Q.P. Code: 59962

(1)

(3 Hours) [Total Marks: 100]

Note: (i) All questions are compulsory.
(ii) Figures to the right indicate marks for respective parts.

					(20)
Q.1	Choose correct alternative in each of the following				
$i.$	If $T:U \rightarrow V$ is a linear transformation such that ker $T=\{0\}$ then				
	(a)	Always surjective	(b)	Always injective	
	(c)	Always bijective	(d)	None the above	
	Ans	b			
$ii.$	Which of the following is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$				
	(a)	$T(x,y)=(x,y)$	(b)	$T(x,y)=(x, y+1)$	
	(c)	$T(x,y)=(x^2+y, x-y)$	(d)	All the above	
	Ans	a			
$iii.$	Let $f:V \rightarrow W$ be an isomorphism of vector spaces V and W. Then, which of the following is true?				
	(a)	Dim V = dim W	(b)	f is bijection	
	(c)	f^{-1} exist	(d)	All the above	
	Ans	d			
$iv.$	Consider the system $AX=b$, Where $A \in M_n(\mathbb{R})$ an invertible matrix and b a non zero column vector. The solution of above system obtained by using cramer's rule is				
	(a)	Always unique	(b)	May not be unique	
	(c)	Always Zero	(d)	None of these.	
	Ans	(a)			
$v.$	Dimension of Solution space of a $m \times n$ homogeneous system of linear equations $AX=0$ is				
	(a)	m	(b)	n	
	(c)	$m - Rank\ A$	(d)	$n - Rank\ A$	
	Ans	(d)			
$vi.$	Let $E=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$ then E^{-1} is				
	(a)	$\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}$	(b)	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	
	(c)	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$	(d)	$\begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}$	
	Ans	(c)			
$vii.$	If A is a non-singular matrix of order n, then the rank of A is				
	(a)	n	(b)	0	
	(c)	$n-1$	(d)	None of these	
	Ans	(a)			
$viii.$	Order of group of prime residue classes modulo n, $U(n)$ is				
	(a)	n	(b)	$n!$	
	(c)	Euler function $\phi(n)$	(d)	None of these	
	Ans	(c)			

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ix.	Let $G_1 = \{\bar{1}, \bar{2}, \bar{3}\}$ mod 4 under multiplication of residue classes modulo 4 and $G_2 = \{\bar{1}, \bar{2}\}$ mod 3 under multiplication of residue classes modulo 3. Then			
	(a)	G_1 and G_2 are groups	(b)	only G_1 is group
	(c)	only G_2 is group	(d)	None of these
	Ans	(c)		
x.	Identity element of group $\mathbb{Q}^*$ under binary operation defined as $a * b = \frac{ab}{2}$ is			
	(a)	1	(b)	2
	(c)	$\frac{1}{2}$	(d)	4
	Ans	(b)		
Q2.	Attempt any **ONE** question from the following: (08)			
a)	i.	Let V, W be vector spaces over $\mathbb{R}$ and $T: V \to W$ be a linear transformation and if V is finite dimensional then show that $\dim V = \dim \operatorname{Ker} T + \dim \operatorname{Im} T$.		

