① Q.P. Code - 54965

(3 Hours) [Total Marks: 100]

Note: (*i*) All questions are compulsory.
(*ii*) Figures to the right indicate marks for respective parts.

Q.1	Choose correct alternative in each of the following (20)			
i.	$T:\mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation if $\forall u, v \in \mathbb{R}^2, \forall \alpha, \beta \in \mathbb{R}$,			
	(a)	$T(\alpha u+\beta v)=\alpha T(u)+\beta T(v)$	(b)	$T(\alpha uv)=\alpha T(u)\bullet T(v)$
	(c)	$T(\alpha u+\beta v)=\alpha T(u)\bullet \beta T(v)$	(d)	None of the above
	Ans	a		
ii.	If $T:U \to V$ is a linear transformation then			
	a	$T(0)=0$	(b)	$T(-u)=-T(u), \forall u \in U$
	c	$T(u_1-u_2)=T(u_1)-T(u_2)$, $\forall u_1, u_2 \in U$	(d)	All of the above
	Ans	d		
iii.	Which of the following is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$?			
	(a)	$T(x,y)=(xy,y)$	(b)	$T(x,y)=(x+1, y+1)$
	(c)	$T(x,y)=(x+y, x-y)$	(d)	All the above
	Ans	c		
iv.	If $\mathbf{A}=\begin{pmatrix}4 & -1\\ 2 & -2\end{pmatrix}$ then to get $\mathbf{EA}=\begin{pmatrix}0 & 3\\ 2 & -2\end{pmatrix}$, **E** is given by			
	(a)	$\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$	(b)	$\begin{pmatrix}0 & -2\\ -1 & 0\end{pmatrix}$
	(c)	$\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}$	(d)	$\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix}$
	Ans	d		
v.	Which one of the following is NOT TRUE			
	(a)	$Det(A^t) = Det\ A$	(b)	$Det(A+B) = DetA + DetB$
	(c)	$DetAB = DetA\ DetB$	(d)	$Det(A^{-1}) = (Det\ A)^{-1}$, when A is invertible.
	Ans	(b)		
vi.	$Det(e_2, 2e_1+3e_2, e_3)$ where e_1, e_2, e_3 are standard basis elements of $\mathbb{R}^3$ is			
	(a)	-1	(b)	0
	(c)	1	(d)	-2
	Ans	(d)		
vii.	Let $A \in M_n(\mathbb{R})$ be an invertible matrix then $det(Adj\ A)$ is			
	(a)	$(\det A)^n$	(b)	$(\det A)^{n-1}$
	(c)	$\det A^{-1}$	(d)	None of these
	Ans	(b)		
viii.	Let V be a finite dimensional inner product space and W be a subspace of V. Then $(W^{\perp})^{\perp}$ is equal to			
	(a)	V	(b)	W

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	(c)	$W^{\perp}$	(d)	$V\backslash W$
	Ans	(b)		
ix.	Which of the following is not an inner product on $\mathbb{R}^2$ for $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$?			
	(a)	$\langle x, y\rangle = 2x_1y_1 + 3x_2y_2$	(b)	$\langle x, y\rangle = x_1y_1 + x_2y_2$
	(c)	$\langle x, y\rangle = x_1y_1 - x_2y_2$	(d)	$\langle x, y\rangle = x_1y_1 + 4x_2y_2$
	Ans	(c)		
x.	If $\{v_1, v_2\}$ is an orthonormal basis for $\mathbb{R}^2$, then for $x = 3v_1 + 7v_2$ and $y = 3v_1 - 7v_2,\ x \cdot y$ is			
	(a)	- 40	(b)	40
	(c)	0	(d)	None of these.
	Ans	(a)		
Q2.	Attempt any **ONE** question from the following: (08)			
a)	i.	State and prove Rank-Nullity Theorem.		
	Ans			1

(3)

Proof : We have $T : V \to W$, be a linear transformation, $\ker T \subseteq V$ is a subspace of V.

Let $\dim V = n$, $\dim \ker T = r$, $\dim W = m$

Let $B = \{u_1, u_2, \cdots, u_r\}$ be basis of ker T As ker T is subspace of V, B is a linearly independent subset of V and hence can be extended to a basis of V.

