(1) Q.P. Code :- 54578

(3 Hours) [Total Marks: 100]

Note: (*i*) All questions are compulsory.

(*ii*) Figures to the right indicate marks for respective parts.

Q.1	Choose correct alternative in each of the following (20)				
i.	The set $S = \{(x,y) \in \mathbb{R}^2 \ / \ 0 \leq x^2 + y^2 \leq 3\}$ is				
	(a)	A closed set	(b)	Open as well as closed set	
	(c)	An open set	(d)	None of these	
	Ans	**A closed set**			
ii.	Let $g: \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x,y) = \begin{cases} \dfrac{x^2 - 4y^2}{x - 2y} & if\ x \neq 2y \\ g(x,y) & if\ x = 2y \end{cases}$ And if f is continuous on the whole plane, then $g(x)$ is				
	(a)	$2xy$	(b)	x	
	(c)	4y	(d)	None of these	
	Ans	$4y$			
iii.	$(x,y) = 100 - x^2 + y^2$. Then the direction along which the directional derivative of f at $(5,6)$ is 0 is				
	(a)	$(-10,12)$	(b)	$\mathbf{(12\,,10)}$	
	(c)	(2,2)	(d)	None of these	
	Ans	$(12\,,10)$			
iv.	Let A: Total derivative is a linear transformation. B: Every differentiable scalar field is continuous. Then which of the following is true?				
	(a)	A is true, B is false.	(b)	A is false, B is true.	
	(c)	Both A & B are true.	(d)	Both A & B are false.	
	Ans	Both A & B are true.			
v.	If $f(x,y) = \lvert xy \rvert, \forall (x,y) \in \mathbb{R}^2$ then				
	(a)	f is differentiable at (0,0)	(b)	f is continuous at (0,0) and $D_u f(0,0)$ exist for any vector u	
	(c)	The partial derivatives f_x, f_y does not exist at (0,0)	(d)	None of these.	
	Ans	f is continuous at (0,0) and $D_u f(0,0)$ exist for any vector u			
vi.	If $f: \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function such that $\frac{\partial f}{\partial y} = 0$, then				
	(a)	f is independent of x and z	(b)	f depends on x and z only	
	(c)	f is constant	(d)	None of these.	
	Ans	f depends on x and z only			
vii.	Which of the following is the level set of $f(x,y) = x^2 + y^2 + z^2$ for $k = 1$?				

	(a)	Sphere of radius 1 centered at origin	(b)	Circle of radius 1 centered at origin
	(c)	Sphere of radius 2 centered at origin	(d)	Sphere of radius 1 centered at (1,0,0)
	Ans	Sphere of radius 1 centered at origin		
viii.	If $u(x,y) = x^2 + y^2, x = r + e^s, \ y = log(s)$ then $\frac{\partial u}{\partial r}$ is			
	(a)	$r + e^s$	(b)	$2r + 2e^s$
	(c)	r	(d)	e^s
	Ans	$2r + 2e^s$		
ix.	A critical point of the function $f(x,y) = x^2y - x - y$ is			
	(a)	$(1,1)$	(b)	$\left(1, \frac{1}{2}\right)$
	(c)	$\left(1, -\frac{1}{2}\right)$	(d)	None of these
	Ans	$\left(1, \frac{1}{2}\right)$		
x.	Saddle point is a point where			
	(a)	the function has maximum value.	(b)	the function has minimum value.
	(c)	the function has zero value.	(d)	the function has neither maximum nor minimum value.
	Ans	the function has neither maximum nor minimum value.		
Q2.	Attempt any **ONE** question from the following: (08)			
a)	i.	Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a real valued function. Let $l \in \mathbb{R}$ such that $\lim_{(x,y)\to(a,b)} f(x,y) = l$. Also assume that the one dimensional limits $\lim_{x\to a} f(x,y)$ and $\lim_{y\to b} f(x,y)$ exists, then prove that $\lim_{x\to a} \lim_{y\to b} f(x,y) = \lim_{y\to b} \lim_{x\to a} f(x,y) = l$.		