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	Ans	**Proof :** We have $T : V \to W$, be a linear transformation, $\ker T \subseteq V$ is a subspace of V.	
		Let $\dim V = n$, $\dim \ker T = r$, $\dim W = m$	
		Let $B = \{u_1, u_2, \cdots, u_r\}$ be basis of ker T As ker T is subspace of V, B is a linearly independent subset of V and hence can be extended to a basis of V.	
		Let $B_1 = \{u_1, u_2, \cdots, u_r, u_{r+1}, \cdots, u_n\}$ be a basis of V, obtained by extension of B.	1
		Let $w_i = T(V_{r+i}), \forall i = 1, \cdots, n-r$.	
		Claim : $B_2 = \{w_1, w_2, \cdots, w_{n-r}\}$ forms a basis of I_mT	
		Let us prove first, B_2 is linearly independent	1
		Let $a_1, a_2, \cdots a_{n-r}$ be scalars such that	
		$a_1w_1 + a_2w_2 + \cdots + a_{n-r}w_{n-r} = 0$	1
		But $T(u_{r+1}) = w_1, T(u_{r+2}) = w_2, \cdots T(u_n) = w_{n-r}$	
		$\therefore a_1T(u_{r+1}) + a_2T(u_{r+2}) + \cdots + a_{n-r}T(u_n) = 0$	
		$\therefore T(a_1u_{r+1} + a_2u_{r+2} + \cdots + a_{n-r}u_n) = 0$	1
		$\Rightarrow T\left(\sum_{i=1}^{n-r} a_i u_{r+i}\right) = 0$	
		$\Rightarrow \sum_{i=1}^{n-r} a_i u_{r+i} \in \ker T$	
		$\Rightarrow \exists\, b_1, b_2, \cdots, b_r$ scalar s.t.	1
		$\sum_{i=1}^{n-r} a_i u_{r+i} = \sum_{j=1}^{r} b_j u \cdots$ as B is basis of ker T	1
		$\Rightarrow b_1u_1 + b_2u_2 + \cdots + b_ru_r - (a_1u_{r+1} + \cdots + a_{n-r}u_n) = 0$	
		As $B_1 = \{u_1, u_2, \cdots, u_r, u_{r+1}, \cdots u_n\}$ is lin independent	1
		$\Rightarrow b_1 = b_2 = b_r = 0$ and	

$a_1 = a_2 = a_{n-r} = 0$ 1

$\Rightarrow \{w_1, w_2, \cdots, w_{n-r}\}$ is linearly independent $\cdots$ (I)

Claim : $\{w_1, w_2, \cdots, w_{n-r}\}$ spans $I_m(T)$

Let $w \in I_m T$

$\Rightarrow \exists\, v \in V$ such that $T(v) = w$.

As $B_1 = \{u_1, u_2, \cdots, u_r, u_{r+1}, u_{r+2}, \cdots u_n\}$ is a basis of V.

$\Rightarrow \exists\, b_1, b_2, \cdots, b_n \in \mathbb{R}$ such that

$v = b_1u_1 + b_2u_2 + \cdots + b_ru_r + b_{r+1}u_{r+1} + \cdots + b_nu_n$

As T is linear,

$\therefore\ T(v) = T(b_1u_1 + \cdots + b_ru_r) + T(b_{r+1}, u_{r+1} + \cdots + b_nu_n)$

$= b_1T(u_1) + b_2T(u_2) + \cdots + b_rT(u_r) + b_r + 1T(u_{r+1}) + \cdots b_nT(u_n) \cdots$

as T is linear

$\therefore\ T(v) = b_{r+1}T(u_{r+1}) + b_{r+2}T(u_{r+2}) + \cdots + b_nT(u_n)$

as $u_1, u_2, \cdots, u_r \in \ker T$

$\Rightarrow T(v) = b_{r+1}w_1 + b_{r+2}w_2 + \cdots + b_nw_{n-r}$

$\Rightarrow w = b_{r+1}w_1 + b_{r+2}w_2 \cdots + b_nw_n$.

$\Rightarrow w \in \text{span}\{w_1, \cdots, w_{n-r}\}$

$\Rightarrow$ If $w \in I_mT \Rightarrow w \in \text{span}\{w_1, w_2, \cdots, w_{n-r}\}$

$\Rightarrow \{w_1, w_2, \cdots, w_{n-r}\}$ spans I_mT. ... (II)

$\Rightarrow \{w_1, w_2, \cdots w_{n-r}\}$ forms a basis of ImT $\cdots$ From (I) and (II)

$\therefore\ \dim(I_mT) = n - r$

$\dim(\ker T) = r$

$\dim v = n = \dim(I_mT) + \dim(\ker T) = r + n - r.$

$\therefore\ \boxed{\dim(V) = \dim(\ker T) + \dim(I_mT)}$

as $\boxed{n = r + n - r}$

$\therefore$ Rank nullity theorem is verified.

ii. Let $T : V \rightarrow V$ be a linear transformation where V is a finite dimensional vector space over then prove that T is injective if and only if T is surjective.

