)	$\int_0^1 \int_1^{1-y} f(x,y)dydx$	(d)	None of these
Ans	(b)	$\int_0^1 \int_0^{1-x} f(x,y)dydx$		

Q2.	Attempt any **ONE** question from the following: (08)	
$a)$	i.	Using Nested Interval Theorem prove that every bounded sequence of real numbers has a convergent subsequence.
	Ans	Let (x_n)be bounded. S=$\{x_n/\ n \in \mathbb{N}\ \}$ is subset of I = [a,b] for some a, b in $\mathbb{R}$. Let n_1=1.Bisect I into two subintervals both of equal length$\frac{b-a}{2}$ say$I_1^{'}{}_{,}I_1^{''}{}_{,}$ Let $A_1 = \{\ n \in \mathbb{N}/\ n > n_1, x_n \in\ I_1^{'}{}_{,}\}$ $B_1 = \{\ n \in \mathbb{N}/\ n > n_1, x_n \in\ I_1^{''}{}_{,}\}$ (2 marks) At least one of $A_{1,}$ B_1 is infinite. Say A_1 is infinite .Then $I_2 = I_1^{'}{}_{,}$ Let n_2 be minimum element of A_1 (2 marks) Else if B_1 is infinite .Then $I_2 = I_1^{''}{}_{,}$ n_2 be minimum element of B_1............ Proceed similarly to get a nested sequence of intervals I_1 □I_2 □……□I_n ……. And a subsequence (x_{n_k}) of (x_n) $such\ that x_{n_k} \in I_k, k \in \mathbb{N}$ and length of I_n = $\frac{b-a}{2^{n_k-1}} \to 0\ as\ n \to \infty$ (2 marks) By Nested Interval Theorem there exists unique element c in$\bigcap_{n=1}^{\infty} I_n$and……$(x_{n_k}) \to c$(2 marks)
	ii.	Using Nested Interval Theorem prove that if $f\ :\ [a,b]\ \to \mathbb{R}$ is a continuous function with $f(a)f(b)\ <\ 0$, then there exists $c\epsilon\ (a,b)$ such that $f(c)\ =\ 0$.
	Ans	Given $f(a)f(b)\ <\ 0$ Say $f(a)\ <\ 0 < f(b)$ Let $I_1\ =\ [a_{1,}b_{1,}]$ a =$a_{1,}$ b = $b_{1,}p_1\ =\ \frac{a_{1+}\ b_{1,}}{2}$ (2 marks) If f (p_1)< 0 then $a_{2=,}p_1, b_{2,}\ =\ b_{1,}$ else if f (p_1)< 0 then $a_{2=,}a_{1,}, b_{2,}\ =\ p_{1,}$ Let $I_2\ =\ [a_{2,}b_2\ ]$,length of $I_2\ =\ \frac{b-a}{2^1}$………………………….. Proceed similarly to get a nested sequence of intervals I_1 □I_2 □……□I_n ……. length of $I_n\ =\ \frac{b-a}{2^n} \to 0\ as\ n \to \infty$ By Nested Interval Theorem there exists unique element c in$\bigcap_{n=1}^{\infty} I_n$ Since f is continuous and f$(a_{n,})$ < 0; f$(b_{n,})$ > 0…………………f(c)=0
Q.2	Attempt any **TWO** questions from the following: (12)	
$b)$	i.	If $I_n\ =\ \left(0, \frac{1}{n}\right)$ for all $n\epsilon\mathbb{N}$ then prove that $\bigcap_{n=1}^{\infty} I_n\ =\ \phi$

