(1) Q. P. Code :- 54967

(3 Hours) [Total Marks: 100]

Note: (*i*) All questions are compulsory. (*ii*)Figures to the right indicate marks for respective parts.

Q.1	Choose correct alternative in each of the following				(20)
i.	If $T: U \to V$ is a linear transformation such that $\ker T = \{0\}$ then				
	(a)	surjective	(b)	injective	
	(c)	bijective	(d)	None of the above	
	Ans	(b)			
ii.	If $\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ then $\mathbf{E}^{-1}$ is				
	(a)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	(b)	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	
	(c)	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	(d)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	
	Ans	d			
iii.	$T: \mathbb{R}^2 \to \mathbb{R}^2$, then which of the following is a linear transformation?				
	(a)	$T(x,y)=(x,y)$	(b)	$T(x,y)=(x, y+1)$	
	(c)	$T(x,y)=(x^2+y, x-y)$	(d)	All the above	
	Ans	a			
iv.	Which of the following matrix is invertible?				
	(a)	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 0 & 0 & 0 \end{pmatrix}$	(b)	$\begin{pmatrix} 1 & 2 & 3 \\ 10 & 10 & 10 \\ 1 & 1 & 1 \end{pmatrix}$	
	(c)	$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	(d)	$\begin{pmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{pmatrix}$	
	Ans	c			
v.	If $A = cI_n, c \in \mathbb{R}$, where I_n is the identity matrix of order $n \times n$ then determinant of A is				
	(a)	c	(b)	1	
	(c)	c^n	(d)	none of these	
	Ans	(c)			
vi.	$Det(e_2, e_1+3e_2, -e_3)$ where e_1, e_2, e_3 are standard basis elements of $\mathbb{R}^3$ is				
	(a)	-1	(b)	0	
	(c)	1	(d)	3	
	Ans	(c)			
vii.	Consider the system $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 4 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ 2 \end{pmatrix}$ Then the value of 'z' satisfying above system by cramer's rule is				

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	(a)	$\dfrac{det\begin{pmatrix}0&0&1\\10&4&0\\2&0&1\end{pmatrix}}{det\begin{pmatrix}1&0&1\\2&4&0\\3&0&1\end{pmatrix}}$	(b)	$\dfrac{det\begin{pmatrix}1&0&1\\2&10&0\\3&2&1\end{pmatrix}}{det\begin{pmatrix}1&0&1\\2&4&0\\3&0&1\end{pmatrix}}$.
	(c)	$\dfrac{det\begin{pmatrix}1&0&0\\2&4&10\\3&0&2\end{pmatrix}}{det\begin{pmatrix}1&0&1\\2&4&0\\3&0&1\end{pmatrix}}$.	(d)	None of these.
	Ans	(c)		
viii.	Let $S = \{(1,-1),(1,1)\}$ in $\mathbb{R}^2$ with usual dot product. Consider the following: (i) S is a linearly independent set. (ii) S is an orthogonal set (iii) S is an orthonormal set			
	(a)	Only (i) is true	(b)	(i) and (ii) are true
	(c)	(i), (ii) and (iii) are all true	(d)	None of these.
	Ans	(b)		
ix.	Let $v = (a,b,c) \neq 0$ in $\mathbb{R}^3$. The set of all vectors orthogonal to v in $\mathbb{R}^3$ represents			
	(a)	A straight line passing through origin and v	(b)	A straight line passing through origin and perpendicular to v
	(c)	A plane through origin with normal v	(d)	None of these
	Ans	(c)		
x.	Let $x = \left(\frac{1}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right), y = \left(\frac{2}{\sqrt{30}}, \frac{3}{\sqrt{30}}\right)$. Then			
	(a)	$\{x,y\}$ is orthogonal in $\mathbb{R}^2$ under usual inner product	(b)	$\{x,y\}$ is not orthogonal under inner product $\langle u,v\rangle = 3u_1v_1 + 2u_2v_2$, where $u = (u_1,u_2), v = (v_1,v_2) \in \mathbb{R}^2$
	(c)	$\{x,y\}$ is orthogonal in $\mathbb{R}^2$ under inner product $\langle u,v\rangle = 3u_1v_1 + 2u_2v_2$, where $u = (u_1,u_2), v = (v_1,v_2) \in \mathbb{R}^2$	(d)	None of these
	Ans	(c)		(08)
Q2.	Attempt any **ONE** question from the following:			
a)	i.	Let V,W be vector spaces over $\mathbb{R}$ and $T:V\to W$ be a linear transformation and if V is finite dimensional then show that		

dim V = dim Ker T + dim Im T.

