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Q.P. Code : 54581

(3 Hours)

[Total Marks: 100]

Note: (i) All questions are compulsory.

(ii) Figures to the right indicate marks for respective parts.

Q.1	Choose correct alternative in each of the following (20)			
i.	Let $S = \{(x, y) \in \mathbb{R}^2 / x > 0, y > 0, x + y < 1\}$, then S is			
	(a)	An open set	(b)	Closed set
	(c)	Neither open nor closed	(d)	None of these
	Ans	An open set		
ii.	The largest directional derivative of $f(x, y) = x^2y^3$ at the point $(2, 3)$ occurs in the direction of			
	(a)	$i + j + k$	(b)	$i - j$
	(c)	$i + j$	(d)	None of these
	Ans	$i + j$		
iii.	$f(x, y) = 1 - x^2 - y^2$. The unit vector in the direction in which f decreases most rapidly at $p = (-1, 2)$ is			
	(a)	$\frac{1}{\sqrt{5}}(i + 2j)$	(b)	$\frac{1}{\sqrt{5}}(i - 2j)$
	(c)	$-\frac{1}{\sqrt{5}}(i + 2j)$	(d)	$\frac{1}{\sqrt{5}}(-i + 2j)$
	Ans	$\frac{1}{\sqrt{5}}(-i + 2j)$		
iv.	If $f(x, y) = x + y , \forall (x, y) \in \mathbb{R}^2$ then			
	(a)	$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist $\forall (x, y)$	(b)	$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ at $(0, 0)$.
	(c)	$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0, \forall (x, y)$	(d)	$\frac{\partial f}{\partial x}$ does not exist at $(0, 0)$.
	Ans	$\frac{\partial f}{\partial x}$ does not exist at $(0, 0)$.		
v.	If $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ is a differentiable function such that $\frac{\partial f}{\partial y} = 0$, then			
	(a)	f is independent of x and z	(b)	f depends on x and z only
	(c)	f is constant	(d)	None of these.
	Ans	f depends on x and z only		

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vi.	Gradient of a scalar field $f(x, y, z) = x^2 + xyz$ is			
	(a)	$(2yz, xz, xy)$	(b)	$(2x + yz, xz, xy)$
	(c)	$(2xz, xz, xy)$	(d)	$(x + y, xz, xy)$
	Ans	$(2x + yz, xz, xy)$		
vii.	Let $g : [0, 1] \rightarrow \mathbb{R}^n$, $g(t) = u_0 + tv_0$ where $u_0, v_0 \in \mathbb{R}^n$ is differentiable function and $F = fog : [0, 1] \rightarrow \mathbb{R}^n$, then $F'(t)$ equals			
	(a)	$Df(t)(v_0)$	(b)	$Df(u_0 + tv_0)(v_0)$
	(c)	$Df(u_0 + tv_0)(v_0 + tv_0)$	(d)	$Df(t)(u_0 + v_0)$
	Ans	$Df(u_0 + tv_0)(v_0)$		
viii.	The linear approximation to $e^x \cos(y + z)$ near the origin is			
	(a)	Independent of x .	(b)	independent of y
	(c)	independent of z	(d)	1
	Ans	Independent of x .		
ix.	$f(x, y) = x^2 - 4xy + y^2$			
	(a)	$(2, 2)$ is a critical point of f	(b)	$(1, 1)$ is a critical point of f .
	(c)	$(0, 0)$ is a critical point of f	(d)	None of these
	Ans	$(0, 0)$ is a critical point of f		
x.	The shortest distance from the origin to the plane $x - 2y - 2z = 3$ is			
	(a)	1	(b)	$\frac{3}{2}$
	(c)	3	(d)	None of these
	Ans	1		
Q2.	Attempt any ONE question from the following:			(08)
a)	i.	Let $f, g: \mathbb{R}^n \rightarrow \mathbb{R}$ be two real valued functions. Let $a \in \mathbb{R}^n$ such that $\lim_{x \rightarrow a} f(x) = l$ and $\lim_{x \rightarrow a} g(x) = m$. Then prove by using $\epsilon - \delta$ definition that $\lim_{x \rightarrow a} (4f + 3g)(x) = 4l + 3m$.		

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Ans

Q.2 a) i) Let $f, g: \mathbb{R}^n \rightarrow \mathbb{R}$ be two real valued function.