	Ans	Suppose $x \in \bigcap_{n=1}^{\infty} I_n$ Hence $x \in I_n \; for \; all \; n \in \mathbb{N}$ since x≠ 0 by A P $\exists n_0 \in \mathbb{N}$s.t. $n_0 > \frac{1}{x}$ x does not belong to I_{n_0} x does not belong to $\bigcap_{n=1}^{\infty} I_n$ Contradiction……..
	ii.	Prove that any real number $x\epsilon\ [0,1]$ can be represented in decimal representation.4
	Ans	Divide [0,1] into 10 equal parts of length $\frac{1}{10}$ Hence $x \in [\frac{b_1}{10}, \frac{b_1+1}{10})$ □$[\frac{b_1}{10}, \frac{b_1+1}{10}]$,for some $b_1 \in \{0, 1, \ldots\ldots 9\}$ If $x \in \frac{b_1}{10}$, $x = 0.b_1$process terminate Else Let $I_1 = [\frac{b_1}{10}, \frac{b_1+1}{10}]$ DivideI_1 into 10 equal parts of length $\frac{1}{10^2}$ Hence $x \in [\frac{b_1}{10} + \frac{b_2}{10^2}, \frac{b_1}{10} + \frac{b_2+1}{10^2})$ □$[\frac{b_1}{10} + \frac{b_2}{10^2}, \frac{b_1}{10} + \frac{b_2+1}{10^2}]$,for some $b_2 \in \{0, 1, \ldots\ldots 9\}$ If $x \in \frac{b_1}{10} + \frac{b_2}{10^2}$, $x = 0.b_1 b_2$ process terminate else let $I_2 = \cdots$…. Proceed similarly we get either $x = 0.b_1 b_2 \ldots b_n$. Or we get a nested sequence of intervals I_1 □I_2 □……□I_n ……. s.t length of $I_n \to 0 \; as \; n \to \infty$ By Nested Interval Theorem there exists unique element c in$\bigcap_{n=1}^{\infty} I_n$ x= c
	iii.	Find a family of open intervals G≡ $\{J_n : n \in N\}$ such that (0, 2] ⊆ $\bigcup_{n=1}^{\infty} J_n$,but there does not exist a finite subset F ={$n_1, n_2, \ldots, n_k$} ⊆ N such that (0, 2] is subset of$\bigcup_{n \in F} J_n$.

	Ans	Let G={J_n=($\frac{1}{n}$,3) / $n \in N$} Part 1: $x \in A$ *implies* ,since x≠ 0 by A P $\exists n_0 \in \mathbb{N}$s.t. $n_0 > \frac{1}{x}$…. Hence ∈ $_0$ A $\cup^{\infty}_{=1}$ (2 marks) Part 2: TPT there does not exist a finite subset F ={$n_1, n_2, \ldots, n_k$} ⊆ such that (0, 2] is subset of∪ ∈ . Suppose there exist a finite subset F ={$n_1, n_2, \ldots, n_k$} ⊆ such that (0, 2] is subset of∪ ∈ . A ∪ $_{=1}$ where $_1$ < $_2$ … … … . . < Hence$\frac{1}{\ _1}$ > ⋯ … … … . . > $\frac{1}{\ }$ $\frac{1}{\ } \neq 0$, h ∃ ∈ s. t $0 <$ < $\frac{1}{\ }$. x does not belong to $\cup_{i=1}^{k} J_{n_i}$ Contradiction…….(4 marks)
	iv.	Using Nested Interval Theorem prove that closed interval[0,1] is uncountable.
	Ans	Suppose that closed interval[0,1] is countable. We can enumerate I= [0,1] as{$x_1, x_2, \ldots\ldots x_n, \ldots$} Select closed interval I_1 of I s.t x_1 does not belong to interval I_1 Select closed interval I_2 of I s.t x_2 does not belong to interval I_2 and so on Proceed similarly to get a nested sequence of intervals I_1 I_2 ……I_n ……. s.tx_n does not belong to interval I_n(4 marks) By Nested Interval Theorem there exists unique element c in$\bigcap_{n=1}^{\infty} I_n$ But c≠ x_n for all n c does not belong to $\bigcap_{n=1}^{\infty} I_n$ contra…………(2 marks)
Q3 a	Attempt any **ONE** question from the following: (08)	
A	i)	Let $f: [a, b] \to \mathbb{R}$ be a bounded function. Prove that f is R-integrable on $[a, b]$ iff for any $\in > 0$, there exists a partition $P_\in$ of $[a, b]$ such that $U(f, P_\in) - L(f, P_\in) < \in$.
	Ans	Proof: (⇒) Given f is R integrable on $[a, b]$. T.P.T: $\forall \epsilon > 0, \exists$apartition $P_\epsilon$of $[a, b]$ such that, $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$. Let, $\epsilon > 0$ be any real number, as f is R integrable, $\therefore U(f) = L(f)$