Ans

Proof : We have $T : V \to W$, be a linear transformation, ker $T \subseteq V$ is a subspace of V.

Let dim V = n, dim ker T = r, dim W = m

Let $B = \{u_1, u_2, \cdots\cdots, u_r\}$ be basis of ker T As ker T is subspace of V, B is a linearly independent subset of V and hence can be extended to a basis of V.

Let $B_1 = \{u_1, u_2, \cdots\cdots, u_r, u_{r+1}, \cdots\cdots, u_n\}$ be a basis of V, obtained by extension of B. 1

Let $w_i = T(V_{r+i}), \forall i = 1, \cdots\cdots, n-r$.

Claim : $B_2 = \{w_1, w_2, \cdots\cdots, w_{n-r}\}$ forms a basis of I_mT

Let us prove first, B_2 is linearly independent

Let $a_1, a_2, \cdots\cdots a_{n-r}$ be scalars such that

$a_1w_1 + a_2w_2 + \cdots\cdots + a_{n-r}w_{n-r} = 0$

But $T(u_{r+1}) = w_1, T(u_{r+2}) = w_2, \cdots\cdots T(u_n) = w_{n-r}$ 1

$\therefore\ a_1T(u_{r+1}) + a_2T(u_{r+2}) + \cdots\cdots + a_{n-r}T(u_n) = 0$

$\therefore\ T(a_1u_{r+1} + a_2u_{r+2} + \cdots\cdots + a_{n-r}u_n) = 0$ 1

$$\Rightarrow T\left(\sum_{i=1}^{n-r} a_i u_{r+i}\right) = 0$$

$$\Rightarrow \sum_{i=1}^{n-r} a_i u_{r+i} \in \ker T$$ 1

$\Rightarrow \exists\ b_1, b_2, \cdots\cdots, b_r$ scalar s.t.

$$\sum_{i=1}^{n-r} a_i u_{r+i} = \sum_{j=1}^{r} b_j u \cdots\cdots \text{as B is basis of ker T}$$

$\Rightarrow b_1u_1 + b_2u_2 + \cdots\cdots + b_ru_r - (a_1u_{r+1} + \cdots\cdots + a_{n-r}u_n) = 0$

As $B_1 = \{u_1, u_2, \cdots\cdots, u_r, u_{r+1}, \cdots\cdots u_n\}$ is lin independent

$\Rightarrow b_1 = b_2 = b_r = 0$ and

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(4)

		$a_1 = a_2 = a_{n-r} = 0$ $\Rightarrow \{w_1, w_2, \cdots, w_{n-r}\}$ is linearly independent $\cdots$ (I) **Claim :** $\{w_1, w_2, \cdots, w_{n-r}\}$ spans $I_m(T)$ Let $w \in I_m T$ $\Rightarrow \exists\, v \in V$ such that $T(v) = w$. As $B_1 = \{u_1, u_2, \cdots, u_r, u_{r+1}, u_{r+2}, \cdots u_n\}$ is a basis of V. $\Rightarrow \exists\, b_1, b_2, \cdots, b_n \in \mathbb{R}$ such that $v = b_1u_1 + b_2u_2 + \cdots + b_ru_r + b_{r+1}u_{r+1} + \cdots + b_nu_n$ As T is linear, $\therefore\ T(v) = T(b_1u_1 + \cdots + b_ru_r) + T(b_{r+1}, u_{r+1} + \cdots + b_nu_n)$ $= b_1T(u_1) + b_2T(u_2) + \cdots + b_rT(u_r) + b_r + 1T(u_{r+1}) + \cdots b_nT(u_n) \cdots$ as T is linear $\therefore\ T(v) = b_{r+1}T(u_{r+1}) + b_{r+2}T(u_{r+2}) + \cdots + b_nT(u_n)$ as $u_1, u_2, \cdots, u_r \in \ker T$ $\Rightarrow T(v) = b_{r+1}w_1 + b_{r+2}w_2 + \cdots + b_nw_{n-r}$ $\Rightarrow w = b_{r+1}w_1 + b_{r+2}w_2 \cdots + b_nw_n$. $\Rightarrow w \in \text{span}\{w_1, \cdots, w_{n-r}\}$ $\Rightarrow$ If $w \in I_mT \Rightarrow w \in \text{span}\{w_1, w_2, \cdots, w_{n-r}\}$ $\Rightarrow \{w_1, w_2, \cdots, w_{n-r}\}$ spans I_mT. . . . (II) $\Rightarrow \{w_1, w_2, \cdots w_{n-r}\}$ forms a basis of ImT $\cdots$ From (I) and (II) $\therefore\ \dim(I_mT) = n - r$ $\dim(\ker T) = r$ $\dim v = n = \dim(I_mT) + \dim(\ker T) = r + n - r$. $\therefore\ \boxed{\dim(V) = \dim(\ker T) + \dim(I_mT)}$ as $\boxed{n = r + n - r}$ $\therefore$ Rank nullity theorem is verified.	1 1 1
	ii.	Let $T : V \to V$ be a linear transformation where V is a finite dimensional vector space over $\mathbb{R}$ then prove that T is injective if and only if T is surjective.	