Let $a \in \mathbb{R}^n$ such that $\lim_{x \rightarrow a} f(x) = l$ and $\lim_{x \rightarrow a} g(x) = m$.

Prove that by using ϵ - δ definition that
 $\lim_{x \rightarrow a} (4f + 3g)(x) = 4l + 3m$.

Ans: Let $\alpha > 0$ & $\beta > 0$

Claim: $\lim_{x \rightarrow a} (\alpha f + \beta g)(x) = \alpha l + \beta m$

Given $\lim_{x \rightarrow a} f(x) = l$

\therefore for $\frac{\epsilon}{\alpha} > 0$, $\exists \delta_1 > 0$ such that

$$0 < \|x - a\| < \delta_1 \Rightarrow |f(x) - l| < \frac{\epsilon}{\alpha} \quad \text{--- (I)}$$

Also $\lim_{x \rightarrow a} g(x) = m$

\therefore for $\frac{\epsilon}{\beta} > 0$, $\exists \delta_2 > 0$ such that

$$0 < \|x - a\| < \delta_2 \Rightarrow |g(x) - m| < \frac{\epsilon}{\beta} \quad \text{--- (II)}$$

$$\text{Let } \delta = \min\{\delta_1, \delta_2\}$$

Let $0 < \|x - a\| < \delta$

Consider $|(\alpha f + \beta g)(x) - (\alpha l + \beta m)|$

$$\leq \alpha |f(x) - l| + \beta |g(x) - m|$$

$$< \alpha \frac{\epsilon}{\alpha} + \beta \frac{\epsilon}{\beta} \quad \text{--- } \because (0 < \|x - a\| < \delta)$$

$$= \epsilon$$

$$\therefore 0 < \|x - a\| < \delta \Rightarrow |(\alpha f + \beta g)(x) - (\alpha l + \beta m)| < \epsilon$$

$$\therefore \lim_{x \rightarrow a} (\alpha f + \beta g)(x) = \alpha l + \beta m$$

Take $\alpha = 4$ & $\beta = 3$

(04) marks

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ii.	<p>Let S be a non-empty subset of \mathbb{R}^n. Let $f, g: S \rightarrow \mathbb{R}$ be two scalar fields. Let $a = (a_1, a_2, \dots, a_n) \in S$. If $\frac{\partial f}{\partial x_i}(a)$ and $\frac{\partial g}{\partial x_i}(a)$ exist for $i = 1, 2, \dots, n$, Then prove that,</p> $\frac{\partial}{\partial x_i}(f \cdot g)(a) = f(a) \frac{\partial g}{\partial x_i}(a) + g(a) \frac{\partial f}{\partial x_i}(a)$
Ans	$\begin{aligned} \frac{\partial}{\partial x_i}(f \cdot g)(a) &= \lim_{h \rightarrow 0} \frac{(f \cdot g)(a_1, a_2, \dots, a_i + h, \dots, a_n) - (f \cdot g)(a_1, a_2, \dots, a_n)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(f)(a_1, a_2, \dots, a_i + h, \dots, a_n)(g)(a_1, a_2, \dots, a_i + h, \dots, a_n) - (f)(a_1, a_2, \dots, a_n)(g)(a_1, a_2, \dots, a_n)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(f)(a_1, \dots, a_i + h, \dots, a_n)(g)(a_1, \dots, a_i + h, \dots, a_n) - (f)(a_1, \dots, a_i + h, \dots, a_n)(g)(a_1, \dots, a_n) + (f)(a_1, \dots, a_i + h, \dots, a_n)(g)(a_1, \dots, a_n) - (f)(a_1, \dots, a_n)(g)(a_1, \dots, a_n)}{h} \\ &= f(a) \frac{\partial g}{\partial x_i}(a) + g(a) \frac{\partial f}{\partial x_i}(a) \end{aligned}$
Q.2	<p>Attempt any TWO questions from the following: (12)</p>
b)	<p>i. Use definition of limit to prove that $\lim_{(x,y) \rightarrow (-1,-1)} x + y = 2$</p>
Ans	<p>Q.2 b i) Use definition of limit to prove that</p> $\lim_{(x,y) \rightarrow (-1,-1)} x + y = 2$ <p>Ans: consider</p> $\begin{aligned} x + y - 2 &= (x-(-1)) + (y-(-1)) \\ &\leq x+1 + y+1 \\ &\leq \ (x,y) - (-1,-1)\ + \ (x,y) - (-1,-1)\ \\ &= 2 \ (x,y) - (-1,-1)\ \end{aligned}$ <p style="text-align: right;">(04) marks</p> <p>\therefore for $\epsilon > 0$, choose $\delta = \frac{\epsilon}{2} > 0$</p> <p>then</p> $0 < \ (x,y) - (-1,-1)\ < \delta \Rightarrow x + y - 2 < 2 \times \frac{\epsilon}{2} = \epsilon$ <p>$\therefore \lim_{(x,y) \rightarrow (-1,-1)} x + y = 2$ (02) marks</p>
ii.	<p>Prove that every linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous on \mathbb{R}^n.</p>