		Where, $U(f) = \inf\{U(f,P): P$ is any partion of $[a,b]\}$ And $L(f) = \sup\{L(f,P): P$ is any partion of $[a,b]\}$ $\therefore$ for given $\epsilon > 0, \exists$ a partition P_1 of $[a,b]$ such that, $U(f) \le U(f,P_1) < U(f) + \frac{\epsilon}{2}$ (1) Also, for given $\epsilon > 0, \exists$ a partition P_2 of $[a,b]$ such that, $L(f) - \frac{\epsilon}{2} < L(f,P_2) \le L(f)$ or $-L(f) \le -L(f,P_2) < -L(f) + \frac{\epsilon}{2}$ (2) from (1) and (2) $U(f) - L(f) \le U(f,P_1) - L(f,P_2) < U(f) - L(f) + \epsilon$ $\therefore 0 \le U(f,P_1) - L(f,P_2) < \epsilon$ (3) $(\because U(f) = L(f))$ Now taking $P_\epsilon = P_1 \cup P_2$, $\therefore U(f,P_\epsilon) \le U(f,P_1) \& L(f,P_\epsilon) \ge L(f,P_2) (\because P_1 \subseteq P_\epsilon \& P_2 \subseteq P_\epsilon)$ $\therefore U(f,P_\epsilon) - L(f,P_\epsilon) \le U(f,P_1) - L(f,P_2) < \epsilon$ by (3) $\therefore U(f,P_\epsilon) - L(f,P_\epsilon) < \epsilon$ $(\Leftarrow)$ Given: $\forall \epsilon > 0, \exists$apartition $P_\epsilon$of $[a,b]$ such that, $U(f,P_\epsilon) - L(f,P_\epsilon) < \epsilon$. T.P.T: f is R integrable on $[a,b]$. i.e. T.P.T: $L(f) = U(f)$ $\because \forall \epsilon > 0, \exists$apartition $P_\epsilon$of $[a,b]$ such that, $U(f,P_\epsilon) - L(f,P_\epsilon) < \epsilon$ We know that, $U(f) \le U(f,P_\epsilon) \& L(f) \ge L(f,P_\epsilon)$ $\therefore 0 \le U(f) - L(f) \le U(f,P_\epsilon) - L(f,P_\epsilon) < \epsilon (\because U(f) \ge L(f))$ $\therefore 0 \le U(f) - L(f) < \epsilon \therefore U(f) = L(f)$
	ii	Let $f: [a,b] \to \mathbb{R}$ be a monotonic increasing function. Show that f is Riemann integrable on $[a,b]$.
	Ans	Claim: if f is increasing function on [a,b] then f is R integrable. Let P={x0, x1, …, xn] be a partition of [a,b] As f is increasing on $[x_{i-1}, xi]$ such that Mi=f(x_i) and mi=f(x_{i-1}) where M_i =sup{f(x)/x∈$[x_{i-1}, x_i]$ } &m_i =inf {f(x)/x∈$[x_{i-1}, x_i]$ }(2 marks) U(P,f)—L(P,f)=$\sum_{i=1}^{n}(Mi - mi)(xi - \text{xi} - 1) \le \sum_{i=1}^{n}(\text{f(xi)} - \text{f}(xi - 1))\|P\|$ = (f(b)-f(a)) $\|P\|$ Select P such that $\|P\| < \frac{\epsilon}{\text{f(b)}-\text{f(a)}+1}$ (2 marks)