3

	Ans	Q.1 ⓐ Let $\lim_{x\to a} f(x,y) = g(y)$ and $\lim_{y\to b} f(x,y) = h(x)$ T.P.T $\lim_{x\to a}\left(\lim_{y\to b} f(x,y)\right) = \lim_{x\to a} h(x) = L$ and $\lim_{y\to b}\left(\lim_{x\to a} f(x,y)\right) = \lim_{y\to b} g(y) = L$. Let $\epsilon > 0$ be given $\lim_{(x,y)\to(a,b)} f(x,y) = L$ $\Rightarrow$ for $\epsilon>0$, $\exists\ \delta_1>0$ such that $0<\|(x,y)-(a,b)\|<\delta$ gives $\|f(x,y)-L\| < \epsilon/2$ — ① $\lim_{y\to b} f(x,y) = h(x)$ $\Rightarrow$ for $\epsilon>0$, $\exists\ \delta_2>0$ s.t. $0<\|y-b\|<\delta_2$ gives $\|f(x,y)-h(x)\|<\frac{\epsilon}{2}$ — ② $\lim_{x\to a} f(x,y) = g(y)$ $\Rightarrow$ for $\epsilon>0$, $\exists\ \delta_3>0$ s.t. $0<\|x-a\|<\delta_3$ gives $\|f(x,y)-g(y)\|<\epsilon/2$ — ③ Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. (04) marks Then for $0<\|(x,y)-(a,b)\|<\delta$, $0<\|x-a\|<\delta$, $0<\|y-b\|<\delta$ we have from ①, ②, ③ $\|h(x)-L\| \le \|h(x)-f(x,y)\| + \|f(x,y)-L\|$ $< \epsilon/2 + \epsilon/2$ $= \epsilon$ & $\|g(y)-L\| < \epsilon$. Given any $\epsilon>0$, $\exists\ \delta>0$ such that (02) marks $0<\|x-a\|<\delta \Rightarrow \|h(x)-L\|<\epsilon$ & $0<\|y-b\|<\delta \Rightarrow \|g(y)-L\|<\epsilon$. $\Rightarrow \lim_{x\to a} h(x) = \lim_{y\to b} g(y) = L$ (02) marks
ii.	If $\langle x_n\rangle$ and $\langle y_n\rangle$ are convergent sequences in $\mathbb{R}^n$ and α, β are real constant, show that $\langle \alpha x_n + \beta y_n\rangle$ is also convergent in $\mathbb{R}^n$ and $$\lim_{n\to\infty} \alpha x_n + \beta y_n = \alpha \lim_{n\to\infty} x_n + \beta \lim_{n\to\infty} y_n.$$	
	Ans	Let $x_n \to p$ *and* $y_n \to q$ *as* $n \to \infty$ Let $\epsilon > 0$ *be arbitrary* $\exists n_1 \in \mathbb{N}$ *such that* $n \ge n_1 \Rightarrow \|x_n - p\| < \frac{\epsilon}{2\|\alpha\|}$ $\exists n_2 \in \mathbb{N}$ *such that* $n \ge n_2 \Rightarrow \|y_n - q\| < \frac{\epsilon}{2\|\beta\|}$ *Choose* $n_0 = Max\{n_1, n_2\}$, *then* $n \ge n_0 \Rightarrow n \ge n_1$ *and* $n \ge n_2$ Consider,

		$\lvert(\alpha x_n + \beta y_n) - (\alpha p + \beta q)\rvert = \lvert\alpha(x_n - p) + \beta(y_n - q)\rvert$ $\leq \lvert\alpha\rvert\lvert x_n - p\rvert + \lvert\beta\rvert\lvert y_n - q\rvert$ $< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ Hence, $\lim(\alpha x_n + \beta y_n) = \alpha \lim x_n + \beta \lim y_n$.
Q.2	Attempt any **TWO** questions from the following: (12)	
b)	i.	Using $\epsilon - \delta$ definition show that f is continuous at (0,0). where $$f(x,y) = \begin{cases} x^{\frac{4}{3}} \sin\left(\frac{1}{x}\right) + y^{\frac{4}{3}} \sin\left(\frac{1}{y}\right) & if\ xy \neq 0 \\ 0 & if\ xy = 0 \end{cases}$$
	Ans	Q2 b) i) use ε-δ definition to show that f is continuous at (0,0) where $f(x,y) = \{ x^{4/3}\sin(1/x) + y^{4/3}\sin(1/y),\ xy \neq 0;\ 0,\ xy = 0$ Ans consider $\lvert f(x,y) - 0\rvert = \lvert x^{4/3}\sin(1/x) + y^{4/3}\sin(1/y) - 0\rvert$ $\leq \lvert x\rvert^{4/3}\lvert\sin(1/x)\rvert + \lvert y\rvert^{4/3}\lvert\sin(1/y)\rvert$ $\leq \lvert x\rvert \times (1) + \lvert y\rvert \times (1)$ $\leq \lvert x\rvert + \lvert y\rvert$ $\leq \lVert (x,y) - (0,0)\rVert + \lVert (x,y) - (0,0)\rVert$ $\leq 2 \lVert (x,y) - (0,0)\rVert$ (04) marks ∴ for $\epsilon > 0$, choose $\delta = \epsilon/2$ $0 < \lVert (x,y) - (0,0)\rVert < \delta \Rightarrow \lvert f(x,y) - 0\rvert < 2 \times \epsilon/2 = \epsilon$ ∴ $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$ (02) marks ∴ f is continuous at (0,0).
	ii.	Prove that every linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is continuous on $\mathbb{R}^n$.