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<p>Ans</p>	<p>Q.2 b) Prove that every linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous on \mathbb{R}^n.</p> <p>Ans: Let $x = (x_1, x_2, \dots, x_n)$ & $y = (y_1, y_2, \dots, y_n)$ let $T \neq 0$ consider</p> $\ T(x) - T(y)\ _2 = \left\ \sum_{i=1}^n x_i T(e_i) - \sum_{i=1}^n y_i T(e_i) \right\ _2$ $= \sum_{i=1}^n x_i - y_i \ T(e_i)\ _2$ $\leq \sum_{i=1}^n \ x - y\ _1 \ T(e_i)\ _2 \quad [x_i - y_i \leq \ x - y\ _1]$ <p>Let $M = \max \{ \ T(e_i)\ _2 \mid i=1, 2, \dots, n \}$ $\leq nM \ x - y\ _1$</p> <p>$\therefore \ T(x) - T(y)\ _2 \leq nM \ x - y\ _1$ (04) marks</p> <p>\therefore for $\epsilon > 0$, choose $\delta = \frac{\epsilon}{nM} > 0$</p> <p>$\ x - y\ _1 < \delta = \frac{\epsilon}{nM} \Rightarrow \ T(x) - T(y)\ _2 < \epsilon$</p> <p>$\therefore T$ is continuous at $y \in \mathbb{R}^n$</p> <p>If $T=0$ then it is constant function hence it is continuous on \mathbb{R}^n. (02) marks</p>
<p>iii.</p>	<p>Define directional derivative of a scalar field f at a point a in the domain in the direction of u. Calculate the directional derivative of the function f, $f(x, y, z) = 3x^2 - 3y^2 + 3z^2$ at $(1, 2, 3)$ in the direction of $(0, 1, 0)$ using the definition and also using the relationship between directional derivative and partial derivative.</p>
<p>Ans</p>	<p>$D_u f(a) = \lim_{t \rightarrow 0} \frac{f(a + tu) - f(a)}{t}$</p> <p>$= \lim_{t \rightarrow 0} \frac{f((1, 2, 3) + t(0, 1, 0)) - f(1, 2, 3)}{t}$</p> <p>$= \lim_{t \rightarrow 0} \frac{(3(1)^2 - 3(2+t)^2 + 3(3)^2) - (3(1)^2 - 3(2)^2 + 3(3)^2)}{t}$</p> <p>$= \lim_{t \rightarrow 0} \frac{-12t - 3t^2}{t}$</p> <p>$= -12$</p> <p>$D_u f(a) = \langle \nabla f(a), u \rangle = -12$</p>

