① Q.P. Code : 54581

(3 Hours) [Total Marks: 100]

**Note:** (*i*) All questions are compulsory.

(*ii*) Figures to the right indicate marks for respective parts.

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| Q.1 | Choose correct alternative in each of the following (20) | | | | |
| *i.* | Let $S = \{(x,y) \in \mathbb{R}^2 \ / \ x > 0,\ y > 0,\ x + y < 1\}$, then $S$ is | | | | |
| | (a) | An open set | (b) | Closed set | |
| | (c) | Neither open nor closed | (d) | None of these | |
| | Ans | An open set | | | |
| *ii.* | The largest directional derivative of $f(x,y) = x^2y^3$ at the point $(2,3)$ occurs in the direction of | | | | |
| | (a) | $i + j + k$ | (b) | $i - j$ | |
| | (c) | $i + j$ | (d) | None of these | |
| | Ans | $i + j$ | | | |
| *iii.* | $f(x,y) = 1 - x^2 - y^2$. The unit vector in the direction in which $f$ decreases most rapidly at $p = (-1,2)$ is | | | | |
| | (a) | $\frac{1}{\sqrt{5}}(i + 2j)$ | (b) | $\frac{1}{\sqrt{5}}(i - 2j)$ | |
| | (c) | $-\frac{1}{\sqrt{5}}(i + 2j)$ | (d) | $\frac{1}{\sqrt{5}}(-i + 2j)$ | |
| | Ans | $\frac{1}{\sqrt{5}}(-i + 2j)$ | | | |
| *iv.* | If $f(x,y) = \lvert x\rvert + \lvert y\rvert, \forall (x,y) \in \mathbb{R}^2$ then | | | | |
| | (a) | $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ exist $\forall (x,y)$ | (b) | $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ at $(0,0)$. | |
| | (c) | $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0, \quad \forall (x,y)$ | (d) | $\frac{\partial f}{\partial x}$ does not exist at $(0,0)$. | |
| | Ans | $\frac{\partial f}{\partial x}$ does not exist at $(0,0)$. | | | |
| *v.* | If $f: \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function such that $\frac{\partial f}{\partial y} = 0$, then | | | | |
| | (a) | $f$ is independent of $x$ and $z$ | (b) | $f$ depends on $x$ and $z$ only | |
| | (c) | $f$ is constant | (d) | None of these. | |
| | Ans | $f$ depends on $x$ and $z$ only | | | |

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| *vi.* | Gradient of a scalar field $f(x,y,z) = x^2 + xyz$ is …….. | | | |
| | (a) | $(2yz, xz, xy)$ | (b) | $(2x + yz, xz, xy)$ |
| | (c) | $(2xz, xz, xy)$ | (d) | $(x + y, xz, xy)$ |
| | Ans | $(2x + yz, xz, xy)$ | | |
| *vii.* | Let $g : [0,1] \rightarrow \mathbb{R}^n$ , $g(t) = u_0 + tv_0$ where $u_0$ , $v_0 \in \mathbb{R}^n$ is differentiable function and $F = fog : [0,1] \longrightarrow \mathbb{R}^n$ , then $F'(t)$ equals | | | |
| | (a) | $Df(t)(v_0)$ | (b) | $Df(u_0 + tv_0)(v_0)$ |
| | (c) | $Df(u_0 + tv_0)(v_0 + tv_0)$ | (d) | $Df(t)(u_0 + v_0)$ |
| | Ans | $Df(u_0 + tv_0)(v_0)$ | | |
| *viii.* | The linear approximation to $e^x cos(y + z)$ near the origin is | | | |
| | (a) | Independent of $x$. | (b) | independent of $y$ |
| | (c) | independent of $z$ | (d) | 1 |
| | Ans | Independent of $x$. | | |
| *ix.* | $f(x,y) = x^2 - 4xy + y^2$ | | | |
| | (a) | $(2,2)$ is a critical point of $f$ | (b) | $(1,1)$ is a critical point of $f$. |
| | (c) | $(0,0)$ is a critical point of $f$ | (d) | None of these |
| | Ans | $(0,0)$ is a critical point of $f$ | | |
| *x.* | The shortest distance from the origin to the plane $x - 2y - 2z = 3$ is | | | |
| | (a) | 1 | (b) | 2 |
| | (c) | 3 | (d) | None of these |
| | Ans | 1 | | |
| Q2. | Attempt any **ONE** question from the following: (08) | | | |
| *a)* | i. | Let $f, g: \mathbb{R}^n \rightarrow \mathbb{R}$ be two real valued functions. Let $a \in \mathbb{R}^n$ such that $\lim_{x \rightarrow a} f(x) = l$ and $\lim_{x \rightarrow a} g(x) = m$. Then prove by using $\epsilon - \delta$ definition that $\lim_{x \rightarrow a} (4f + 3g)(x) = 4l + 3m$. | | |