Let $B_1 = \{u_1, u_2, \cdots, u_r, u_{r+1}, \cdots, u_n\}$ be a basis of V, obtained by extension of B.

Let $w_i = T(V_{r+i}), \forall i = 1, \cdots, n-r$.

Claim : $B_2 = \{w_1, w_2, \cdots, w_{n-r}\}$ forms a basis of $I_m T$

Let us prove first, B_2 is linearly independent

Let $a_1, a_2, \cdots a_{n-r}$ be scalars such that

$a_1 w_1 + a_2 w_2 + \cdots + a_{n-r} w_{n-r} = 0$

But $T(u_{r+1}) = w_1, T(u_{r+2}) = w_2, \cdots T(u_n) = w_{n-r}$

$\therefore a_1 T(u_{r+1}) + a_2 T(u_{r+2}) + \cdots + a_{n-r} T(u_n) = 0$

$\therefore T(a_1 u_{r+1} + a_2 u_{r+2} + \cdots + a_{n-r} u_n) = 0$

$$\Rightarrow T\left(\sum_{i=1}^{n-r} a_i u_{r+i}\right) = 0$$ 3

$$\Rightarrow \sum_{i=1}^{n-r} a_i u_{r+i} \in \ker T$$

$\Rightarrow \exists\, b_1, b_2, \cdots, b_r$ scalar s.t.

$$\sum_{i=1}^{n-r} a_i u_{r+i} = \sum_{j=1}^{r} b_j u \cdots \text{as B is basis of ker T}$$

$\Rightarrow b_1 u_1 + b_2 u_2 + \cdots + b_r u_r - (a_1 u_{r+1} + \cdots + a_{n-r} u_n) = 0$

As $B_1 = \{u_1, u_2, \cdots, u_r, u_{r+1}, \cdots u_n\}$ is lin independent

$\Rightarrow b_1 = b_2 = b_r = 0$ and

(4)

$a_1 = a_2 = a_{n-r} = 0$

$\Rightarrow \{w_1, w_2, \cdots, w_{n-r}\}$ is linearly independent $\cdots$ (I)

Claim : $\{w_1, w_2, \cdots, w_{n-r}\}$ spans $I_m(T)$

Let $w \in I_mT$

$\Rightarrow \exists\, v \in V$ such that $T(v) = w$.

As $B_1 = \{u_1, u_2, \cdots, u_r, u_{r+1}, u_{r+2}, \cdots u_n\}$ is a basis of V.

$\Rightarrow \exists\, b_1, b_2, \cdots, b_n \in \mathbb{R}$ such that

$v = b_1u_1 + b_2u_2 + \cdots + b_ru_r + b_{r+1}u_{r+1} + \cdots + b_nu_n$

As T is linear,

$\therefore\ T(v) = T(b_1u_1 + \cdots + b_ru_r) + T(b_{r+1}, u_{r+1} + \cdots + b_nu_n)$

$= b_1T(u_1) + b_2T(u_2) + \cdots + b_rT(u_r) + b_r + 1T(u_{r+1}) + \cdots b_nT(u_n) \cdots$

as T is linear

$\therefore\ T(v) = b_{r+1}T(u_{r+1}) + b_{r+2}T(u_{r+2}) + \cdots + b_nT(u_n)$

as $u_1, u_2, \cdots, u_r \in \ker T$

$\Rightarrow T(v) = b_{r+1}w_1 + b_{r+2}w_2 + \cdots + b_nw_{n-r}$

$\Rightarrow w = b_{r+1}w_1 + b_{r+2}w_2 \cdots + b_nw_n.$

$\Rightarrow w \in \text{span}\,\{w_1, \cdots, w_{n-r}\}$

$\Rightarrow$ If $w \in I_mT \Rightarrow w \in \text{span}\,\{w_1, w_2, \cdots, w_{n-r}\}$

$\Rightarrow \{w_1, w_2, \cdots, w_{n-r}\}$ spans I_mT. ... (II)

$\Rightarrow \{w_1, w_2, \cdots w_{n-r}\}$ forms a basis of ImT $\cdots$ From (I) and (II)

$\therefore\ \dim(I_mT) = n - r$

$\dim(\ker T) = r$

$\dim v = n = \dim(I_mT) + \dim(\ker T) = r + n - r.$ 4

$\therefore$ $\boxed{\dim(V) = \dim(\ker T) + \dim(I_mT)}$

as $\boxed{n = r + n - r}$

$\therefore$ Rank nullity theorem is verified.

	ii.	Let $A \in M_n(\mathbb{R})$. Prove that the system $AX=B$ of n non-homogenous linear equations in n unknowns has a unique solution if and only if rank$(A) = n$.
	Ans	

Proof : Given A is $n \times n$ matrix.

and suppose rank of $A = n$.