		Hence U(P,f)—L(P,f) $< \frac{\epsilon}{f(b)-f(a)+1}(f(b)-f(a)) < \epsilon$ (2 marks)
b)	Attempt any TWO questions from the following:	
	i.	Let $f : [a,b] \to \mathbb{R}$ be a bounded function with $m = Inf\,(f)$ and $M = Sup(f)$on $[a,b]$. With usual notations, define $L(P,f)$ and $U(P,f)$ where$P$ is a partition of $[a,b]$. Hence prove that $m\,(b-a) \le L(P,f) \le U(P,f) \le M\,(b-a)$.
	Ans	Let f be a bounded function on [a,b] . Let $P = \{x_0, x_1, \cdots\cdots\cdots, x_n\}$be a partition of [a,b] As f is continuous by boundedness be a partition of [a,b] Define i) upper sum ,ii) lower sum (3 marks) let $M_i = sup${f(x) / x∈$[x_{i-1}, x_i]$ } &$m_i = inf$ {f(x)/x∈$[x_{i-1}, x_i]$ }- $m \le m_i \le M_i \le M$ for i=1,2,3…,n hence m(b-a) ≤ L(P,f) ≤ U(P,f) ≤ M(b-a) (3 marks)
	ii	Let $f: [a,b] \to \mathbb{R}$ be a bounded function if f is R-integrable on $[a,c]$ and $[c,b]$ then prove that f is R-integrable on $[a,b]$and$\int_a^b f = \int_a^c f + \int_c^b f$
	Ans	Given f is integrable on $[a,c]$ and $[c,b]$. for any $\in > 0$ ∃ a partition P_1 of $[a,c]$ such that $U(f,P_1) - L(f,P_1) < \in$ for any $\in > 0$ ∃ a partition P_2 of $[c,b]$ such that $U(f,P_2) - L(f,P_2) < \in$ since $U(f,P) = U(f,P_1) + U(f,P_2)$ *and* $L(f,P) = L(f,P_1) + L(f,P_2)$ 2 marks Take $P = P_1 \cup P_2 \Longrightarrow U(f,P) \le U(f,P_1)$ *and* $L(f,P) \ge L(f,P_1)$ $\therefore U(f,P) - L(f,P) \le U(f,P_1) - L(f,P_1) < \in$ $\therefore f$ is R − integrable on $[a,b]$ for any $\in > 0$ ∃ partitions P_1 and P_2 of $[a,c]$ and $[c,b]$ then one can find a partition of $[a,b]$ such that $U(f,P) - L(f,P) < \in \Longrightarrow f$ is R − integrable on $[a,b]$. 1 marks Claim : $\int_a^b f = \int_a^c f + \int_c^b f$ LHS = $\int_a^b f = \int_c^{\bar{b}} f \le U(f,P) < \in + L(f,P) < \in + L(f,P_1) + L(f,P_2) < \in + \int_a^c f + \int_c^b f$ therefore $\int_a^b f \le \int_a^c f + \int_c^b f$ 2 marks Similarly one can show $\int_a^b f \ge \int_a^c f + \int_c^b f$ 1 marks Hence $\int_a^b f = \int_a^c f + \int_c^b f$

<table>
<tr><td></td><td>iii.</td><td>By stating properties of Riemann integrability justify which of the following functions are Riemann integrable ?