Ans

Q:2 a) i) Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two real valued function.

Let $a \in \mathbb{R}^n$ such that $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$.

Prove that by using $\epsilon$-$\delta$ definition that $\lim_{x\to a} (4f+3g)(x) = 4l + 3m$.

Ans: Let $\alpha > 0$ & $\beta > 0$

claim: $\lim_{x\to a} (\alpha f + \beta g)(x) = \alpha l + \beta m$

Given $\lim_{x\to a} f(x) = l$

$\therefore$ for $\frac{\epsilon}{\alpha} > 0$, $\exists \delta_1 > 0$ such that

$0 < \|x-a\|_1 < \delta_1 \Rightarrow |f(x) - l| < \frac{\epsilon}{2\alpha}$ — (I)

Also $\lim_{x\to a} g(x) = m$

$\therefore$ for $\frac{\epsilon}{\beta} > 0$, $\exists \delta_2 > 0$ such that

$0 < \|x-a\|_1 < \delta_2 \Rightarrow |g(x) - m| < \frac{\epsilon}{2\beta}$ — (II)

Let $\delta = \min\{\delta_1, \delta_2\}$ (04) marks

Let $0 < \|x-a\|_1 < \delta$

Consider $|(\alpha f + \beta g)(x) - (\alpha l + \beta m)|$

$\leq \alpha(|f(x) - l|) + \beta(|g(x) - m|)$

$< \alpha \frac{\epsilon}{2\alpha} + \beta \frac{\epsilon}{2\beta}$ $(\because 0 < \|x-a\|_1 < \delta)$

$= \epsilon$

$\therefore$ $0 < \|x-a\|_1 < \delta \Rightarrow |(\alpha f + \beta g)(x) - (\alpha l + \beta m)| < \epsilon$