$\therefore$ A is row equivalent to I. 1

i.e. A is invertible.

Observe A is $n \times n$ matrix with rank of $A = n$.

rank of $A|B = n$ or $n + 1$ 1

But augemented matrix $[A|B]$ is of order $n \times n + 1$

$\Rightarrow$ rank of $[A : B] = n =$ rank of A

$\therefore$ The nonhomogenous system of linear equations $Ax = B$ is consistent. 1

i.e. solution of $Ax = B$ exists.

Let x be any solution of $Ax = B$

then left multiplication by A^{-1} concludes

$$A^{-1}Ax = A^{-1}B.$$

$$\Rightarrow x = A^{-1}B$$

$\therefore$ If x is a solution then $x = A^{-1}B$. 1

$\therefore$ The solution exists uniquely.

Conversly, if $Ax = B$ has a unique solution i.e. whenever x_1, x_2 are solutions such that

$$Ax_1 = B$$

$$Ax_2 = B$$ 2

$$\Rightarrow Ax_1 = Ax_2$$

$$\Rightarrow A(x_1 - x_2) = 0$$ 1

i.e. whenever $A(x_1 - x_2) = 0$

$$\Rightarrow x_1 - x_2 = 0$$

$$\Rightarrow x_1 = x_2$$ 1

i.e. $Ax = 0$ has a unique solution.

rather $Ax = 0$ has no nontrivial solution.

$\Rightarrow$ By earlier result A must be of rank n.

$\Rightarrow$ rank $A = n$.

Hence the theorem.

Q.2	Attempt any **TWO** questions from the following:	(12)

b)	i.	Show that F is non-singular where $F : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $F(x,y,z) = (x+y-2z, x+2y+z, 2x+2y-3z)$.	
	Ans	Let $(x,y,z) \in ker T$ so that $x+y-2z=0, x+2y+z=0, 2x+2y-3z=0$	1
		The matrix corresponding to this system $\begin{pmatrix} 1 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & 2 & -3 \end{pmatrix}$ whose row reduced	2
		form is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Thus $x = y = z = 0 \Rightarrow ker F = \{0\}$	2
		Therefore F is non-singular.	1
	ii.	$T : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $T(x,y,z) = (x+2y-z, y+z, x+y-2z)$. Find the basis for Ker T and Nullity T.	
	Ans	Let $(x,y,z) \in ker T$ so that $x+2y-z=0, y+z=0, x+y-2z=0$	1
		The matrix corresponding to this system $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix}$ whose row reduced	2
		form is $\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Thus putting $z = t$, we get $y = -t$, and $x = 3t$	1
		$\Rightarrow ker F = \{t(3,-1,1) / t \in \mathbb{R}\}$. Therefore the basis of $ker F$ is $(3,-1,1)$	1
		and is of dimension 1.	1
	iii.	Show that any n-dimensional real vector space is isomorphic to $\mathbb{R}^n$.	
	Ans	Let $x_1, x_2 \ldots\ldots x_n$ be a basis of V of dim n. Therefore for	1
		$x \in V, x = \sum c_i x_i \quad \forall c_i \in \mathbb{R}$	1
		Thus a LT is defined by $T(x) = (c_1, \ldots\ldots\ldots c_n)$ is one-one.	2
		Since Dim V = n = dim $\mathbb{R}^n$, T is onto. Therefore T is invertible. Therefore	1
		$V \cong \mathbb{R}^n$.	1
	iv.	Test for consistency and if consistent solve the system: $2x-3y+7z=5, 3x+y-3z=13, 2x+19y-47z=32$	
	Ans	$\left(\begin{array}{ccc|c} 2 & -3 & 7 & 5 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{array}\right)$	1

After successive row transformation

$$\left(\begin{array}{ccc|c} 1 & 4 & -10 & 8 \\ 0 & -11 & 27 & -11 \\ 0 & 0 & 0 & 5 \end{array}\right).$$

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Rank (A) = 2 and Rank (A|B) = 3. Therefore system is inconsistent.