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iv.	<p>If $\sin u = \frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$; $(x, y, z) \neq (0, 0, 0)$. Then prove that</p> $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} + 3 \tan u = 0$		
Ans	$\cos u \frac{\partial u}{\partial x} = \frac{\sqrt{x^8+y^8+z^8} - (x+2y+3z) \frac{8x^7}{2\sqrt{x^8+y^8+z^8}}}{(\sqrt{x^8+y^8+z^8})^2}$ $\cos u \frac{\partial u}{\partial y} = \frac{\sqrt{x^8+y^8+z^8}(2) - (x+2y+3z) \frac{8y^7}{2\sqrt{x^8+y^8+z^8}}}{(\sqrt{x^8+y^8+z^8})^2}$ $\cos u \frac{\partial u}{\partial z} = \frac{\sqrt{x^8+y^8+z^8}(3) - (x+2y+3z) \frac{8z^7}{2\sqrt{x^8+y^8+z^8}}}{(\sqrt{x^8+y^8+z^8})^2}$ $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = \frac{1}{\cos u} (\sin u - 4 \sin u)$ $\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} + 3 \tan u = 0 \quad (3)$	3	
Q3. Attempt any ONE question from the following:		(08)	
a)	i.	<p>When do you say that a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at $a \in \mathbb{R}^n$? Show that such a function is necessarily continuous at a. Is the converse true? Justify your answer.</p>	
Ans		<p>Definition: A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at $a \in \mathbb{R}^n$ if \exists a linear transformation $T_a: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f(a+v) = f(a) + T_a(v) + \ v\ E(a,v)$ where $E(a,v) \rightarrow 0$ as $\ v\ \rightarrow 0$</p> <p>continuity: $f(a+v) - f(a) = T_a(v) + \ v\ E(a,v)$</p> <p>As $\ v\ \rightarrow 0, E(a,v) \rightarrow 0$ and $T_a(v) = T_a(0) = 0 \Rightarrow f(a+v) - f(a) \rightarrow 0$</p> <p>$\Rightarrow \lim_{v \rightarrow 0} f(a+v) = f(a) \Rightarrow f$ is continuous.</p> <p>Converse: Let $f(x,y) = x + y , \forall (x,y) \in \mathbb{R}^2$ then f is continuous at $(0,0)$ but not differentiable. (Any other example can be given)</p>	
	ii.	State and prove sufficient condition for the equality of mixed partial derivatives.	
Ans		<p>Statement.....(3 Marks)</p> <p>Steps in the Proof: Considering the rectangle with corners</p>	

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		$(a_1, a_2), (a_1 + h, a_2), (a_1 + h, a_2 + k), (a_1, a_2 + k),$ Define $G(h, k) = f(a_1 + h, a_2 + k) - f(a_1 + h, a_2) - f(a_1, a_2 + k) + f(a_1, a_2)$ Let $\phi(x) = f(x, a_2 + k) - f(x, a_2)$ then $\phi(x)$ is continuous and differentiable on rectangle. Apply Lagrange's mean value theorem to $\phi(x)$ in the interval $[a_1, a_1 + h]$ $G(h, k) = \phi(a_1 + h) - \phi(a_1) = \phi'(\theta_1)h$ where $a_1 < \theta_1 < a_1 + h$ Define $\Psi(y) = f_x(\theta_1, y)$ Apply Lagrange's mean value theorem to $\Psi(y)$ in the interval $[a_2, a_2 + k]$ Thus $G(h, k) = hkf_{xy}(\theta_1, \theta_2)$ Apply the same procedure to $u(y) = f(a_1 + h, y) - f(a_1, y)$ And show that mixed partial derivatives are equal.
Q3.	Attempt any TWO questions from the following: (12)	
b)	i.	Find total derivative as linear transformation T for the function $f(x, y) = x^2 + 2y^2 + 3z$ at point $a = (1, -1, 0)$
	Ans	Total derivative as linear transformation is $T_a(v) = \nabla f(a) \cdot v$ let $v = (x, y, z)$ and $\nabla f(a) = (f_x(a), f_y(a), f_z(a)) = (2, -4, 0) \Rightarrow T_a(v) = 2x - 4y$
	ii.	Find directional derivative of $f(x, y) = x^2 - 3xy$ at $(1, 2)$ along the parabola $y = x^2 - x + 2$
	Ans	Formula is $D_u f(a) = \nabla f(a) \cdot T$. where T is unit tangent vector to the surface. $\nabla f(a) = (f_x(a), f_y(a)) = (-4, -3)$ Tangent vector $= (1, 2t - 1) \Rightarrow T = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \Rightarrow D_u f(a) = \frac{-7}{\sqrt{2}}$
	iii.	Define level set of $f: \mathbb{R}^n \rightarrow \mathbb{R}$ for $k \in \mathbb{R}$. Find level sets of the following scalar fields for the given constants. 1. $f(x, y) = x^2 + 4y^2$ for $k = 1, 4$. 2. $f(x, y, z) = x^2 + y^2 + z^2$ for $k = 1, 9$.
	Ans	1. Ellipse with semi major axis 1 and semi minor axis $\frac{1}{2}$ Ellipse with semi major axis $\frac{1}{2}$ and semi minor axis 1