$\therefore \lim_{x\to a} (\alpha f + \beta g)(x) = \alpha l + \beta m$

Take $\alpha = 4$ & $\beta = 3$ (04) marks

| | | |
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| | ii. | Let S be a non-empty subset of $\mathbb{R}^n$. *Let* $f, g: S \to \mathbb{R}$ be two scalar fields. Let $a = (a_1, a_2, \dots, a_n) \in S$. If $\frac{\partial f}{\partial x_i}(a)$ *and* $\frac{\partial g}{\partial x_i}(a)$ exist for $i = 1,2, \dots, n$, Then prove that, $\frac{\partial}{\partial x_i}(f.g)(a) = f(a)\frac{\partial g}{\partial x_i}(a) + g(a)\frac{\partial f}{\partial x_i}(a)$ |
| | Ans | $\frac{\partial}{\partial x_i}(f\cdot g)(a) = \lim_{h\to 0}\frac{(f\cdot g)(a_1,a_2,\dots,a_i+h,\dots,a_n) - (f\cdot g)(a_1,a_2,\dots,a_n)}{h}$ <br> $= \lim_{h\to 0}\frac{(f)(a_1,a_2,\dots,a_i+h,\dots,a_n)(g)(a_1,a_2,\dots,a_i+h,\dots,a_n) - (f)(a_1,a_2,\dots,a_n)(g)(a_1,a_2,\dots,a_n)}{h}$ <br> $= \lim_{h\to 0}\frac{(f)(a_1,\dots,a_i+h,\dots,a_n)(g)(a_1,\dots,a_i+h,\dots,a_n) - (f)(a_1,\dots,a_i+h,\dots,a_n)(g)(a_1,,\dots,a_n) + (f)(a_1,\dots,a_i+h,\dots,a_n)(g)(a_1,\dots,a_n) - (f)(a_1,\dots,a_n)(g)(a_1,\dots,a_n)}{h}$ <br> $= f(a)\frac{\partial g}{\partial x_i}(a) + g(a)\frac{\partial f}{\partial x_i}(a)$ |
| | | |
| Q.2 | Attempt any **TWO** questions from the following: (12) | |
| *b)* | i. | Use definition of limit to prove that $\lim_{(x,y)\to(-1,-1)} \lvert x\rvert + \lvert y\rvert = 2$ |
| | Ans | Q:2 b i) Use definition of limit to prove that $\lim_{(x,y)\to(-1,-1)} \lvert x\rvert+\lvert y\rvert=2$ <br> Ans: consider <br> $\big\lvert\lvert x\rvert+\lvert y\rvert-2\big\rvert = \big\lvert(\lvert x\rvert-\lvert -1\rvert)+(\lvert y\rvert-\lvert -1\rvert)\big\rvert$ <br> $\leq \lvert x+1\rvert + \lvert y+1\rvert$ <br> $\leq \lVert(x,y)-(-1,-1)\rVert + \lVert(x,y)-(-1,-1)\rVert$ <br> $= 2\lVert(x,y)-(-1,-1)\rVert$ (4) marks <br> $\therefore$ for $\epsilon>0$, choose $\delta = \epsilon/2 > 0$ <br> then <br> $0 < \lVert(x,y)-(-1,-1)\rVert < \delta \Rightarrow \big\lvert\lvert x\rvert+\lvert y\rvert-2\big\rvert < 2\times\epsilon/2 = \epsilon$ <br> $\therefore \lim_{(x,y)\to(-1,-1)} \lvert x\rvert+\lvert y\rvert = 2$ (2) marks |
| | ii. | Prove that every linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is continuous on $\mathbb{R}^n$. |

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| | Ans | Q:2 b (ii) Prove that every linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is continuous on $\mathbb{R}^n$.<br>Ans: let & $x=(x_1,x_2,\dots x_n)$ & $y=(y_1,y_2,\dots,y_n)$<br>Let $T\neq 0$<br>Consider<br>$\|T(x)-T(y)\|_2 = \|\sum_{i=1}^{n} x_i T(e_i) - \sum_{i=1}^{n} y_i T(e_i)\|_2$<br>$= \sum_{i=1}^{n} \|x_i - y_i\| \, \|T(e_i)\|_2$<br>$\leq \sum_{i=1}^{n} \|x-y\|_1 \|T(e_i)\|_2$ $[\because \|x_i - y_i\| \leq \|x-y\|]$<br>Let $M = \max\{\|T(e_i)\|_2 \mid i=1,2,\dots n\}$<br>$\leq nM\|x-y\|_1$<br>$\therefore \|T(x)-T(y)\|_2 \leq nM\|x-y\|_1$ (04) marks<br>$\therefore$ for $\epsilon>0$, choose $\delta = \frac{\epsilon}{nM} > 0$<br>$\|x-y\|_1 < \delta = \frac{\epsilon}{nM} \Rightarrow \|T(x)-T(y)\| < \epsilon$<br>$\therefore$ T is continuous at $y \in \mathbb{R}^n$<br>If $T=0$ then it is constant function hence it is continuous on $\mathbb{R}^n$. (02) marks |
| | iii. | Define directional derivative of a scalar field $f$ at a point $a$ in the domain in the direction of $u$. Calculate the directional derivative of the function $f$, $f(x,y,z)=3x^2-3y^2+3z^2$ at (1,2,3) in the direction of (0,1,0) using the definition and also using the relationship between directional derivative and partial derivative. |
| | Ans | $D_u f(a) = \lim_{t\to 0}\frac{f(a+tu-f(a)}{t}$<br>$= \lim_{t\to 0}\frac{f((1,2,3)+t(0,1,0))-f(1,2,3)}{t}$<br>$= \lim_{t\to 0}\frac{(3(1)^2-3(2+t)^2+3(3)^2)-(3(1)^2-3(2)^2+3(3)^2)}{t}$<br>$= \lim_{t\to 0}\frac{-12t-3t^2}{t}$<br>$= -12$<br>$D_u f(a) = \langle \nabla f(a), u\rangle = -12$ |