i) $f(x) = e^{\sin|x|}$ on [-π.π]

ii) $f(x) = 0 if x = 0$
$= \frac{1}{n} if \frac{1}{n+1} < x \leq \frac{1}{n}$, n ∈ ℕ</td></tr>
<tr><td></td><td>Ans</td><td>1) e^x ,$\sin x, |x|, x^2$ all are continuous functions on ℝ and composition of continuous functions is continuous hence Riemann intrgrable .
f *has fintely* many discontinuities hence is Riemann Integarable.</td></tr>
<tr><td></td><td>iv.</td><td>Using Riemann Criterion, show that the function : [2, 3] ⟶ ℝ defined by $f(x) = x + 3$ is Riemann integrable.</td></tr>
<tr><td></td><td>Ans</td><td>For any $\in > 0$ Claim : $U(P,f) - L(P,f) < \in$
By Archimedean property, $\exists n \in \mathbb{N}$ such that $n > 1/\in \Rightarrow 1/n < \in$
Let $P = \{0, 1/n, 2/n, \cdots\cdots\cdots, 1\}$ be a partition of [0, 1].
$x_k - x_{k-1} = 1/n$ and $x_k = k/n$
Since f is increasing, hence $M_k = x_k + 3$ and $m_k = x_{k-1} + 3$ (3 marks)
$U(P,f) - L(P,f) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) - \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n}(x_k - x_{k-1})(x_k - x_{k-1})$
$= \sum_{k=1}^{n} \frac{1}{n}\frac{1}{n} < \frac{1}{n^2} \times n < \frac{1}{n} < \in$ (3 marks)
$\therefore f$ is R-integrable.</td></tr>
<tr><td>Q4.</td><td colspan="2">Attempt any **ONE** question from the following: (08)</td></tr>
<tr><td>$a)$</td><td>i.</td><td>If the function $f: [a,b] \to \mathbb{R}$ is a continuous function and let$F(x) = \int_a^x f(t)dt \, \forall x \in [a,b]$, then prove that $F(x)$ is differentiable and $F'(x) = f(x) \forall x \in (a,b)$.</td></tr>
<tr><td></td><td>Ans</td><td>Let $x \in (p - \delta, p + \delta)$
Case 1: $x \in (p - \delta, p)$. $F(x) - F(p) = -\int_p^x f(t)dt$.
$\left|\frac{F(x)-F(p)}{x-p} - f(p)\right| = \left|\frac{1}{p-x}\int_x^p |f(t) - f(p)|dt\right| \leq \left|\frac{1}{p-x}\right| \left|\int_x^p |f(t) - f(p)|dt\right| \leq$
$\frac{1}{p-x}\int_x^p \frac{\epsilon}{2} dt \leq \epsilon$ (4 marks)</td></tr>
</table>