Ans	Let $T : V \to V$ .Suppose T is 1-1.		
	$\therefore \ker T = \{0\}$		
	$\therefore \dim(\ker T) = 0.$		
	By rank nullity theorem.		
	$\dim V = \text{nullity } T + \text{Rank of } T$		
	$\dim V = 0 + \text{Rank of } T$		
	$\dim V = \dim(I_m T)$	1	
	As $I_m T \subseteq V$ subspace of V such that	1	
	$\dim V = \dim I_m T$		
	$\Rightarrow I_m T = V$	1	
	$\therefore T(V) = V$		
	$\therefore$ T is onto.		
	Conversly, suppose T is onto.	1	
	$I_m T = V$		
	$T(V) = V.$		
	$\therefore \dim(I_m T) = \dim V$	1	
	$\therefore$ Rank nullity theorem,		
	$\text{din}(V) = \dim(\ker T) + \dim I_m T$	1	
	$\Rightarrow \dim V = \dim(\ker T) + \dim V$		
	$\Rightarrow \dim \ker T = 0.$	1	
	$\Rightarrow \ker T = \{0\}$	1	
	$\Rightarrow$ T is $1-1$		
	$\therefore$ T is $1-1$ if and only if T is onto.		
			(12)
Q.2	Attempt any **TWO** questions from the following:		
b)	i.	If $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T(x,y,z) = (x,y)$, show that T is linear. Find $\ker T$, basis of $\ker T$ and nullity T.	

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	Ans	Let $(x,y,z)\in ker\,T$ so that $x=0,y=0$ Thus we put $z=t$, $\Rightarrow ker\,T=\{t(0,0,1)/t\in\mathbb{R}\}$. Therefore the basis of $ker\,F$ is $(0,0,1)$ and nullity $T=1$.	1 1 2 2
	ii.	Check whether $T:\mathbb{R}^3\to\mathbb{R}^3$ given by $T(x,y,z)=(x+y,x-z,y+2z)$ is an isomorphism?	
	Ans	Let $(x,y,z)\in ker\,T$ so that $x+y=0,x-z=0,y+2z=0$ The matrix corresponding to this system $\begin{pmatrix}1&1&0\\1&0&-1\\0&1&2\end{pmatrix}$ whose row reduced form is $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$. Thus $x=y=z=0\Rightarrow ker\,F=\{0\}$ Since Dim V = 3 = dim $\mathbb{R}^3$, T is onto. Therefore T is invertible. Therefore T is an isomorphism	1 1 2 2
	iii.	Show that the following equations does not have a solution unless $a+c=2b$. $3x+4y+5z=a,4x+5y+6z=b,5x+6y+7z=c$	
	Ans	$\left(\begin{array}{ccc\|c}3&4&5&a\\4&5&6&b\\5&6&7&c\end{array}\right)$ After successive row transformation $\left(\begin{array}{ccc\|c}1&1&1&c-b\\0&1&2&5b-4c\\0&0&0&a-2b+c\end{array}\right)$ Therefore Rank (A) = 2 = Rank (A|B) if and only if $a+c=2b$.	1 3 2
	iv.	Test for consistency and if consistent solve the following system. $x+y+z=3,x+2y+3z=4,x+4y+9z=6$	
	Ans	$\left(\begin{array}{ccc\|c}1&1&1&3\\1&2&3&4\\1&4&9&6\end{array}\right)$ After successive row transformation	1 2 2