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	Ans	Let $T: V \to V$ .Suppose T is 1-1. $\therefore$ ker T = {0} $\therefore$ dim (ker T) = 0. By rank nullity theorem. dim V = nullity T + Rank of T dim V = 0 + Rank of T dim V = dim (I_mT) As $I_m T \subseteq V$ subspace of V such that dim V = dim I_mT $\Rightarrow I_m T = V$ $\therefore$ T(V) = V $\therefore$ T is onto. Conversly, suppose T is onto. I_mT = V T(V) = V. $\therefore$ dim (I_mT) = dim V $\therefore$ Rank nullity theorem, din (V) = dim (ker T) + dim I_mT $\Rightarrow$ dim V = dim (ker T) + dim V $\Rightarrow$ dim ker T = 0. $\Rightarrow$ ker T = {0} $\Rightarrow$ T is 1 − 1 $\therefore$ T is 1 − 1 if and only if T is onto.	1 1 1 1 1 1 1 1
Q.2	Attempt any **TWO** questions from the following: (12)		
b)	i.	If $T: \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(x,y,z)=(x,y)$, show that T is linear. Find ker T, basis of ker T and nullity T.	
	Ans	Let $(x,y,z) \in ker T$ so that $x=0, y=0$ Thus we put $z=t$, $\Rightarrow ker T = \{t(0,0,1)/t \in \mathbb{R}\}$. Therefore the basis of $ker F$ is $(0,0,1)$ and nullity $T=1$.	1 1 2 2
	ii.	Check whether $T: \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(x,y,z)=(x+y, x-z, y+2z)$ is an isomorphism?	

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Ans Let $(x,y,z)\in ker\,T$ so that 1

$x+y=0, x-z=0, y+2z=0$ 1

The matrix corresponding to this system $\begin{pmatrix}1 & 1 & 0\\ 1 & 0 & -1\\ 0 & 1 & 2\end{pmatrix}$ whose row reduced form is 2

$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$. Thus $x=y=z=0 \Rightarrow ker\,F=\{0\}$ 2

Since Dim V = 3 = $\dim\mathbb{R}^3$, T is onto. Therefore T is invertible. Therefore T is an isomorphism

iii. Show that any n-dimensional vector space is isomorphic to $\mathbb{R}^n$.

Ans Let $x_1, x_2 \ldots\ldots x_n$ be a basis of V of dim n. Therefore for 2

$x\in V, x=\sum c_i x_i \quad \forall c_i\in\mathbb{R}$ 2

Thus a LT is defined by $T(x)=(c_1,\ldots\ldots\ldots c_n)$ is one-one.

Since Dim V = n = $\dim\mathbb{R}^n$, T is onto. Therefore T is invertible. Therefore $V\cong\mathbb{R}^n$. 2

iv. $P_3[\mathbb{R}]$ denote the vector space of all polynomials over $\mathbb{R}$ of degree 3 or less an $D(f(x)) = \frac{df(x)}{dx}, \forall f(x)\in P_3[\mathbb{R}]$ denote the differentiation mapping. Let $B=\{1, 1+x, 1+x^2, 1+x^3\}$ be the basis. Find $[m(D)]_B^B$.

Ans

$Dv_1 = 0 = 0v_1+0v_2+0v_3+0v_4$

$Dv_2 = 1 = 1v_1+0v_2+0v_3+0v_4$ 4

$Dv_3 = 2x = -2v_1+2v_2+0v_3+0v_4$

$Dv_4 = 3x^2 = -3v_1+0v_2+3v_3+0v_4$

$$\Rightarrow [m(D)]_B^B = \begin{pmatrix}0 & 1 & -2 & -3\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 3\\ 0 & 0 & 0 & 0\end{pmatrix}$$ 2

Q3. Attempt any **ONE** question from the following: (08)

a) i. Let $A^1, A^2\in\mathbb{R}^2$ and $c\in\mathbb{R}$. Show that

i) $\det(A^1, A^2)=0$ iff $\{A^1, A^2\}$ is linearly dependent.

ii) $\det(A^1+cA^2, A^2)=\det(A^1, A^2)$.