	Ans	Q:2 b:ii) Prove that every linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is continuous on $\mathbb{R}^n$. Ans: Let $x=(x_1,x_2,\dots x_n)$ & $y=(y_1,y_2,\dots,y_n)$ Let $T \neq 0$ Consider $\|T(x)-T(y)\|_2 = \|\sum_{i=1}^{n} x_i T(e_i) - \sum_{i=1}^{n} y_i T(e_i)\|_2$ $= \sum_{i=1}^{n} \|x_i - y_i\| \|T(e_i)\|_2$ $\leq \sum_{i=1}^{n} \|x-y\|_1 \|T(e_i)\|_2$ $[\because \|x_i-y_i\| \leq \|x-y\|]$ Let $M = \max\{\|T(e_i)\|_2 \mid i=1,2,\dots n\}$ $\leq nM\|x-y\|_1$ $\therefore \|T(x)-T(y)\|_2 \leq nM\|x-y\|_1$ (04) marks $\therefore$ for $\epsilon>0$, choose $\delta = \frac{\epsilon}{nM} > 0$ $\|x-y\|_1 < \delta = \frac{\epsilon}{nM} \Rightarrow \|T(x)-T(y)\| < \epsilon$ $\therefore$ T is continuous at $y \in \mathbb{R}^n$ If $T=0$ then it is constant function hence it is continuous on $\mathbb{R}^n$. (02) marks
	iii.	Let $f:\mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$. Define $D_i f(a)$, the i-th partial derivative of f at $a, 1 \leq i \leq n$. Determine whether the partial derivatives of f exist at $(0,0)$ for the following function. In case they exist, find them. $f(x,y) = \|(x,y)\|^4$
	Ans	Let $f:\mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$. Define $D_i f(a)$, the i-th partial derivative of f at $a, 1 \leq i \leq n$. Determine whether the partial derivatives of f exist at $(0,0)$ for the following function. In case they exist, find them. $f(x,y) = \|(x,y)\|^4 = (x^2+y^2)^2$ $D_1 f(a) = \lim_{t\to 0} \frac{f(a+te_1)-f(a)}{t}$ $D_1 f(0,0) = \lim_{t\to 0} \frac{f(0+t,0)-f(0,0)}{t}$ $= \lim_{t\to 0} \frac{t^4-0}{t}$ $= \lim_{t\to 0} t^3$ $= 0$

		$D_2f(a) = \lim_{t\to 0}\frac{f(a+te_2)-f(a)}{t}$ $D_2f(0,0) = \lim_{t\to 0}\frac{f(0,0+t)-f(0,0)}{t}$ $= \lim_{t\to 0}\frac{t^4-0}{t}$ $= \lim_{t\to 0} t^3$ $= 0$
	iv.	Let $f:\mathbb{R}^2\to\mathbb{R}, a=(-1,2), u=(3,-4), v=(12,5)$ and $w=(15,1)$. If $D_uf(a)=8$, $D_vf(a)=1$ find $D_wf(a)$.
	Ans	$f:\mathbb{R}^2\to\mathbb{R}$. We know that , $D_uf(a)=\langle\nabla f(a),u\rangle$ and $D_vf(a)=\langle\nabla f(a),v\rangle$ Observe that, $w=(15,1)=(3,-4)+(12,5)=u+v$ $\therefore D_wf(a) = \langle\nabla f(a),w\rangle$ $=\langle\nabla f(a),u+v\rangle$ $=\langle\nabla f(a),u\rangle+\langle\nabla f(a),v\rangle$ $=D_uf(a)+D_vf(a)$ $=8+1$ $=9$
Q3.	Attempt any **ONE** question from the following: (08)	
a)	i.	Let U be an open set in $\mathbb{R}^n$ and $f:U\to\mathbb{R}$ be differentiable at $a\in U$ .Prove that $D_if(a)$ exists for each $i=1,2,\ldots\ldots,n$.Explain with an example that converse of this is not true.
	Ans	Since $f'(a,y)=\lim_{h\to 0}\frac{f(a+hy)-f(a)}{h}$ exist $\forall y\in\mathbb{R}^n$. Take $y=\bar{e_1}$ then $f'(a,\bar{e_1})$ exists and $f'(a,\bar{e_1})=\lim_{h\to 0}\frac{f((a_1,\ldots\ldots,a_n)+(h,0,0,\ldots,0))-f(a)}{h}$ $f'(a,\bar{e_1})=\lim_{h\to 0}\frac{f((a_1+h,\ldots\ldots,a_n))-f(a)}{h}=D_1f(a)$ exists. Similarly we can prove for $D_if(a)$, $\forall i$ exist. Counter example Let $f(x,y)=\begin{cases}\frac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$ is not differentiable at (0,0) but $f_x(0,0)=0$ *and* $f_y(0,0)=0$
	ii.	State and prove sufficient condition for the equality of mixed partial derivatives.