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		$\left(\begin{array}{ccc\|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \end{array}\right)$. Rank (A) = 3 =Rank (A|B). Therefore system is consistent. $x = 2, y = 1, z = 0$	1
Q3.	Attempt any **ONE** question from the following:		(08)
$a)$	i.	Let $v_1, v_2, \dots, v_n \in \mathbb{R}^n$. Show that I) $\det(v_1, v_2, \dots, v_n) = 0$ $if\ v_i = 0\ for\ some\ 1 \le i \le n$ II) If$\{v_1, v_2, \dots, v_n\}$ is linearly dependent then $\det(v_1, v_2, \dots, v_n) = 0$.	
	Ans	I) $\det(v_1, v_2, \dots, v_i, \dots, v_n)$ $= \det(v_1, v_2, \dots, 0, \dots, v_n)$ $= \det(v_1, v_2, \dots, 0.0, \dots, v_n)$ $= 0.\det(v_1, v_2, \dots, 0, \dots, v_n)$ $= 0$	4
		II) $\{v_1, v_2, \dots, v_n\}$ is linearly dependent $\exists v_i \in \{v_1, v_2, \dots, v_n\}$ such that $v_i = \sum_{j=1, j\neq i}^{n} \alpha_j v_j$ $\det(v_1, v_2, \dots, v_i, \dots, v_n)$ $= \det\left(v_1, v_2, \dots, v_i - \sum_{j=1, j\neq i}^{n} \alpha_j v_j, \dots, v_n\right)$ $= \det(v_1, v_2, \dots, 0, \dots, v_n)$ $= 0$	4
	ii.	State and prove Cramer's Rule .	
	Ans	Let $A \in M_n(\mathbb{R})$,det $A \neq 0$ then the Solution $x = (x_1, x_2, \dots, x_n)$ of the system $Ax = b$ is given by $x_j = \frac{\det(A_1, \dots, A_{j-1}, b, A_{j+1}, \dots, A_n)}{\det A}$ $\forall\ 1 \le j \le n$	2
		Consider $X_j = (e_1, \dots\dots, x, \dots\dots, e_n)$ which is obtained b replacing j^{th}Column of identity matrix by column vector $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$	2

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$$\therefore X_j = \begin{pmatrix} 1 & 0 & x_1 & 0 \\ 0 & 1 & x_2 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & x_n & 1 \end{pmatrix} \quad \forall\, j = 1\ to\ n$$

$$\begin{aligned} detX_j &= \det(e_1, e_2, \dots, x_1e_1 + x_2e_2 + \cdots . x_ne_n, \dots\dots, e_n) \\ &= \det(e_1, e_2, \dots, x_je_j, \dots\dots, e_n) \\ &= x_j \det(e_1, e_2, \dots, e_j, \dots\dots, e_n) \\ &= x_j \end{aligned}$$ 2

$$\begin{aligned} x_j &= detX_j \\ &= detA^{-1}AX_j \\ &= \det(A^{-1})\det(AX_j) \\ &= \frac{1}{\mathrm{detA}}\det(\mathrm{Ae_1}, \mathrm{Ae_2}, \dots, \mathrm{Ax}, \dots \mathrm{Ae_n}) \\ &= \frac{1}{\mathrm{detA}}\det(\mathrm{A^1}, \mathrm{A^2}, \dots, \mathrm{b}, \dots \mathrm{A^n}) \end{aligned}$$ 2

Q3. Attempt any **TWO** questions from the following: (12)

b) i. Define adjoint of a matrix. Find $\boldsymbol{A^{-1}}$ for $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$ using adjoint.

Ans

For $A \in M_n(\mathbb{R})$,
Let A_{ij} be the matrix obtained from A by deleting its ith row and jth column
Let $c_{ij} = (-1)^{i+j}\det A_{ij}$
$C = (c_{ij})$ is called matrix of cofactors
$adj(A) := C^t$ 2

Given matrix is $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$ $detA = -2$

Matrix of cofactors is $C = \begin{pmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{pmatrix}$ 2

$\mathrm{Adj}(A) = C^t = \begin{pmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{pmatrix}$

$A^{-1} = \frac{1}{\det A} Adj(A) = \frac{1}{-2}\begin{pmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{pmatrix}$ 2

ii. I) Use determinant to check whether the set $\{(3,1,2,0), (0,2,1,0), (0,0,-1,1), (0,0,1,1)\}$ is linearly dependent or independent? State the result used.