iv. If $\sin u = \frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$; $(x,y,z) \neq (0,0,0)$. Then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3\tan u = 0$$

Ans

$$\cos u\frac{\partial u}{\partial x} = \frac{\sqrt{x^8+y^8+z^8} - (x+2y+3z)\frac{8x^7}{2\sqrt{x^8+y^8+z^8}}}{\left(\sqrt{x^8+y^8+z^8}\right)^2}$$

$$\cos u\frac{\partial u}{\partial y} = \frac{\sqrt{x^8+y^8+z^8}(2) - (x+2y+3z)\frac{8y^7}{2\sqrt{x^8+y^8+z^8}}}{\left(\sqrt{x^8+y^8+z^8}\right)^2}$$

$$\cos u\frac{\partial u}{\partial x} = \frac{\sqrt{x^8+y^8+z^8}(3) - (x+2y+3z)\frac{8z^7}{2\sqrt{x^8+y^8+z^8}}}{\left(\sqrt{x^8+y^8+z^8}\right)^2}$$ 3

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{1}{\cos u}(\sin u - 4\sin u)$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3\tan u = 0$$

(3)

Q3. Attempt any **ONE** question from the following: (08)

a) i. When do you say that a function $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}^n$? Show that such a function is necessarily continuous at $a$.Is the converse true? Justify your answer.

Ans Definition: A function $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}^n$ if $\exists$ a linear transformation $T_a: \mathbb{R}^n \to \mathbb{R}$ such that $f(a+v) = f(a) + T_a(v) + \|v\|E(a,v)$

where $E(a,v) \to 0$ $as$ $\|v\| \to 0$

continuity: $|f(a+v) - f(a)| = |T_a(v) + \|v\|E(a,v)|$

As $\|v\| \to 0$ ,$E(a,v) \to 0$ and $T_a(v) = T_a(0) = 0 \Rightarrow |f(a+v) - f(a)| \to 0$

$\Rightarrow \lim_{v \to 0} f(a+v) = f(a) \Rightarrow$ f is continuous.

Converse: Let $f(x,y) = |x| + |y|$ , $\forall (x,y) \in \mathbb{R}^2$ then $f$ is continuous at (0,0) but not differentiable. (Any other example can be given)

ii. State and prove sufficient condition for the equality of mixed partial derivatives.

Ans **Statement**.......................(3 Marks)