Q3. Attempt any **ONE** question from the following: (08)

a) i. Let $A \in M_n(\mathbb{R})$ then prove that A is invertible if and only if Columns of A are linearly independent. Hence, prove that if $\det A = 0$ then columns of A are linearly dependent.

Ans

($\Rightarrow$) Given: A is invertible

T.P.T: Columns of A are linearly independent.

Consider $\alpha_1 A^1 + \alpha_2 A^2 + \cdots + \alpha_n A^n = 0$

$\alpha_1 Ae_1 + \alpha_2 Ae_2 + \cdots + \alpha_n Ae_n = 0$ $\quad (\because Ae_i = i^{th}\text{column of A})$

$A\alpha_1 e_1 + A\alpha_2 e_2 + \cdots + A\alpha_n e_n = 0$

$A(\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n) = 0$

$\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n = 0$ $\quad (\because$ A is invertible)

$\therefore \alpha_i = 0 \;\forall\, i = 1 \text{ to } n$

$(\because \{e_1, e_2, \ldots . e_n\}$is linearly independent)

$\alpha_1 A^1 + \alpha_2 A^2 + \cdots + \alpha_n A^n = 0 \Rightarrow \alpha_i = 0 \;\forall\, i = 1 \text{ to } n$

$\{A^1, A^2, \ldots, A^n\}$is linearly indpendent subset of $\mathbb{R}^n$

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($\Leftarrow$) Given: Columns of A are linearly independent

T.P.T: A is invertible

$\{A^1, A^2, \ldots, A^n\}$is linearly indpendent & $|\{A^1, A^2, \ldots, A^n\}| = n$

$\because \{A^1, A^2, \ldots, A^n\}$ is the basis of $\mathbb{R}^n$

$$e_j = \sum_{i=1}^{n} b_{ij} A^i \quad \ldots \forall j = i \text{ to } n$$

$$AB = (AB^1, AB^2, \ldots, AB^n) = \left(\sum_{i=1}^{n} b_{i1} A^i, \sum_{i=1}^{n} b_{i2} A^i, \ldots \sum_{i=1}^{n} b_{in} A^i\right)$$

$= (e_1, e_2, \ldots . e_n) = I_n$

Hence, A is invertible.

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Now,

$\det A = 0$

T. P.T: Columns of A are linearly dependent.

Let if Columns of A are linearly independent

$\therefore A$ is invertible

$\therefore AA^{-1} = I$

$\therefore \det A \neq 0$ which is a contradiction.

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ii. Let $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$. Show that

I) $\det(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) = \det(v_1, \ldots, v_i + \alpha v_j, \ldots, v_j, \ldots, v_n)$ for $1 \leq i \neq j \leq n$ and $\alpha \in \mathbb{R}$

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II) $\det(v_1,..,v_i,..,v_j,..,v_n) = -\det(v_1,..,v_j,..,v_i,..,v_n)$ $for\ 1 \le i \ne j \le n$

Ans

(I) Consider

$$\alpha \det(v_1,..,v_i,..,v_j,..,v_n)$$
$$= \det(v_1,..,v_i,..,\propto v_j,..,v_n)$$
$$= \det(v_1,..,v_i + \propto v_j,..,\propto v_j,..,v_n)$$
$$= \propto \det(v_1,..,v_i + \propto v_j,..,v_j,..,v_n)$$

As $\propto \ne 0$

$$\therefore \det(v_1,..,v_i,..,v_j,..,v_n) = \det(v_1,..,v_i + \propto v_j,..,v_j,..,v_n)$$

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(II)

As $\det(v_1,..,v_i + v_j,..,v_i + v_j,..,v_n) = 0$

$$\det(v_1,..,v_i,..,v_j,..,v_n) + \det(v_1,..,v_i,..,v_j,..,v_n)$$
$$+ \det(v_1,..,v_j,..,v_i,..,v_n) + \det(v_1,..,v_j,..,v_j,..,v_n) = 0$$

$$\det(v_1,..,v_i,..,v_j,..,v_n) + \det(v_1,..,v_j,..,v_i,..,v_n) = 0$$
$$\det(v_1,..,v_j,..,v_i,..,v_n) = -\det(v_1,..,v_i,..,v_j,..,v_n)$$

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Q3. Attempt any **TWO** questions from the following: (12)

b) i. Let $A \in M_n(\mathbb{R})$. show that $A.\,adj(A) = \det A\,.\,I_n$, where I_n is $n \times n$ identity matrix.