II) Use determinant to find volume of the parallelepiped spanned by vectors. $x = (1,1,0), y = (0,1,0)$ and $z=(0,0,1)$. State the result used

Ans

I)
Result:
Columns of matrix A are linearly dependent iff $\det A = 0$.

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		Since $\det\begin{pmatrix}3&0&0&0\\1&2&0&0\\2&1&-1&1\\0&0&1&1\end{pmatrix} = -12 \neq 0$ $\{(3,1,2,0),(0,2,1,0),(0,0,-1,1),(0,0,1,1)\}$ is linearly independent. II) Result: Volume of a parallelopiped spanned by vectors $x=(x_1,x_2,x_3), y=(y_1,y_2,y_3)$ and $z=(z_1,z_2,z_3)$ of $\mathbb{R}^3$ is $det\begin{pmatrix}x_1&y_1&z_1\\x_2&y_2&z_2\\x_3&y_3&z_3\end{pmatrix}$. Required volume = $\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix} = 1$ sq.units	3 3
	iii.	Solve the following system of linear equations using Cramer's rule $2x-y+z=1, x+3y-2z=1, 4x-3y+z=0$	
	Ans	$2x-y+z=1, x+3y-2z=1, 4x-3y+z=0$The corresponding non-homogeneous system is $\begin{pmatrix}2&-1&1\\1&3&-2\\4&-3&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}$ $x=\dfrac{det\begin{pmatrix}1&-1&1\\1&3&-2\\0&-3&1\end{pmatrix}}{\det\begin{pmatrix}2&-1&1\\1&3&-2\\4&-3&1\end{pmatrix}}=\dfrac{-5}{-12}$ $y=\dfrac{det\begin{pmatrix}2&1&1\\1&1&-2\\4&0&1\end{pmatrix}}{\det\begin{pmatrix}2&-1&1\\1&3&-2\\4&-3&1\end{pmatrix}}=\dfrac{-11}{-12}$ $z=\dfrac{det\begin{pmatrix}2&-1&1\\1&3&1\\4&-3&0\end{pmatrix}}{\det\begin{pmatrix}2&-1&1\\1&3&-2\\4&-3&1\end{pmatrix}}=\dfrac{-13}{-12}$	2 2 2
	iv.	For $A \in M_n(\mathbb{R})$, Prove that $\det A = \det A^t$ Further show that $\det(A^tB^t) = \det A.\det B$	

(10)