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Ans	I) (⇒) Given: $\det(A^1, A^2) = 0$ T.P.T : $\{A^1, A^2\}$ is linearly dependent. Suppose $\{A^1, A^2\}$ is linearly independent $\therefore \{A^1, A^2\}$ is the basis of $\mathbb{R}^2$ Let $E^1 = \alpha_1 A^1 + \alpha_2 A^2$ and $E^2 = \beta_1 A^1 + \beta_2 A^2$ $\det(E^1, E^2) = \det(\alpha_1 A^1 + \alpha_2 A^2, \beta_1 A^1 + \beta_2 A^2)$ $= (\alpha_1\beta_2 - \alpha_2\beta_1)\det(A^1, A^2)$ $= 0$. which is a contradiction.	3
	(⇐) Given: $\{A^1, A^2\}$ is linearly dependent. T.P.T : $\det(A^1, A^2) = 0$ As $\{A^1, A^2\}$ is linearly dependent $\therefore$ Let $A^2 = cA^1, c \neq 0, c \in \mathbb{R}$ $\det(A^1, A^2) = \det(A^1, cA^1)$ $= c\det(A^1, A^1)$ $= 0$	3
	II) $\det(A^1 + cA^2, A^2) = \det(A^1, A^2) + \det(cA^2, A^2)$ $= \det(A^1, A^2) + c\det(A^2, A^2)$ $= \det(A^1, A^2)$	2
ii.	Prove that the general solution of the non homogeneous system is sum of a particular solution of the system and the solutions of the associated homogeneous system.	
Ans	Let $AX = b$ be a non-homogeneous system of linear equations And $AX = 0$ be its associated homogeneous system of linear equations. Let $x_0$be the particular solution of non-homogeneous system $AX = b$ T.P.T : Set of solutions of the non-homogeneous system $AX = b$ is given by, $\{x_0 + x \mid x$ is a solution of associated homogeneous system $AX = 0\}$ ... (*) Claim 1: Every element of (*) is a solution of non-homogeneous system $AX = b$ Let x is be a solution of associated homogeneous system $AX = 0 \therefore Ax = 0$ Then, $A(x_0 + x) = Ax_0 + Ax = b + 0 = b$ $\therefore x_0 + x$ is the solution of non-homogeneous system $AX = b$ Claim 2: Every solution of non-homogeneous system $AX = b$ is element of (*) Let x' be the solution of non-homogeneous system $AX = b$ $x' = x_0 + (x' - x_0)$	4
	Now, $A(x' - x_0) = Ax' - Ax_0 = b - b = 0$ Hence, x' is an element of (*) $\therefore$By claim 1 and claim 2 Set of solution of the non homogeneous system is precisely the sum of a particular solution of the system and the solutions of the associated homogeneous system.	4

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Q3.	Attempt any **TWO** questions from the following:		(12)
b)	i.	Define adjoint of a matrix. Find A^{-1} for $A = \begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$ using adjoint.	
	Ans	For $A \in M_n(\mathbb{R})$, Let A_{ij} be the matrix obtained from A by deleting its ith row and jth column Let $c_{ij} = (-1)^{i+j} \det A_{ij}$ $C = (c_{ij})$ is called matrix of cofactors $adj(A) := C^t$	2
		Given matrix is $A = \begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$ Matrix of cofactors is $C = \begin{pmatrix} 2 & -2 & 2 \\ 2 & 3 & -3 \\ 0 & 10 & 0 \end{pmatrix}$	2
		$\text{Adj}(A) = C^t = \begin{pmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{pmatrix}$ $A^{-1} = \frac{1}{\det A} Adj(A) = \frac{1}{10} \begin{pmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{pmatrix}$	2
	ii.	Solve the following system using Cramer's rule. $2x + y + z = 1, x - y + 4z = 0, x + 2y - 2z = 3$	
	Ans	$2x + y + z = 1, x - y + 4z = 0, x + 2y - 2z = 3$The corresponding non-homogeneous system is $\begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 4 \\ 1 & 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$	2
		$x = \frac{det \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 4 \\ 3 & 2 & -2 \end{pmatrix}}{\det \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 4 \\ 1 & 2 & -2 \end{pmatrix}} = \frac{9}{-3} = -3$	
		$y = \frac{det \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 3 & -2 \end{pmatrix}}{\det \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 4 \\ 1 & 2 & -2 \end{pmatrix}} = \frac{-15}{-3} = 5$	2
		$z = \frac{det \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 3 \end{pmatrix}}{\det \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 4 \\ 1 & 2 & -2 \end{pmatrix}} = \frac{-6}{-3} = 2$	2