		Case 2: $x \in (p, p+\delta)$. $F(x) - F(p) = \int_p^x f(t)dt$. $\left\|\frac{F(x)-F(p)}{x-p} - f(p)\right\| = \left\|\frac{1}{x-p}\int_p^x \|f(t) - f(p)\|dt\right\| \le \left\|\frac{1}{x-p}\right\| \left\|\int_p^x \|f(t) - f(p)\|dt\right\| \le \frac{1}{x-p}\int_p^x \frac{\epsilon}{2} dt \le \epsilon$ (4 marks)
	ii.	If a function $f: [a,b] \to \mathbb{R}$ is R-integrable on $[a,b]$ and function $F: [a,b] \to \mathbb{R}$ is defined by $F(x) = \int_a^x f(t)dt \ \forall x \in [a,b]$then prove that F is continuous on $[a,b]$.
	Ans	Case 1: $x \ge p$. $\|F(x) - F(p)\| = \left\|\int_p^x f(t)dt\right\| \le \int_p^x \|f(t)\|dt \le M\int_p^x dt \le \epsilon$ (4 marks) Case 2: $x < p$. $\|F(x) - F(p)\| = \left\|-\int_x^p f(t)dt\right\| \le \int_x^p \|f(t)\|dt \le M\int_x^p dt \le \epsilon$ (4 marks)
Q4.	Attempt any **TWO** questions from the following: (12)	
$b)$	i.	State and prove First Mean Value theorem of integral calculus.
	Ans	Statement: Let $f: [a,b] \to \mathbb{R}$ be a continuous function . Then show that $\exists\ c \in [a,b]$ such that$\int_a^b f(x)dx = f(c)(b-a)$. (2 marks) $$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$ $m \le \frac{1}{b-a}\int_a^b f(x)dx \le M$ (2 marks) f is continuous hence attains bounds. By intermedite value property f takes a value between m and M. So $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$ (2 marks)
	ii.	If $\int_a^\infty \|f(x)\|dx$converges then show that $\int_a^\infty f(x)dx$converges.
	Ans	$$\left\|\int_x^y \|f(t)\|\, dt\right\| < \epsilon \forall x, y \ge x_0$$ $$\left\|\int_x^y f(t)dt\right\| \le \int_x^y \|f(t)\|\, dt \le \left\|\int_x^y \|f(t)\|\, dt\right\| < \epsilon \forall x, y \ge x_0$$
	iii.	Prove that $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ converges if and only if m and n are both positive.
	Ans	For $m \ge 0, n \ge 0$ the integral is proper. When $m \le 1$, infinite discontinuity at 0 and when $n \le 1$, infinite discontinuity at 1. $\int_0^1 x^{m-1}(1-x)^{n-1}dx = \int_0^{1/2} x^{m-1}(1-x)^{n-1}dx + \int_{1/2}^1 x^{m-1}(1-x)^{n-1}dx$=$I_1$+$I_2$ For I_1, $f(x) = \frac{(1-x)^{n-1}}{x^{1-m}}, g(x) = \frac{1}{x^{1-m}}$. $\lim_{x\to 0}\frac{f(x)}{g(x)} = 1$. By comparison test $\int_0^{\frac{1}{2}} f(x)dx$, $\int_0^{\frac{1}{2}} g(x)dx$ converge and diverge together. $\int_0^{\frac{1}{2}} g(x)dx = \int_0^{\frac{1}{2}} \frac{1}{x^{1-m}}dx$converges iff $1-m < 1$ $i.e.\ m > 0$. (3 marks)

		For $I_2(x) = \frac{x^{m-1}}{(1-x)^{1-n}}, g(x) = \frac{1}{(1-x)^{1-n}}$. Same approach as above. (3 marks)
	iv.	Sketch the region of integration and calculate the integral $\int_0^1 \int_{4x}^4 x^2 dydx$.
	Ans	Sketching of area (3 marks) Calculation $\int_0^1 \int_{4x}^4 x^2 dydx = 1/3$. (3 marks)
Q5.	Attempt any **FOUR** questions from the following: (20)	
$a)$	Show that 0.21 and 0.2099…. represent the same rational number.	
Ans	$0.21 = \frac{2}{10} + \frac{1}{100} = \frac{21}{100}$(2 marks) 0.2099………..$= \frac{2}{10} + \frac{0}{100} + \frac{9}{10^3}\left(1 + \frac{1}{10} + \cdots \ldots\ldots\ldots\right)$ $= \frac{20}{100} + \frac{9}{10^3}\lim_{n\to\infty}\left(\frac{1-\frac{1}{10^{n-2}}}{1-\frac{1}{10}}\right) == \frac{20}{100} + \frac{1}{10^2}(1-0) = \frac{21}{100}$(3 marks) 2	
$b)$	Show that f(x) $= x^6 - 4x^4 - x + 1$ has a zero in each of the intervals $(-1, 0)$, $(0, 1)$ and $(1, 2)$.	
Ans	Since f is continuous on IR, by I VP since (2 marks) f(-1) f(0)< 0 ,f has zero in (-1 , 0) f(0) f(1)< 0 ,f has zero in (0 , 1) f(1) f(2)< 0 ,f has zero in (1 , 2)(3 marks)	
c)	Let P = { 2, 2.1, 2.3, 2.5, 2.9, 3 } be a partition of [2, 3] and f : [2, 3] ⟶ ℝ is a function such that $f(x) = x + 1$ then verify that $L(P, f) \leq U(P, f)$	
Ans	Solution not required.	
d)	Show that the function f : [1, 3]⟶ ℝ is Riemann integrable, where $f(x) = 5 for\ 1 \leq x < 2$ $= -9 for\ 2 \leq x \leq 3$	