Steps in the Proof: Considering the rectangle with corners

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| | | $(a_1, a_2), (a_1 + h, a_2), (a_1 + h, a_2 + k), (a_1, a_2 + k)$, <br> Define $G(h,k) = f(a_1 + h, a_2 + k) - f(a_1 + h, a_2) - f(a_1, a_2 + k) + f(a_1, a_2)$ <br> Let $\phi(x) = f(x, a_2 + k) - f(x, a_2)$ then $\phi(x)$ is continuous and differentiable on rectangle. <br> Apply Lagrange's mean value theorem to $\phi(x)$ in the interval $[a_1, a_1 + h]$ <br> $G(h,k) = \phi(a_1 + h) - \phi(a_2) = \phi'(\theta_1)h$ where $a_1 < \theta_1 < a_1 + h$ <br> Define $\Psi(y) = f_x(\theta_1, y)$ <br> Apply Lagrange's mean value theorem to $\Psi(x)$ in the interval $[a_2, a_2 + k]$ <br> Thus $G(h,k) = hkf_{xy}(\theta_1, \theta_2)$ <br> Apply the same procedure to $u(y) = f(a_1 + h, y) - f(a_1, y)$ <br> And show that mixed partial derivatives are equal. |
| | | |
| Q3. | Attempt any **TWO** questions from the following: (12) | |
| $b)$ | i. | Find total derivative as linear transformation $T$ for the function <br> $f(x,y) = x^2 + 2y^2 + 3z$ at point $a = (1,-1,0)$ |
| | Ans | Total derivative as linear transformation is $T_a(v) = \nabla f(a) \cdot v$ <br> let $v = (x,y,z)$ *and* $\nabla f(a) = \left(f_x(a), f_y(a), f_z(a)\right) = (2,-4,0) \Rightarrow T_a(v) = 2x - 4y$ |
| | ii. | Find directional derivative of $f(x,y) = x^2 - 3xy$ at $(1,2)$ along the parabola $y = x^2 - x + 2$ |
| | Ans | Formula is $D_u f(a) = \nabla f(a) \cdot T$. where $T$ is unit tangent vector to the surface. <br> $\nabla f(a) = \left(f_x(a), f_y(a)\right) = (-4,-3)$ <br> Tangent vector $= (1, 2t-1) \Rightarrow T = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \Rightarrow D_u f(a) = \frac{-7}{\sqrt{2}}$ |
| | iii. | Define level set of $f: \mathbb{R}^n \to \mathbb{R}$ for $k \in \mathbb{R}$. Find level sets of the following scalar fields for the given constants. <br> 1. $f(x,y) = x^2 + 4y^2$ for $k = 1, 4$. <br> 2. $f(x,y,z) = x^2 + y^2 + z^2$ for $k = 1, 9$. |
| | Ans | 1. Ellipse with semi major axis 1 and semi minor axis $\frac{1}{2}$ <br> Ellipse with semi major axis $\frac{1}{2}$ and semi minor axis 1 |

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| | | 2. Circles centered at origin with radii 1 and 3. |
| | iv. | Using chain rule, evaluate the total derivative of $w = \sqrt{xy+yz+zx}$ where $x = t,\ y = \sin t$ and $z = \cos t$. |
| | Ans | $\frac{1}{2\sqrt{xy+yz+zx}}[(y+z)+(x+z)\cos t-(x-y)\sin t]$ |
| | | |
| Q4. | Attempt any **ONE** question from the following: (08) | |
| $a)$ | i. | Let $U$ be an open set in $\mathbb{R}^n$ and $f: U \to \mathbb{R}^m$ be given by $f(x) = (f_1(x), f_2(x), \dots f_m(x))$, $\forall x \in U$. Prove that $f$ is differentiable at $a \in U$ if and only if each $f_i$ is differentiable at $a$ and for any $u \in \mathbb{R}^n$. $Df(a)(u) = (Df_1(a)(u), Df_2(a)(u), \dots, Df_m(a)(u))$ |
| | Ans | Let $f$ is differentiable at $a \in \mathbb{R}^n$ with total derivative $Df(a)$.<br>$\Leftrightarrow f(a+h) = f(a) + Df(a) + \lVert h \rVert\, E(h)$<br>where $E(h) \to 0$ $as$ $\lVert h \rVert \to 0$ 1M<br>$\Leftrightarrow (f_1(a+h), \cdots\cdots, f_m(a+h))$<br>$= (f_1(a), \cdots\cdots, f_m(a)) + D(f_1(a), \cdots\cdots, f_m(a)) + \lVert h \rVert (E_1(a), \cdots\cdots, E_m(a))$<br>where $E_i(h) \to 0$ $as$ $h \to 0$<br>$\Leftrightarrow (f_1(a+h), \cdots\cdots, f_m(a+h))$<br>$= (f_1(a) + Df_1(a) + \lVert h \rVert E_1(a), \cdots\cdots\cdots, f_m(a) + Df_m(a) + E_m(a))$<br>$\Leftrightarrow f_i(a+h) = f_i(a) + Df_i(a) + \lVert h \rVert E_i(a) \quad \forall\ i = 1\ to\ m$<br>$\Leftrightarrow$ each $f_i$ is differentiable at $a \in \mathbb{R}^n$. 6M<br>Further, $Df_i(a)(h) = T_i(h)$<br>therefore, $Df(a)(u) = T(u) = (T_1(u),\ T_2(u), \cdots\cdots, T_m(u))$<br>$= (Df_1(u),\ Df_2(u), \cdots\cdots, Df_m(u))$ 1M |
| | ii. | Let $f: S \subseteq \mathbb{R}^n$ be a scalar field where $S$ is a non-empty open subset of $\mathbb{R}^n$. Let $a \in S$ and $f$ is differentiable at $a$. Prove that if $f$ has a local maximum or local minimum at $a$ then $\nabla f(a) = 0$. |
| | Ans | Let $a = (a_1, a_2, , \dots . a_n) \in S$. Assume $f$ has a local maxima at $a \in S$.<br>To show $\frac{\partial f}{\partial x_i}(a) = 0,\ i = 1,2, \dots n$. Let $g_1(t) = f(t, a_2, \dots . . a_n)$ then $g_1(t)$ is a function of single variable and has local maximum at $t = a_1$.<br>Also $g_1(t)$ is differentiable as $f$ is differentiable. By the theorem of one |