	Ans	$\det A$ $= \sum_{\sigma \epsilon S_n} sgn\sigma \; a_{1\,\sigma(1)}\, a_{2\,\sigma(2)} \cdots a_{n\,\sigma(n)}$ $= \sum_{\sigma^{-1} \epsilon S_n} sgn\sigma \; a_{\sigma^{-1}(1),1}\, a_{\sigma^{-1}(2),2} \cdots a_{\sigma^{-1}(n),n}$ $= \sum_{\sigma^{-1} \epsilon S_n} sgn\sigma^{-1} \; a_{\sigma^{-1}(1),1}\, a_{\sigma^{-1}(2),2} \cdots a_{\sigma^{-1}(n),n}$ $(\because sgn\sigma = sgn\sigma^{-1})$ $= \sum_{\tau \epsilon S_n} sgn\tau \; a_{\tau(1),1}\, a_{\tau(2),2} \cdots a_{\tau(n),n}$ $(\tau = \sigma)$ $= det A^t$	4
		$\det(A^t B^t) = det A^t . det B^t$ $= \det A . \det B$	2
Q4.	Attempt any **ONE** question from the following: (08)		
$a)$	i.	Let W be a subspace of a finite dimensional inner product space V over $\mathbb{R}$. Define $W^\perp$. Show that $V = W \oplus W^\perp$	
	Ans	Definition of orthogonal compliment of a set.	1M
		Let $\dim V = n, \dim W = r < n$ Let $B = \{u_1, u_2, \ldots u_r\}$ be an orthonormal basis of W. Let $v \in V$. Consider $w' = v - \langle v, u_1\rangle u_1 - \langle v, u_2\rangle u_2 - \cdots - \langle v, u_r\rangle u_r$ ----(1)	1M
		$\langle w', u_i\rangle = \langle v - \langle v, u_1\rangle u_1 - \cdots - \langle v, u_r\rangle u_r, u_i\rangle, \quad 1 \le i \le r$ $= \langle v, u_i\rangle - \langle v, u_1\rangle\langle u_1, u_i\rangle - \cdots - \langle v, u_r\rangle\langle u_r, u_i\rangle$ $= \langle v, u_i\rangle - \langle v, u_i\rangle = 0$ $\therefore \langle w', u_i\rangle = 0 \forall i = 1,2, \ldots r$ $\therefore \langle w', u\rangle = 0 \forall u \in W$ $\therefore w' \in W^\perp$	3M
		From (1), $v = w' + \langle v, u_1\rangle u_1 + \langle v, u_2\rangle u_2 + \cdots + \langle v, u_r\rangle u_r$ $\Rightarrow v = w' + w$ $\Rightarrow v \in W^\perp + W$ $\Rightarrow V \subseteq W^\perp + W$ Since W is a subspace of V, $W^\perp + W \subseteq V$ Hence $V = W^\perp + W$. Let $\alpha \in W^\perp \cap W \Rightarrow \alpha \in W^\perp \& \alpha \in W$ $\Rightarrow \langle \alpha, \alpha\rangle = 0$	2M

		$\Rightarrow \alpha = 0$ $\therefore V = W \oplus W^{\perp}$	1M
	ii.	Prove that $\{sinx, cosx\}$ is an orthogonal set of $V = C[-\pi,\pi]$ where $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$. Find the corresponding orthonormal set.	
	Ans	Let $f(x) = sinx, g(x) = cosx$ $\langle f, g\rangle = \int_{-\pi}^{\pi} sinxcosx\ dx$ $= \frac{1}{2}\int_{-\pi}^{\pi} 2sinxcosx\ dx$ $= \frac{-1}{4}[cos2x]_{-\pi}^{\pi} = 0$ $\therefore$ $\{sinx, cosx\}$is orthogonal set.	2M
		$\langle f, f\rangle = \int_{-\pi}^{\pi} sin^2x\ dx$ $= \int_{-\pi}^{\pi} \frac{1-cos2x}{2}\ dx$ $= \frac{1}{2}\left[x - \frac{sin2x}{2}\right]_{-\pi}^{\pi} = \pi$ $\lVert f\rVert = \sqrt{\pi}$ $\langle g, g\rangle = \int_{-\pi}^{\pi} cos^2x\ dx$ $= \int_{-\pi}^{\pi} \frac{1+cos2x}{2}\ dx$ $= \frac{1}{2}\left[x + \frac{sin2x}{2}\right]_{-\pi}^{\pi} = \pi$ $\lVert g\rVert = \sqrt{\pi}$	3M
		Hence the corresponding orthonormal set is $\left\{\frac{sinx}{\sqrt{\pi}}, \frac{cosx}{\sqrt{\pi}}\right\}$	3M
Q4.	Attempt any **TWO** questions from the following:		(12)
b)	i.	Define angle between two vectors in an inner product space. Find angle between p and q where $p(x) = x + 1$ and $q(x) = x^2$ with respect to inner product $\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$.	
	Ans	Definition of angle $\langle p, q\rangle = 2$ $\quad\langle p, p\rangle = 5$ $\quad\langle q, q\rangle = 2$ $\lVert p\rVert = \sqrt{5}$ $\quad\lVert q\rVert = \sqrt{2}$ $\cos\theta = \frac{\langle p, q\rangle}{\lVert p\rVert\lVert q\rVert} = \sqrt{\frac{2}{5}}$ $\therefore \theta = \cos^{-1}\sqrt{\frac{2}{5}}$	1 3 1 1
	ii.	Define inner product. Check whether the function given by $\langle x, y\rangle = x_1^2y_1^2 + x_2^2y_2^2$ is an inner product on $\mathbb{R}^2$ where $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$.	
	Ans	Definition of inner product Let $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ and $a \in \mathbb{R}$. Then $\langle ax, y\rangle = (ax_1)^2y_1^2 + (ax_2)^2y_2^2 = a^2(x_1^2y_1^2 + x_2^2y_2^2)$ $= a^2\langle x, y\rangle \neq a\langle x, y\rangle$ For example, let $a = 2, x = (1,1)$ and $y = (1,-1)$	2 2