Ans

Claim:

$$a_{i1}C_{k1} + a_{i2}C_{k2} + \cdots + a_{in}C_{kn} = 0 \qquad \text{if } i \ne k$$
$$= \det A \qquad \text{if } i = k$$

If $i = k$ then by Laplace expansion

$a_{i1}C_{k1} + a_{i2}C_{k2} + \cdots + a_{in}C_{kn} = \det A$

If $i \ne k$

Consider the matrix B obtained from matrix A by replacing k^th of A by i^th row.

$\therefore a_{i1}C_{k1} + a_{i2}C_{k2} + \cdots + a_{in}C_{kn} = 0 \qquad \text{if } i \ne k$

Hence, the claim is proved.

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$$\sum_{j=1}^{n} a_{ij}(-1)^{k+j} \det A_{kj} = \delta_{ik} \det A$$

$A.\,Adj\ A = \delta_{ik} \det A$

A. $Adj\ A = \det A\,.\,I$ (As $\delta_{ik} = 1$ if $i = k$
$= 0$ if $i \ne k$)

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ii. Solve the following system of linear equations using Cramer's rule

$2x - y + z = 1, x + 3y - 2z = 1, 4x - 3y + z = 0$

Ans

$2x - y + z = 1, x + 3y - 2z = 1, 4x - 3y + z = 0$

The corresponding non- homogeneous system is

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & -2 \\ 4 & -3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$x = \frac{det\begin{pmatrix} 1 & -1 & 1 \\ 1 & 3 & -2 \\ 0 & -3 & 1 \end{pmatrix}}{\det\begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & -2 \\ 4 & -3 & 1 \end{pmatrix}} = \frac{5}{12}$$ [2]

$$y = \frac{det\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & -2 \\ 4 & 0 & 1 \end{pmatrix}}{\det\begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & -2 \\ 4 & -3 & 1 \end{pmatrix}} = \frac{11}{12}$$ [2]

$$z = \frac{det\begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & 1 \\ 4 & -3 & 0 \end{pmatrix}}{\det\begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & -2 \\ 4 & -3 & 1 \end{pmatrix}} = \frac{13}{12}$$ [2]

iii. For $A, B \in M_n(\mathbb{R})$, if A is invertible show that

I) $\det(A^{-1}) = (\det A)^{-1}$

II) $\det(ABA^{-1}) = \det B$

III) $\det(adj\ A) = (\det A)^{n-1}$

Ans

I)

$AA^{-1} = I$

$\det(A.A^{-1}) = \det I$

$\det(A).\det(A^{-1}) = 1$ $(\because \det(AB) = detA.\det B)$

$\det(A^{-1}) = (\det A)^{-1}$ $(As\ A$ is invertible $\therefore \det A \neq 0)$ [2]

II)

$\det(ABA^{-1})$

$= \det(A)\det(B)\det(A^{-1})$ $(\because \det(AB) = detA.\det B)$

$= \det A\ \det B\ (detA)^{-1}$ $(\because \det(A^{-1}) = (\det A)^{-1})$

$= \det B$ [2]

III)

$A.Adj\ A = \det A.I$

$\det(A.\ Adj\ A) = \det(\det A.I)$

$\det A.\det(Adj\ A) = (detA)^n$ $(\because \det(AB) = detA.\det B)$

$\det(adj\ A) = (\det A)^{n-1}$ $(As\ A$ is invertible $\therefore \det A \neq 0)$ [2]

(C)

iv. Define adjoint of a matrix. Find A^{-1} for $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ using adjoint.