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| | | variable $\frac{d}{dt}g_1(t) = 0\ at\ t = a_1$. But $\frac{d}{dt}g_1(a_1) = \frac{\partial f}{\partial x_1}(a) = 0$.<br>Similarly considering $g_2(t) = f(a_1, t, a_3, \ldots .. a_n)$ we get $\frac{\partial f}{\partial x_2}(a)$=0.<br>In general $\frac{\partial f}{\partial x_i}(a) = 0, i = 1,2, \ldots n$.<br>So $\nabla f(a) = \left(\frac{\partial f}{\partial x_1}(a), \frac{\partial f}{\partial x_2}(a), \ldots \ldots, \frac{\partial f}{\partial x_n}(a)\right) = (0,0, \ldots \ldots,0)$.<br>Similarly the proof for minima. |
| | | |
| Q4. | Attempt any **TWO** questions from the following: (12) | |
| b) | i. | If $f(x,y,z) = xi + yj + zk$ then prove that the Jacobian matrix $Df(x,y,z)$ is the identity matrix of order 3. Also find all differentiable vector fields $f: \mathbb{R}^3 \to \mathbb{R}^3$ for which the Jacobian matrix $Df(x,y,z)$ is the identity matrix of order 3. |
| | Ans | a) $f(x,y,z) = x\,\bar{i} + y\,\bar{j} + z\,\bar{k}$<br>$Df(x,y,z) = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ which is identity matrix of order 3.<br>2M<br>b) Let $f(x,y,z) = (f_1, f_2, f_3)$ where $f_1, f_2, f_3$ are functions of $x, y, z$.<br>Since $Df(x,y,z)$ is identity matrix of order 3.<br>$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y} = \frac{\partial f_3}{\partial z} = 1 \Rightarrow f_1 = x + a, f_2 = y + b, f_3 = z + c\ , a, b, c \in \mathbb{R}$<br>$\therefore f(x,y,z) = (x + a, y + b, z + c)$<br>4M |
| | ii. | Determine the second order Taylor formula for the function $f(x,y) = e^x\, cosy$ at $(0, \frac{\pi}{2})$ |
| | Ans | $f(x,y) = e^x \cos(y) \qquad p = \left(0, \frac{\pi}{2}\right) = (a, b)$<br>$f_x = e^x \cos y\ ,\ f_y = -e^x \sin y\ ,$<br>$f_{xx} = e^x \cos y\ ,\ f_{xy} = -e^x \sin y\ ,\ f_{yy} = -e^x \cos y$<br>2M<br>$f(p) = f_x(p) = f_{xx}(p) = f_{yy}(p) = 0\ , f_y(p) = f_{xy}(p) = -1$ 1M<br>Using Taylor's formula<br>$f(x,y) = f(p) + (x-a)f_x(p) + (y-b)f_y(p) +$<br>$\frac{1}{2!}\left[(x-a)^2 f_{xx}(p) + 2(x-a)(y-b)f_{xy}(p) + (y-b)^2 f_{yy}(p)\right]$ 2M |