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		2. Circles centered at origin with radii 1 and 3.	
	iv.	Using chain rule, evaluate the total derivative of $w = \sqrt{xy + yz + zx}$ where $x = t$, $y = \sin t$ and $z = \cos t$.	
	Ans	$\frac{1}{2\sqrt{xy + yz + zx}} [(y + z) + (x + z) \cos t - (x - y) \sin t]$	
Q4.	Attempt any ONE question from the following:		(08)
a)	i.	Let U be an open set in \mathbb{R}^n and $f: U \rightarrow \mathbb{R}^m$ be given by $f(x) = (f_1(x), f_2(x), \dots, f_m(x))$, $\forall x \in U$. Prove that f is differentiable at $a \in U$ if and only if each f_i is differentiable at a and for any $u \in \mathbb{R}^n$. $Df(a)(u) = (Df_1(a)(u), Df_2(a)(u), \dots, Df_m(a)(u))$	
	Ans	<p>Let f is differentiable at $a \in \mathbb{R}^n$ with total derivative $Df(a)$.</p> <p>$\Leftrightarrow f(a + h) = f(a) + Df(a)h + \ h\ E(h)$ where $E(h) \rightarrow 0$ as $\ h\ \rightarrow 0$ 1M</p> <p>$\Leftrightarrow (f_1(a + h), \dots, f_m(a + h))$ $= (f_1(a), \dots, f_m(a)) + D(f_1(a), \dots, f_m(a))h + \ h\ (E_1(a), \dots, E_m(a))$ where $E_i(h) \rightarrow 0$ as $h \rightarrow 0$</p> <p>$\Leftrightarrow (f_1(a + h), \dots, f_m(a + h))$ $= (f_1(a) + Df_1(a)h + \ h\ E_1(a), \dots, f_m(a) + Df_m(a)h + \ h\ E_m(a))$</p> <p>$\Leftrightarrow f_i(a + h) = f_i(a) + Df_i(a)h + \ h\ E_i(a) \quad \forall i = 1 \text{ to } m$ \Leftrightarrow each f_i is differentiable at $a \in \mathbb{R}^n$. 6M</p> <p>Further, $Df_i(a)(h) = T_i(h)$ therefore, $Df(a)(u) = T(u) = (T_1(u), T_2(u), \dots, T_m(u))$ $= (Df_1(u), Df_2(u), \dots, Df_m(u))$ 1M</p>	
	ii.	Let $f: S \subseteq \mathbb{R}^n$ be a scalar field where S is a non-empty open subset of \mathbb{R}^n . Let $a \in S$ and f is differentiable at a . Prove that if f has a local maximum or local minimum at a then $\nabla f(a) = 0$.	
	Ans	<p>Let $a = (a_1, a_2, \dots, a_n) \in S$. Assume f has a local maxima at $a \in S$.</p> <p>To show $\frac{\partial f}{\partial x_i}(a) = 0$, $i = 1, 2, \dots, n$. Let $g_1(t) = f(t, a_2, \dots, a_n)$ then $g_1(t)$ is a function of single variable and has local maximum at $t = a_1$.</p> <p>Also $g_1(t)$ is differentiable as f is differentiable. By the theorem of one</p>	



		<p>variable $\frac{d}{dt}g_1(t) = 0$ at $t = a_1$. But $\frac{d}{dt}g_1(a_1) = \frac{\partial f}{\partial x_1}(a) = 0$.</p> <p>Similarly considering $g_2(t) = f(a_1, t, a_3, \dots, a_n)$ we get $\frac{\partial f}{\partial x_2}(a) = 0$.</p> <p>In general $\frac{\partial f}{\partial x_i}(a) = 0, i = 1, 2, \dots, n$.</p> <p>So $\nabla f(a) = \left(\frac{\partial f}{\partial x_1}(a), \frac{\partial f}{\partial x_2}(a), \dots, \frac{\partial f}{\partial x_n}(a) \right) = (0, 0, \dots, 0)$.</p> <p>Similarly the proof for minima.</p>	
Q4.	Attempt any TWO questions from the following:		(12)
b)	i.	If $f(x, y, z) = xi + yj + zk$ then prove that the Jacobian matrix $Df(x, y, z)$ is the identity matrix of order 3. Also find all differentiable vector fields $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ for which the Jacobian matrix $Df(x, y, z)$ is the identity matrix of order 3.	
	Ans	<p>a) $f(x, y, z) = x\bar{i} + y\bar{j} + z\bar{k}$</p> $Df(x, y, z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ <p>which is identity matrix of order 3.</p> <p style="text-align: right;">2M</p> <p>b) Let $f(x, y, z) = (f_1, f_2, f_3)$ where f_1, f_2, f_3 are functions of x, y, z.</p> <p>Since $Df(x, y, z)$ is identity matrix of order 3.</p> $\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y} = \frac{\partial f_3}{\partial z} = 1 \implies f_1 = x + a, f_2 = y + b, f_3 = z + c, a, b, c \in \mathbb{R}$ <p>$\therefore f(x, y, z) = (x + a, y + b, z + c)$</p> <p style="text-align: right;">4M</p>	
	ii.	Determine the second order Taylor formula for the function $f(x, y) = e^x \cos y$ at $(0, \frac{\pi}{2})$	
	Ans	<p>$f(x, y) = e^x \cos(y) \quad p = (0, \frac{\pi}{2}) = (a, b)$</p> <p>$f_x = e^x \cos y, f_y = -e^x \sin y,$</p> <p>$f_{xx} = e^x \cos y, f_{xy} = -e^x \sin y, f_{yy} = -e^x \cos y$</p> <p style="text-align: right;">2M</p> <p>$f(p) = f_x(p) = f_{xx}(p) = f_{yy}(p) = 0, f_y(p) = f_{xy}(p) = -1$</p> <p style="text-align: right;">1M</p> <p>Using Taylor's formula</p> <p>$f(x, y) = f(p) + (x - a)f_x(p) + (y - b)f_y(p) +$</p> $\frac{1}{2!} [(x - a)^2 f_{xx}(p) + 2(x - a)(y - b)f_{xy}(p) + (y - b)^2 f_{yy}(p)]$ <p style="text-align: right;">2M</p>	