Ans

For $A, \in M_n(\mathbb{R})$,

Let A_{ij} be the matrix obtained from A by deleting its ith row and jth column

Let $c_{ij} = (-1)^{i+j} \det A_{ij}$

$C = (c_{ij})$ is called matrix of cofactors

$adj(A) := C^t$

Given matrix is $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ — 2

Matrix of cofactors is $C = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix}$

$\text{Adj}(A) = C^t = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{pmatrix}$ — 2

$A^{-1} = \frac{1}{\det A} Adj(A) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{pmatrix}$ — 2

Q4. Attempt any **ONE** question from the following: **(08)**

a) i. Define inner product and inner product space over $\mathbb{R}$. Show that $(P_2[x], \langle\ ,\ \rangle)$ is an inner product space over $\mathbb{R}$ where $\langle p, q\rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$ for $p(x) = a_0 + a_1x + a_2x^2$, $q(x) = b_0 + b_1x + b_2x^2$.

Ans Definition of inner product and inner product space. — 2 M

(i) $\langle p, q\rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$
$= q(0)p(0) + q(1)p(1) + q(2)p(2)$
$= \langle q, p\rangle$ — 1 M

(ii) $\langle p+q, r\rangle = (p+q)(0)r(0) + (p+q)(1)r(1) + (p+q)(2)r(2)$
$= p(0)r(0) + p(1)r(1) + p(2)r(2) + q(0)r(0) + q(1)r(1) + q(2)r(2)$
$= \langle p, r\rangle + \langle q, r\rangle$ — 1 M

(iii) $\langle kp, q\rangle = k\,p(0)q(0) + kp(1)q(1) + kp(2)q(2)$
$= k[p(0)q(0) + p(1)q(1) + p(2)q(2)]$
$= k\langle p, q\rangle$ — 1 M

(iv) $\langle p, p\rangle = p(0)^2 + p(1)^2 + p(2)^2 \geq 0$ — 1M

(v) $\langle p, p\rangle = 0 \Rightarrow p(0) = 0, p(1) = 0, p(2) = 0$

(11)

		$\Rightarrow a_0 = 0, a_0 + a_1 = 0, a_0 + 2a_1 + 4a_2 = 0$ $\Rightarrow a_0 = a_1 = a_2 = 0$ $\Rightarrow p = 0$	2M
	ii.	Define orthogonal and orthonormal sets. Let $\{x_1, x_2, \cdots, x_n\}$ be an orthonormal basis of an inner product space V. Let $x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$. Then prove the following: (p) $\alpha_i = \langle x, x_i \rangle$ for $i = 1,2,..,n$ (q) $\| x \|^2 = \sum_{i=1}^{n} \langle x, x_i \rangle^2$	
	Ans	Definition of orthogonal and orthonormal sets. $x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ $(p) \quad \langle x, x_i \rangle = \langle \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, x_i \rangle \quad ; i = 1,2,..n$ $= \alpha_1 \langle x_1, x_i \rangle + \alpha_2 \langle x_2, x_i \rangle + \cdots + \alpha_n \langle x_n, x_i \rangle$ $= \alpha_i \langle x_i, x_i \rangle = \alpha_i \quad$ ($\{x_1, x_2, \cdots, x_n\}$ is orthonormal) $(q) \quad \|x\|^2 = \langle x, x \rangle$ $= \langle x, \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \rangle$ $= \alpha_1 \langle x, x_1 \rangle + \alpha_2 \langle x, x_2 \rangle + \cdots + \alpha_n \langle x, x_n \rangle$ $= \langle x, x_1 \rangle \langle x, x_1 \rangle + \langle x, x_2 \rangle \langle x, x_2 \rangle + \cdots + \langle x, x_n \rangle \langle x, x_n \rangle$ $= \sum_{i=1}^{n} \langle x, x_i \rangle^2$ (from(i))	2M 3M 3M
Q4.	Attempt any **TWO** questions from the following: (12)		
b)	i.	Define angle between two vectors in a real inner product space. Find angle between $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ with respect to the inner product $\langle A, B \rangle = tr(AB^t)$ on $M_2(\mathbb{R})$.	
	Ans	Definition of angle $\langle A, B \rangle = 1 \qquad \langle A, A \rangle = 2 \qquad \langle B, B \rangle = 2$ $\|A\| = \sqrt{2} \qquad \|B\| = \sqrt{2}$ $\cos\theta = \frac{\langle A, B \rangle}{\|A\|\|B\|} = \frac{1}{2}$ $\therefore \theta = \frac{\pi}{3}$	1 3 1 1
	ii.	Prove that an orthogonal set in a real inner product space V is linearly independent.	
	Ans	Let $S = \{x_1, x_2, ..., x_n\}$ be an orthogonal set in real inner product space V. Hence $x_1, x_2, ..., x_n$ are all non-zero vectors of V such that $\langle x_i, x_j \rangle = 0$ for $i \neq j, \quad i, j = 1,2, ..., n$ ....(*) Let $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$ where $a_1, a_2, ..., a_n \in \mathbb{R}$. For any $i = 1,2, ..., n$, consider $\langle x_i, a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \rangle = \langle x_i, 0 \rangle = 0$ $\therefore \langle x_i, \sum_{j=1}^{n} a_j x_j \rangle = 0 \Rightarrow \sum_{j=1}^{n} \langle x_i, a_j x_j \rangle = 0$	 1 3