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| | | $f(x,y) = -\left(y-\frac{\pi}{2}\right) - x\left(y-\frac{\pi}{2}\right) = -xy + \frac{\pi}{2}x - y + \frac{\pi}{2}$ | 1M |
| | iii. | Find the critical points, saddle points and local extrema if any for the function $f(x,y) = y^2 - y^3 - x^2 + xy$. | |
| | Ans | (see table below) | |
| | iv. | If $2x+3y+4z=a$ then prove that the maximum value of $x^2y^3z^4$ is $\left(\frac{a}{9}\right)^9$. | |
| | Ans | $h(x,y,z) = x^2y^3z^4 + \lambda(2x+3y+4z-a). \frac{\partial h}{\partial x} = 2xy^3z^4 + 2\lambda = 0,$ $\frac{\partial h}{\partial y} = 3x^2y^2z^4 + 3\lambda = 0, \frac{\partial h}{\partial z} = 4x^2y^3z^3 + 4\lambda = 0. x = y = z.$ Put in $2x+3y+4z=a$ we will get $x=y=z=a/9$ . $f\left(\frac{a}{9},\frac{a}{9},\frac{a}{9}\right) = \left(\frac{a}{9}\right)^9$. | |
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| Q5. | Attempt any **FOUR** questions from the following: (20) | | |
| $a)$ | Evaluate the limit of the following functions, if it exists, by converting to polar Coordinates. (i) $\lim_{(x,y)\to(0,0)} y\log(x^2+y^2)$ (ii) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{\sqrt{x^2+y^2}}$ | | |

Table for iii (Ans):

| Critical points | $\Delta$ | $f_{xx}$ | Extrema | $f(x,y)$ |
|---|---|---|---|---|
| (0,0) | −5 | | Saddle point | |
| (5/3, 5/6) | 5 | −2 | maxima | 8 |

(11)