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	$f(x, y) = -\left(y - \frac{\pi}{2}\right) - x\left(y - \frac{\pi}{2}\right) = -xy + \frac{\pi}{2}x - y + \frac{\pi}{2}$	1M
iii.	Find the critical points, saddle points and local extrema if any for the function $f(x, y) = y^2 - y^3 - x^2 + xy.$	
Ans	Critical points	Δ
	(0,0)	-5
	(5/3, 5/6)	5
	f_{xx}	Extrema
	-2	Saddle point
	8	maxima
iv.	If $2x + 3y + 4z = a$ then prove that the maximum value of $x^2y^3z^4$ is $\left(\frac{a}{9}\right)^9$.	
Ans	$h(x, y, z) = x^2y^3z^4 + \lambda(2x + 3y + 4z - a).$ $\frac{\partial h}{\partial x} = 2xy^3z^4 + 2\lambda = 0,$ $\frac{\partial h}{\partial y} = 3x^2y^2z^4 + 3\lambda = 0, \frac{\partial h}{\partial z} = 4x^2y^3z^3 + 4\lambda = 0. x = y = z.$ Put in $2x + 3y + 4z = a$ we will get $x = y = z = a/9. f\left(\frac{a}{9}, \frac{a}{9}, \frac{a}{9}\right) = \left(\frac{a}{9}\right)^9.$	
Q5.	Attempt any FOUR questions from the following: (20)	
a)	Evaluate the limit of the following functions, if it exists, by converting to polar Coordinates.	
	(i) $\lim_{(x,y) \rightarrow (0,0)} y \log(x^2 + y^2)$ (ii) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y^2}{\sqrt{x^2+y^2}}$	

(11)