		$\Rightarrow \langle x_i, a_i x_i\rangle + \sum_{\substack{j=1 \\ j\neq i}}^{n} \langle x_i, a_j x_j\rangle = 0$ $\Rightarrow a_i\langle x_i, x_i\rangle + 0 = 0$ ......from (*) Since $x_i \neq 0, \langle x_i, x_i\rangle \neq 0 \quad \Rightarrow a_i = 0$ for all $i = 1,2,\dots,n$ Hence, S is linearly independent set.	1 1
	iii.	Let W be a subspace of a real inner product space V. Define $W^\perp$, the orthogonal complement of W. Show that $W^\perp$ is a subspace of V.	
	Ans	Definition of $W^\perp$ Since $\langle 0, w\rangle = 0$ for all $w \in W$, $0 \in W^\perp$. Hence $W^\perp$ is non-empty. Let $u, v \in W^\perp$ and $\alpha, \beta \in \mathbb{R}$ Since $u, v \in W^\perp, \langle u, w\rangle = 0 = \langle v, w\rangle \ \ \forall w \in W$ Consider $\langle \alpha u + \beta v, w\rangle = \langle \alpha u, w\rangle + \langle \beta v, w\rangle$ $= \alpha\langle u, w\rangle + \beta\langle v, w\rangle$ $= \alpha.0 + \beta.0 = 0 \quad \forall w \in W$ $\therefore \alpha u + \beta v \in W^\perp$. Hence $W^\perp$ is a subspace of V	2 1 1 2
	iv.	Apply Gram-Schmidt process to obtain orthogonal set corresponding to $\{(0,1,1), (1,-1,0), (2,0,1)\}$ in $\mathbb{R}^3$ with dot product.	
	Ans	Let $x_1 = (0,1,1),\ x_2 = (1,-1,0),\ x_3 = (2,0,1)$ Let $y_1 = x_1 = (0,1,1)$ $y_2 = x_2 - \frac{\langle x_2, y_1\rangle}{\langle y_1, y_1\rangle} y_1 = \left(1, -\frac{1}{2}, \frac{1}{2}\right)$ $y_3 = x_3 - \frac{\langle x_3, y_1\rangle}{\langle y_1, y_1\rangle} y_1 - \frac{\langle x_3, y_2\rangle}{\langle y_2, y_2\rangle} y_2 = \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)$ By Gram-Schmidt process, $\left\{(0,1,1), \left(1, -\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)\right\}$ is the orthogonal set corresponding to the given set.	1 2 2 1
Q5.	Attempt any **FOUR** questions from the following: (20)		
a)	Prove that if $T : V \to V'$ is a linear transformation then T is injective if and only if $\ker T = \{0\}$.		
Ans	**Proof:** Suppose T is injective and $x \in \ker T$. Then $Tx = o$. However $To = o$. $\therefore$ $Tx = To$ forcing x to be o. Thus $\ker T = \{o\}$. Conversely, suppose $\ker T = \{o\}$. If $Tx = Ty$, then since T is l.t. we have $T(x - y) = Tx - Ty = o$, implying that $x - y \in \ker T$. $\therefore \quad x - y = o$ or $x = y$. This proves that T is injective.		1 1 2 1

b) Find the rank of $A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -1 & 1 \\ 4 & -1 & 2 \\ -1 & 1 & -1 \end{pmatrix}$.