iii.		Define Elementary matrix. Which of the following are elementary matrices? Justify. $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$	
Ans		Elementary matrix is the matrix obtained from the identity matrix by applying any of the elementary row operations.	2
		A is obtained from identity matrix by applying row operation $R_1 \to R_1 + 2R_3$ $\therefore A$ is elementary matrix. B is obtained from identity matrix by applying row operation $R_2 \to R_3$ $\therefore B$ is elementary matrix. C can not be obtained from identity matrix by applying any of the row operation. $\therefore C$ is not elementary matrix.	4
iv.		$A \in M_n(\mathbb{R})$. Prove that the non-homogeneous system of linear equations $AX = b$ has a solution if and only if rank A=rank (A, b)	
Ans		The non-homogeneous system of linear equations $AX = b$ has a solution $\Leftrightarrow \exists c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ *such that* $Ac = b$	2
		$\Leftrightarrow \exists\, c_1, c_2, \ldots c_n \in \mathbb{R}$ (not all zero) such that $c_1A^1 + c_2A^2 + .. + c_n A^n = b$	2
		$\Leftrightarrow$ Column vector b is linearly dependent with 'n' columns of A	2
		$\Leftrightarrow$ rank A=rank (A, b)	
			(08)
Q4.		Attempt any **ONE** question from the following:	
$a)$	i.	Discuss the group of symmetries of a rectangle which is not a square. Identify this group.	
	Ans	Let 1,2,3,4 denote the vertices of a rectangle which is not a square and let O denote the centroid of the rectangle. The group of symmetries of G consists of 2 rotations and 2 reflections. (1) e = Rotation of 0° about O in the anticlockwise direction = $\begin{pmatrix} 1,2,3,4 \\ 1,2,3,4 \end{pmatrix}$. (draw figure) (2) σ_1= Rotation of 180° about O in the anticlockwise direction = $\begin{pmatrix} 1,2,3,4 \\ 3,4,1,2 \end{pmatrix}$ = (1,3)(2,4). (draw figure)	

(10)

		(3) σ_2= Reflection about the line passing through O which is the perpendicular bisector of side (1, 2) = $\begin{pmatrix}1,2,3,4\\2,1,4,3\end{pmatrix}$ = (1,2)(3,4). (draw figure) (4) σ_3= Reflection about the line passing through O which is the perpendicular bisector of side (1, 4) = $\begin{pmatrix}1,2,3,4\\2,3,2,1\end{pmatrix}$ = (1,4)(2,3). (draw figure) Thus $G = \{e, \sigma_1, \sigma_2, \sigma_3\}$ is a group of order 4. Further, since the square of every element of G gives the identity element e (this can also be seen geometrically). Therefore G is actaually the Klein-4 group V_4.	6 2
	ii.	Define Subgroup. Let G be a group and H, K be subgroups of G. Prove that $H \cap K$ is a subgroup of G but $H \cup K$ may not be a subgroup of G.	
	Ans	Let H be a non-empty subset of a group G. We say, $H$is a subgroup of G if Hitself is a group under the same binary operation as of G. Let H,K be subgroups of G. $\therefore e \in H$ and $e \in K \Rightarrow e \in H \cap K \Rightarrow H \cap K \neq \phi$ Consider any $a, b \in H \cap K \Rightarrow a, b \in H$ and $a, b \in K$ Since H ,$K$are subgroups of G, we get $ab^{-1} \in H$ and $ab^{-1} \in K$. $\therefore ab^{-1} \in H \cap K$ $\therefore H \cap K$ is a subgroup of G Further, $H \cup K$ may not be a subgroup of G. For example, Consider the group $U(8) = \{\bar{1}, \bar{3}, \bar{5}, \bar{7}\}$ under multiplication modulo 8. Let $H = \{\bar{1}, \bar{3}\}$ and $K = \{\bar{1}, \bar{5}\}$, then H and K are subgroups of G which follows from the following composition tables: (table 1 below) (table 2 below) But $H \cup K = \{\bar{1}, \bar{3}, \bar{5}\}$ is not a subgroup of G since $\bar{3}, \bar{5} \in H \cup K$ and$\bar{3} \cdot \bar{5} = \bar{7} \notin H \cup K$.	2 3 3
			(12)
Q4.		Attempt any **TWO** questions from the following:	
$b)$	i.	Let G be a group with identity e. (p) Show that: $(ab)^{-1} = b^{-1}a^{-1}, \forall a, b \in G$. (q) If $a^2 = e, \forall a \in G$, then G is abelian.	