(12)

		$\langle ax,y\rangle = 4+4 = 8$ But $a\langle x,y\rangle = 2(1+1) = 4 \quad \therefore \langle ax,y\rangle \neq a\langle x,y\rangle$ Hence, given function is not an inner product.	1 1
	iii.	Using Gram-Schmidt Process, find orthogonal set corresponding to the set $\{(1,1,1),(1,0,3),(1,2,3)\}$ in $\mathbb{R}^3$.	
	Ans	Let $x_1 = (1,1,1),\ x_2 = (1,0,3),\ x_3 = (1,2,3)$ Let $y_1 = x_1 = (1,1,1)$ $y_2 = x_2 - \frac{\langle x_2,y_1\rangle}{\langle y_1,y_1\rangle} y_1 = \left(-\frac{1}{3},-\frac{4}{3},\frac{5}{3}\right)$ $y_3 = x_3 - \frac{\langle x_3,y_1\rangle}{\langle y_1,y_1\rangle} y_1 - \frac{\langle x_3,y_2\rangle}{\langle y_2,y_2\rangle} y_2 = \left(-\frac{6}{7},\frac{4}{7},\frac{2}{7}\right)$ By Gram-Schmidt process, $\left\{(1,1,1),\left(-\frac{1}{3},-\frac{4}{3},\frac{5}{3}\right),\left(-\frac{6}{7},\frac{4}{7},\frac{2}{7}\right)\right\}$ is the orthogonal set corresponding to the given set.	1 2 2 1
	iv.	Let V be an inner product space over $\mathbb{R}$ and x,y be non-zero vectors in V. Prove that x and y are orthogonal vectors in V if and only if $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.	
	Ans	Assume x and y are orthogonal vectors in V. Hence $\langle x,y\rangle = 0$. Then $\|x+y\|^2 = \langle x+y,x+y\rangle = \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle$ $= \langle x,x\rangle + 0 + \langle y,y\rangle = \|x\|^2 + \|y\|^2$ Conversely assume $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ $\therefore \langle x+y,x+y\rangle = \langle x,x\rangle + \langle y,y\rangle$ $\langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle = \langle x,x\rangle + \langle y,y\rangle$ Hence $\langle x,y\rangle = 0$. Hence x and y are orthogonal vectors in V.	3 3
			(20)
Q5.		Attempt any **FOUR** questions from the following:	
a)		Let $T: V \to W$ be a linear transformation such that $\ker T = \{0\}$. Prove that if $v_1, v_2 \ldots\ldots v_n$ are linearly independent in V then $T(v_1), T(v_2) \ldots\ldots\ldots T(v_n)$ are linearly independent in W.	
Ans		Proof: Let $c_1Tv_1 + c_2Tv_2 + \ldots\ldots + c_nTv_n = o$ Since T is l.t. it follows that $T(c_1v_1 + \ldots\ldots + c_nv_n) = o$. As $\ker T = \{o\}$, this implies that $c_1v_1 + \ldots\ldots + c_nv_n = o$ which further implies that $c_1, c_2, \ldots\ldots c_n$ are all zero, proving that $Tv_1, Tv_2, \ldots\ldots Tv_n$ are l.i.	1 2 1 1
b)		Find the rank of $A = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 2 & 1 & 1 & 1 \\ 4 & -3 & 1 & 7 \end{pmatrix}$.	5
Ans		Rank $A = 2$	
c)		Determine whether the homogeneous system $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 6 & 1 \\ 3 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ has non trivial solution. When does the non homogeneous system $AX = b$ have unique solution? What will be that	