Ans	<p>Q. 5) Evaluate the limit of the following functions, if it exists by using polar co-ordinates</p> <p>i) $\lim_{(x,y) \rightarrow (0,0)} y \log(x^2+y^2)$ ii) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{\sqrt{x^2+y^2}}$</p> <p>Sol Ans i) $\lim_{(x,y) \rightarrow (0,0)} y \log(x^2+y^2) = \lim_{r \rightarrow 0} r \sin \theta \log(r^2)$ As $x = r \cos \theta$ $y = r \sin \theta$</p> $= \lim_{r \rightarrow 0} 2 r \sin \theta \log r \quad \left(\frac{\infty}{\infty} \text{ form}\right)$ $= 2 \sin \theta \lim_{r \rightarrow 0} \frac{1/r}{-1/r^2} \quad (\text{By L'Hospital's rule})$ $= 2 \sin \theta \lim_{r \rightarrow 0} (-r)$ <p style="text-align: right;">(03) marks</p> $= 0$ <p>ii) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{\sqrt{x^2+y^2}} = \lim_{r \rightarrow 0} \frac{r^4 \cos^2 \theta \sin^2 \theta}{\sqrt{r^2}}$</p> $= \lim_{r \rightarrow 0} r^3 \cos^2 \theta \sin^2 \theta = 0$ <p style="text-align: right;">(*: $x = r \cos \theta, y = r \sin \theta$) (02) marks</p>
b)	<p>Find the real value of $\theta \in (0, 1)$ if it exists, satisfying,</p> $f(b) - f(a) = \langle \nabla f(a + \theta(b-a)), b-a \rangle$ for the following functions: $f(x, y, z) = xyz, \quad a = (0, 0, 0), \quad b = \left(1, \frac{1}{2}, \frac{1}{3}\right)$
Ans	<p>To find $\nabla f(a + \theta(b-a))$:</p> $f_x(x, y, z) = yz; \quad f_y(x, y, z) = xz; \quad f_z(x, y, z) = xy$ $a + \theta(b-a) = (0, 0, 0) + \theta \left(1, \frac{1}{2}, \frac{1}{3}\right) = \left(\theta, \frac{\theta}{2}, \frac{\theta}{3}\right) \Rightarrow \nabla f(a + \theta(b-a)) = \left(\frac{\theta^2}{6}, \frac{\theta^2}{3}, \frac{\theta^2}{2}\right)$ <p>Substituting this in $f(b) - f(a) = \nabla f(a + \theta(b-a)) \cdot (b-a)$</p> <p>We have, $f\left(1, \frac{1}{2}, \frac{1}{3}\right) - f(0, 0, 0) = \left\langle \left(\frac{\theta^2}{6}, \frac{\theta^2}{3}, \frac{\theta^2}{2}\right), \left(1, \frac{1}{2}, \frac{1}{3}\right) \right\rangle$</p> $\frac{1}{6} = \frac{\theta^2}{2} \Rightarrow \theta = \frac{1}{\sqrt{3}}$

c)	Find the maximum rate of change of the function $f(x, y, z) = \log(x + y + z)$ at $(1, 2, 3)$. Also find the direction in which maximum rate of change occurs.
Ans	$\nabla f(x, y, z) = \left(\frac{1}{x+y+z}, \frac{1}{x+y+z}, \frac{1}{x+y+z} \right) \quad (2 \text{ Marks})$ $\nabla f(1, 2, 3) = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6} \right) \quad (1 \text{ Mark})$ $\text{Maximum rate of change} = \ \nabla f(1, 2, 3)\ = \frac{\sqrt{3}}{6} \quad (1 \text{ Mark})$ $\text{Direction} \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6} \right) \quad (1 \text{ Mark})$
d)	Let $f(x, y) = x^3 + 9xy^2$, find $f_x, f_y, f_{xy}, f_{xx}, f_{yy}$.
Ans	$f_x = 3x^2 + 9y^2 \quad f_y = 18xy \quad f_{xy} = 18y \quad f_{xx} = 6x \quad f_{yy} = 18x$
e)	Given $z = f(x, y)$ where f has continuous partial derivatives of second order, $x = u + v, y = u - v$, show that $\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}$
Ans	$\frac{\partial x}{\partial u} = \frac{\partial x}{\partial v} = 1, \quad \frac{\partial y}{\partial u} = 1, \quad \frac{\partial y}{\partial v} = -1 \quad 1M$ <p>using chain rule, $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \quad 1M$</p> <p>diff $\frac{\partial z}{\partial v}$ w. r. t 'u'</p> $\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial}{\partial u} \left(\frac{\partial z}{\partial x} \right) - \frac{\partial}{\partial u} \left(\frac{\partial z}{\partial y} \right)$ $= \frac{\partial^2 z}{\partial x^2} \frac{\partial x}{\partial u} + \frac{\partial^2 z}{\partial x \partial y} \frac{\partial y}{\partial u} - \frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial u} - \frac{\partial^2 z}{\partial y^2} \frac{\partial y}{\partial u} \quad 1M$ $= \frac{\partial^2 z}{\partial x^2} (1) + \frac{\partial^2 z}{\partial x \partial y} (1) - \frac{\partial^2 z}{\partial x \partial y} (1) - \frac{\partial^2 z}{\partial y^2} (1) \quad 1M$ $= \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} \quad 1M$
f)	Find the Hessian matrix of $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ at $(1, 1, 1)$ where $f(x, y, z) = x^3 y^2 (12 - 3x - 4y)$.
Ans	$\begin{bmatrix} 12 & 12 & 0 \\ 12 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$