Table 1:

$\times_8$	$\bar{1}$	$\bar{3}$
$\bar{1}$	$\bar{1}$	$\bar{3}$
$\bar{3}$	$\bar{3}$	$\bar{1}$

Table 2:

$\times_8$	$\bar{1}$	$\bar{5}$
$\bar{1}$	$\bar{1}$	$\bar{5}$
$\bar{5}$	$\bar{5}$	$\bar{1}$

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Ans	(p) To show that for any group G, $(ab)^{-1} = b^{-1}a^{-1}$ We have $(ab)(ab)^{-1} = e$. Left-multiplying both sides of this equation first by a and then by then by b^{-1} yields $(ab)^{-1} = b^{-1}a^{-1}$. (q) Let $a, b \in G$, then $a^2 = e$ and $b^2 = e$ as also $(ab)^2 = e$, which implies $(ab)(ab) = aa \underset{\text{LCL}}{\Rightarrow} bab = a \Rightarrow bab^2 = ab \Rightarrow ba = ab$. $\therefore G$ is abelian	3 3
ii.	Construct composition table of $\mathbb{Z}_5^*$ under multiplication modulo 5. Also find order of its each element.	
Ans	$\mathbb{Z}_5^* = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ Composition table $\bar{1}$ is the identity. $\therefore\ o(\bar{1}) = 1$ $\bar{2}^2 = \bar{4}$, $\bar{2}^3 = \bar{3}$, $\bar{2}^4 = \bar{1}$ $\therefore o(\bar{2}) = 4$ $\bar{3}^2 = \bar{4}$, $\bar{3}^3 = \bar{2}$, $\bar{3}^4 = \bar{1}$ $\therefore o(\bar{3}) = 4$ $\bar{4}^2 = \bar{1}$ $\therefore o(\bar{4}) = 2$	2 1 1 1 1
iii.	Let G be a group and $a \in G$. Show that $H = \{a^n \mid n \in \mathbb{Z}$ is the smallest subgroup of G containing a.	
Ans	$H \neq \phi$, since $e = a^0 \in H$. Also, for $a^n, a^m \in H$, $a^n(a^m)^{-1} = a^{n-m} \in H$. Therefore, H is a subgroup of G. Also, if K is a subgroup of G containing a, then $a, a^{-1} \in K$. Now, since K is a group, $\therefore a^n, a^{-n} = (a^{-1})^n \in K, \forall n \in \mathbb{N}$. Whence $a^n \in K, \forall n \in \mathbb{Z}$. $\therefore H \subseteq K$, so that H is the smallest subgroup of G containing a.	3 3
iv.	Prove that $G = \left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R}\right\}$ is a group under matrix addition.	
Ans	Closure : $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ -b-d & a+c \end{pmatrix} \in G$ Prove Associative Prove Identity is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ Prove inverse of $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is $\begin{pmatrix} -a & -b \\ b & -a \end{pmatrix}$	1 2 1 2
	(20)	
Q5.	Attempt any **FOUR** questions from the following:	
a)	Let $T : V \to W$ be a linear transformation such that $\ker T = \{0\}$. Prove that if $v_1, v_2 \ldots\ldots v_n$ are linearly independent then $T(v_1), T(v_2) \ldots\ldots\ldots T(v_n)$ are linearly independent elements of W.	
Ans	Proof: Let $c_1Tv_1 + c_2Tv_2 + \ldots\ldots + c_nTv_n = o$ Since T is l.t. it follows that $T(c_1v_1 + \ldots\ldots + c_nv_n) = o$. As $\ker T = \{o\}$, this implies that $c_1v_1 + \ldots\ldots + c_nv_n = o$ which further implies that $c_1, c_2, \ldots\ldots c_n$ are all zero, proving that $Tv_1, Tv_2, \ldots\ldots Tv_n$ are l.i.	1 2 1 1
b)	Find the map $T : \mathbb{R}^2 \to \mathbb{R}^2$, such that $T(3, 1) = (2, -4)$ and $T(1, 1) = (0, 2)$	