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	unique solution?	
Ans	Result: For $A \in M_n(\mathbb{R})$, homogeneous system $AX = 0$ has a non trivial solution if and only if $det\, A = 0$. Given homogeneous system is $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 6 & 1 \\ 3 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 6 & 1 \\ 3 & 0 & -1 \end{pmatrix} = -52 \neq 0$ $\therefore \begin{pmatrix} 1 & 2 & 3 \\ 1 & 6 & 1 \\ 3 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ does not have non trivial solution.	3
	The non homogeneous system $AX = b$ has unique solution iff $detA \neq 0$ That unique solution is $X = A^{-1}b$	2
d)	Define n-linear map. Further, prove that $\det\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & a_{nn} \end{pmatrix} = a_{11}a_{22}\dots a_{nn}$.	
Ans	$f: \mathbb{R}^n X \mathbb{R}^n X \dots \mathbb{R}^n \to \mathbb{R}$ is n-linear if for $v_1, v_2, \dots, v_i, v_i', \dots v_n \in \mathbb{R}^n$ and for $\alpha \in \mathbb{R}$ i) $f(v_1, \dots, v_i + v_i', \dots v_n) = f(v_1, \dots, v_i, \dots v_n) + f(v_1, \dots, v_i', \dots v_n)\ \forall\, 1 \leq i \leq n$ ii) $f(v_1, \dots, \propto v_i, \dots v_n) = \propto f(v_1, \dots, v_i, \dots v_n)\quad \forall\, 1 \leq i \leq n$	2
	$det\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & a_{nn} \end{pmatrix}$ $= \det(a_{11}e_1, a_{22}e_2, \dots a_{nn}e_n)$ $= a_{11}\det(e_1, a_{22}e_2, \dots a_{nn}e_n)$ ($\because$ determinant is n linear map) $= a_{11}a_{22}\det(e_1, e_2, \dots a_{nn}e_n)$ ($\because$ determinant is n linear map) $\vdots$ $= a_{11}a_{22} \cdot\cdot a_{nn}\det(e_1, e_2, \dots e_n)$ ($\because$ determinant is n linear map) $= a_{11}a_{22} \cdot\cdot a_{nn}$	3
e)	If $\|u + v\| = 4$, $\|u - v\| = 6$, $\|u\| = 1$, find $\langle u, v \rangle$ and $\|v\|$. Also verify triangle inequality.	
Ans	$\|u + v\|^2 = 4^2 = \langle u + v, u + v \rangle$ $16 = \langle u, u \rangle + 2\langle u, v \rangle + \langle v, v \rangle$----(1) $\|u - v\|^2 = 6^2 = \langle u - v, u - v \rangle$ $36 = \langle u, u \rangle - 2\langle u, v \rangle + \langle v, v \rangle$----(2)	2M
	(1)-(2), $-20 = 4\langle u, v \rangle$ $\langle u, v \rangle = -5$ Substituting the values of $\langle u, v \rangle$, $\|u\|$ in (1),	1M

	$16 = 1 + 2 \times -5 + \langle v, v\rangle$ $\langle v, v\rangle = 25$ $\|v\| = 5$ $\|u + v\| = 4, \|u\| + \|v\| = 6$ $\therefore \|u + v\| \leq \|u\| + \|v\|$,Triangle inequality verified.	1M 1M
$f)$	Verify Cauchy-Schwarz inequality for the vectors $u = (1,0,0,1), v = (1,-1,1,0) \in \mathbb{R}^4$ where $\langle u, v\rangle = 3x_1y_1 + 4x_2y_2 + 2x_3y_3 + x_4y_4$, $u = (x_1, x_2, x_3, x_4)$ & $v = (y_1, y_2, y_3, y_4)$ and find angle between them.	
Ans	$\langle u, v\rangle = 3 \times 1 \times 1 + 4 \times 0 \times -1 + 2 \times 0 \times 1 + 1 \times 0 = 3$ $\|u\|^2 = 3 \times 1 \times 1 + 4 \times 0 \times 0 + 2 \times 0 \times 0 + 1 \times 1 = 4$ $\|u\| = 2$ $\|v\|^2 = 3 \times 1 \times 1 + 4 \times -1 \times -1 + 2 \times 1 \times 1 + 0 \times 0 = 9$ $\|v\| = 3$ $\langle u, v\rangle = 3, \|u\|\|v\| = 6$ $\|\langle u, v\rangle\| \leq \|u\|\|v\|$, Cauchy-Schwarz inequality verified. $\cos\theta = \frac{\langle u,v\rangle}{\|u\|\|v\|} = \frac{3}{6} = \frac{1}{2}$ $\therefore \theta = cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$	3M 1M 1M

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