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| Ans | Q. 5 i) Evaluate the limit of the following functions, if it exists by using polar coordinates<br>i) $\lim_{(x,y)\to(0,0)} y\log(x^2+y^2)$ ii) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{\sqrt{x^2+y^2}}$<br>Sol Ans i) $\lim_{(x,y)\to(0,0)} y\log(x^2+y^2) = \lim_{r\to 0} r\sin\theta \log(r^2)$ As $x = r\cos\theta$, $y = r\sin\theta$<br>$= \lim_{r\to 0} 2\times\sin\theta\times\frac{\log r}{1/r}$ ($\frac{\infty}{\infty}$ form)<br>$= 2\sin\theta \lim_{r\to 0}\frac{1/r}{-1/r^2}$ (By L'Hospitals rule)<br>$= 2\sin\theta \lim_{r\to 0}(-r)$ (03) marks<br>$= 0$<br>ii) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{\sqrt{x^2+y^2}} = \lim_{r\to 0}\frac{r^4\cos^2\theta\sin^2\theta}{\sqrt{r^2}}$<br>$= \lim_{r\to 0} r^3\cos^2\theta\sin^2\theta = 0$<br>($\because x = r\cos\theta$, $y = r\sin\theta$) (02) Marks |
| b) | Find the real value of $\theta \in (0,1)$ if it exists, satisfying,<br>$f(b) - f(a) =< \nabla f(a+\theta(b-a), b-a >$ for the following functions:<br>$f(x,y,z) = xyz,\quad a = (0,0,0),\quad b = \left(1,\frac{1}{2},\frac{1}{3}\right)$ |
| Ans | To find $\nabla f\big(a+\theta(b-a)\big)$:<br>$f_x(x,y,z) = yz;\ f_y(x,y,z) = xz;\ f_z(x,y,z) = xy$<br>$a+\theta(b-a) = (0,0,0)+\theta\left(1,\frac{1}{2},\frac{1}{3}\right) = \left(\theta,\frac{\theta}{2},\frac{\theta}{3}\right) \Rightarrow \nabla f\big(a+\theta(b-a)\big) = \left(\frac{\theta^2}{6},\frac{\theta^2}{3},\frac{\theta^2}{2}\right)$<br>Substituting this in $f(b)-f(a) = \nabla f\big(a+\theta(b-a)\big)\cdot(b-a)$<br>We have, $f\left(1,\frac{1}{2},\frac{1}{3}\right) - f(0,0,0) = \langle\left(\frac{\theta^2}{6},\frac{\theta^2}{3},\frac{\theta^2}{2}\right),\left(1,\frac{1}{2},\frac{1}{3}\right)\rangle$<br>$\frac{1}{6} = \frac{\theta^2}{2} \Rightarrow \theta = \frac{1}{\sqrt{3}}$ |

| | |
|---|---|
| c) | Find the maximum rate of change of the function $f(x,y,z) = \log(x+y+z)$ at $(1,2,3)$. Also find the direction in which maximum rate of change occurs. |
| Ans | $\nabla f(x,y,z) = \left(\frac{1}{x+y+z}, \frac{1}{x+y+z}, \frac{1}{x+y+z}\right)$ (2 Marks)<br>$\nabla f(1,2,3) = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ (1 Mark)<br>Maximum rate of change = $\lVert \nabla f(1,2,3) \rVert = \frac{\sqrt{3}}{6}$ (1 Mark)<br>Direction $\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$ (1 Mark) |
| d) | Let $f(x,y) = x^3 + 9xy^2$, find $f_x, f_y, f_{xy}, f_{xx}, f_{yy}$. |
| Ans | $f_x = 3x^2 + 9y^2$ $f_y = 18xy$ $f_{xy} = 18y$ $f_{xx} = 6x$ $f_{yy} = 18x$ |
| e) | Given $z = f(x,y)$ where $f$ has continuous partial derivatives of second order, $x = u + v, y = u - v$, show that $\frac{\partial^2 z}{\partial u\, \partial v} = \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}$ |
| Ans | $\frac{\partial x}{\partial u} = \frac{\partial x}{\partial v} = 1$ , $\frac{\partial y}{\partial u} = 1$, $\frac{\partial y}{\partial v} = -1$ 1M<br>using chain rule, $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}$ 1M<br>diff $\frac{\partial z}{\partial v}$ w. r. t '$u$'<br>$\frac{\partial^2 z}{\partial u\, \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right)$<br>$= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial u} + \frac{\partial^2 z}{\partial x\, \partial y}\frac{\partial y}{\partial u} - \frac{\partial^2 z}{\partial x\, \partial y}\frac{\partial x}{\partial u} - \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial u}$ 1M<br>$= \frac{\partial^2 z}{\partial x^2}(1) + \frac{\partial^2 z}{\partial x\, \partial y}(1) - \frac{\partial^2 z}{\partial x\, \partial y}(1) - \frac{\partial^2 z}{\partial y^2}(1)$ 1M<br>$= \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}$ 1M |
| f) | Find the Hessian matrix of $f: \mathbb{R}^3 \to \mathbb{R}$ at $(1,1,1)$ where $f(x,y,z) = x^3y^2(12 - 3x - 4y)$. |
| Ans | $\begin{bmatrix} 12 & 12 & 0 \\ 12 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |

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