Ans	$(x,y)=a(3,1)+b(1,1)$ $\Rightarrow 3a+b=x, a+b=y \Rightarrow a=\frac{x-y}{2}, b=\frac{3y-x}{2}$ $(x,y)=\left(\frac{x-y}{2}\right)(3,1)+\left(\frac{3y-x}{2}\right)(1,1)$ $T(x,y)=\left(\frac{x-y}{2}\right)T(3,1)+\left(\frac{3y-x}{2}\right)T(1,1)$ $T(x,y)=\left(\frac{x-y}{2}\right)(2,3)+\left(\frac{3y-x}{2}\right)(1,4)$ $=(x-y,5y-3x)$	1 2 1 1 1
c)	Check whether the set $\{(2,0,1),(4,1,-1),(-1,0,2)\}$ is linearly dependent or independent using determinants.	
Ans	Columns of A are linearly dependent iff $\det A = 0$ $\det\begin{pmatrix} 2 & 4 & -1 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix} = 23 \neq 0$ $\{(2,0,1),(4,1,-1),(-1,0,2)\}$ is linearly independent	2 2 1
d)	Find Rank of the matrix $A=\begin{pmatrix} 1 & 3 \\ 0 & -1 \\ 3 & 4 \end{pmatrix}$. What can you say about the rank of the matrix B which is obtained from A by multiplying 2nd row of A by 3?	
Ans	Number of linearly independent columns of $A = 2$ $\therefore$Rank A=2. We know that rank of the matrix does not change by applying any of the row or column operations on it. $\therefore$ Rank of the matrix B which is obtained from A by multiplying 2nd row of A by 3 is also equals 2.	2 2 1
e)	Define the order of an element a of a group $G$and show that $\text{order}(a^{-1}) = \text{order}(a)$.	

Ans	The order of an element $a$of a group is the smallest positive integer m such that $a^m = e$, where $e$denotes the identity element of the group.
	If no such $m$exists, a is said to have infinite order.
	If, $a^n = e$, then repeated multiplication by a^{-1} - n times gives $(a^{-1})^n = e$. This shows that if a is of finite order than so is a^{-1} with $\text{order}(a^{-1}) \le \text{order}(a)$.
	//ly, $\because (a^{-1})^{-1} = a$, therefore we see that if a^{-1} is of finite order than so is $a = (a^{-1})^{-1}$ with $\text{order}(a) \le \text{order}(a^{-1})$. Whence, $\text{order}(a^{-1}) = \text{order}(a)$, in this case.
	Also, if a is of infinite order then so is a^{-1} (for if $a^{-1}$is of finite order then again we can as above that a is also of finite order.)
	//ly, $\because (a^{-1})^{-1} = a$, therefore if a^{-1} is of infinite order then so is a.
	Thus both$a$ and a^{-1} are of infinite order or both are of finite order with $\text{order}(a^{-1}) = \text{order}(a)$.
	This proves the result.
		1
		2
		2

f)	Show that H = $\{\overline{1}, \overline{7}, \overline{9}, \overline{23}\}$ modulo 40 is a subgroup of $U(40)$.	

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Ans	Since $(\bar{1}, \overline{40}) = 1, (\bar{7}, \overline{40}) = 1, (\bar{9}, \overline{40}) = 1$ and $(\overline{23}, \overline{40}) = 1$, H is a subset of $U(40)$. Composition table Closure property, Associative law and Identity Inverse $(\bar{1})^{-1} = \bar{1}, (\bar{7})^{-1} = \overline{23}, (\bar{9})^{-1} = \bar{9}, (\overline{23})^{-1} = \bar{7}$ H itself is a group under multiplication modulo 40. Hence, H is a subgroup of $U(40)$.	1 2 1 1

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