Ans Rank $A = 3$

5

c) Use the following expression of determinant

$$\det A = \sum_{\sigma \in S_n} sgn\sigma \; a_{1\,\sigma(1)}\, a_{2\,\sigma(2)} \dots .\, a_{n\,\sigma(n)}$$

to find the determinant of the matrix $\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

Ans

$$\det A = \sum_{\sigma \in S_3} sgn\sigma \; a_{1\,\sigma(1)}\, a_{2\,\sigma(2)} a_{3\,\sigma(3)}$$

$$S_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$$

1

$$\det A = sgn \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} a_{11}a_{22}a_{33} + sgn \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} a_{11}a_{23}a_{32}$$
$$+ sgn \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} a_{12}a_{21}a_{33} + sgn \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} a_{12}a_{23}a_{31}$$
$$+ sgn \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} a_{13}aa_{21}a_{32} + sgn \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} a_{13}a_{22}a_{31}$$

2

$$= (1)(1)(1)(1) + (-1)(1)(3)(0) + (-1)(0)(2)(1) +$$
$$(1)(0)(3)(0) + (1)(2)(2)(0) + (-1)(2)(1)(0)$$
$$= 1 + 0 + 0 + 0 + 0 + 0 \quad = 1$$

2

d) I) Use determinant to check whether the homogeneous system $\begin{pmatrix} 1 & 2 & 3 \\ 1 & -6 & 1 \\ 7 & 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ has non trivial solution. State the result used.

II) Use determinant to find area of the parallelogram spanned by vectors. $x = (5,6)$ and $y = (2,5)$. State the result used.

Ans I)

Result:

For $A \in M_n(\mathbb{R})$, homogeneous system $AX = 0$ has a non trivial solution if and only if $det\ A = 0$.

Given homogeneous system is $\begin{pmatrix} 1 & 2 & 3 \\ 1 & -6 & 1 \\ 7 & 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 1 & -6 & 1 \\ 7 & 3 & 1 \end{pmatrix} = 138 \neq 0$$

$\therefore \begin{pmatrix} 1 & 2 & 3 \\ 1 & -6 & 1 \\ 7 & 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ has only trivial solution.

3

	II) Result: Area of a parallelogram spanned by vectors $x = (x_1, x_2)$ & $y = (y_1, y_2)$ of $\mathbb{R}^2$ is $\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$. Required area = $\begin{vmatrix} 5 & 6 \\ 2 & 5 \end{vmatrix} = 25 - 12 = 13$ sq.units	2
e)	Let V be an inner product space and $u, v \in V$. Let a, b be nonzero elements of $\mathbb{R}$ & $a \neq \pm b$. Prove that $\lVert au + bv \rVert = \lVert bu + av \rVert$ iff $\lVert u \rVert = \lVert v \rVert$	
Ans	$\lVert au + bv \rVert = \lVert bu + av \rVert \Leftrightarrow \lVert au + bv \rVert^2 = \lVert bu + av \rVert^2$ $\Leftrightarrow \langle au + bv, au + bv \rangle = \langle bu + av, bu + av \rangle$ $\Leftrightarrow a^2\langle u, u \rangle + 2ab\langle u, v \rangle + b^2\langle v, v \rangle =$ $b^2\langle u, u \rangle + 2ab\langle u, v \rangle + a^2\langle v, v \rangle$ $\Leftrightarrow (a^2 - b^2)\langle u, u \rangle = (a^2 - b^2)\langle v, v \rangle$ $\Leftrightarrow \langle u, u \rangle = \langle v, v \rangle \qquad a^2 - b^2 \neq 0$ $\Leftrightarrow \lVert u \rVert = \lVert v \rVert$	1M 2M 1M 1M
f)	Find distance between $f(x) = \cos x$ and $g(x) = \sin x$ in $C[-\pi, \pi]$ using $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$	
Ans	$d(f, g) = \lVert f - g \rVert$ $\lVert f - g \rVert^2 = \langle f - g, f - g \rangle$ $= \int_{-\pi}^{\pi} (f - g)(x)(f - g)(x)dx$ $= \int_{-\pi}^{\pi} (cosx - sinx)^2 dx$ $= \int_{-\pi}^{\pi} (cos^2 x + sin^2 x - 2cosxsinx)dx$ $= \int_{-\pi}^{\pi} (1 - sin2x)\, dx$ $= \left[x + \frac{cos2x}{2}\right]_{-\pi}^{\pi} = 2\pi$ $d(f, g) = \sqrt{2\pi}$	1M 2M 2M

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