Ans

Divide the interval [1, 3] into $2n$ equal parts each of length $\frac{3-1}{2n} = \frac{1}{n}$

Let $P = \left\{1, 1+\frac{1}{n}, 1+\frac{2}{n}, \cdots\cdots, 1+\frac{n-1}{n}, 2, 2+\frac{1}{n}, 2+\frac{2}{n}, \cdots\cdots, 2+\frac{n-1}{n}, 3\right\}$

Let m_i and M_i , $i = 1$ to n be infimum and supremum of f respectively on $\left[1, 1+\frac{1}{n}\right]$,

$$\left[1+\frac{1}{n}, 1+\frac{2}{n}\right], \cdots\cdots, \left[1+\frac{n-1}{n}, 2\right]$$

$\Longrightarrow m_i = M_i = 5,\ i = 1\ to\ n-1$ and $m_n = -9,\ M_n = 5$ 1 marks

Let m_i' and M_i' , $i = 1$ to n be infimum and supremum of f respectively on $\left[2, 2+\frac{1}{n}\right]$,

$$\left[2+\frac{1}{n}, 2+\frac{2}{n}\right], \cdots\cdots, \left[2+\frac{n-1}{n}, 3\right]$$

$\Longrightarrow m_i' = M_i' = -9,\ i = 1\ to\ n$ 1 marks

$L(P, f) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) + \sum_{k=1}^{n} m_k'(x_k - x_{k-1}) = \frac{1}{n}[5+5+\cdots+(-9)] + \frac{1}{n}[-9-9-\cdots\cdots-9]$

$= \frac{5(n-1)}{n} + \frac{(-9)(n+1)}{n} = -4 - \frac{14}{n} \Longrightarrow L(f) = -4$

1 marks

$$U(P, f) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) + \sum_{k=1}^{n} M_k'(x_k - x_{k-1})$$

$= \frac{1}{n}[5+5+\cdots+5] + \frac{1}{n}[-9-9-\cdots\cdots-9]$

$= \frac{5n}{n} + \frac{(-9)n}{n} = -4 \Longrightarrow U(f) = -4$ 1 marks

$\therefore L(f) = U(f) \Longrightarrow f$ is R-integrable. 1 marks

e)

Prove that $\sqrt{\pi}\,\Gamma(2m) = 2^{2m-1}\,\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right)$.

Ans

$$\beta(m, m) = \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} (\sin 2\theta)^{2m-1}\, d\theta$$

Substituting $t = 2\theta$. $\beta(m, m) = \frac{2}{2^{2m-1}} \int_0^{\pi} \frac{(\sin t)^{2m-1} dt}{2} = \frac{1}{2^{2m-1}}\left[\int_0^{\frac{\pi}{2}} (\sin t)^{2m-1} dt + \int_0^{\frac{\pi}{2}} (\sin(\pi - t))^{2m-1} dt\right] = \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} (\sin t)^{2m-1} dt$ (2 marks)

Also $\beta\left(m, \frac{1}{2}\right) = \int_0^{\frac{\pi}{2}} (\sin t)^{2m-1} dt$.(1 marks)

Combining above and using beta gamma relationship get the result.(2 marks)

f)	Find the volume of the solid that lies under the hyperbolic paraboloid $z = 4 + x^2 - y^2$ and above the square $R = [-1,1]\ X\ [0,2]$ using Fubini's theorem.
Ans	Volume=$\int_{-1}^{1}\int_{0}^{2}(4 - x^2 - y^2)dydx = 12$.

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