	Ans	**Statement**......................(3 Marks) Steps in the Proof: Considering the rectangle with corners $(a_1, a_2), (a_1 + h, a_2), (a_1 + h, a_2 + k), (a_1, a_2 + k)$, define $G(h,k) = f(a_1 + h, a_2 + k) - f(a_1 + h, a_2) - f(a_1, a_2 + k) + f(a_1, a_2)$ Let $\emptyset(x) = f(x, a_2 + k) - f(x, a_2)$ $\emptyset(x)$ is continuous and differentiable on rectangle. Apply Lagrange's mean value theorem to $\emptyset(x)$ in the interval $[a_1, a_1 + h]$ $G(h,k) = \emptyset(a_1 + h) - \emptyset(a_2) = \emptyset'(\theta_1)h$ where $a_1 < \theta_1 < a_1 + h$ Define $\Psi(y) = f_x(\theta_1, y)$ Apply Lagrange's mean value theorem to $\Psi(x)$ in the interval $[a_2, a_2 + k]$ Thus $G(h,k) = hkf_{xy}(\theta_1, \theta_2)$ Apply the same procedure to $u(y) = f(a_1 + h, y) - f(a_1, y)$ And show that mixed partial derivatives are equal.
Q3.	Attempt any **TWO** questions from the following:	(12)
b)	i.	Find total derivative as linear transformation T for the function $f(x,y) = x^2 + 2xy + y^2$at point $a = (-1,-2)$.
	Ans	total derivative as linear transformation is $T_a(v) = \nabla f(a) \cdot v$ let $v = (x,y)$ and $\nabla f(a) = \left(f_x(a), f_y(a)\right) = (-6,-6) \Rightarrow T_a(v) = -6x \pm 6y$.
	ii.	Find directional derivative of $f(x,y) = x^2 - 3xy$ at $(1,2)$ along the parabola $y = x^2 - x + 2$ at $(1,2)$.
	Ans	Formula is $D_u f(a) = \nabla f(a) \cdot T$.where T is unit tangent vector to the surface. $\nabla f(a) = \left(f_x(a), f_y(a)\right) = (-4,-3)$ Tangent vector=$(1,2t-1) \Rightarrow T = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \Rightarrow D_u f(a) = \frac{-7}{\sqrt{2}}$.
	iii.	Find the equation of the tangent plane and normal line to the surface$x^3 + 7x^2z + z^3 = 4$ at $(2,1,-2)$.
	Ans	Equation of tangent plane is $f_x(x-2) + f_y(y-1) + f_z(z+2) = 0$ $12x - 28y + 12z + 28 = 0$ Equation of normal line is $\frac{x-2}{12} = \frac{y-1}{-28} = \frac{z+2}{12}$.
	iv.	Evaluate the total derivative of $z = 4x^3y + 7x^2y^3$ where $x = 4 + 4t^4$ and $y = 1 - 2t^2$, using chain rule.
	Ans	$(12x^2y + 14xy^3)(16t^3) + (4x^3 + 21x^2y^2)(-4t)$

Q4.	Attempt any **ONE** question from the following: (08)	
a)	i.	State and prove Taylor's Theorem for a real valued function of two variables.
	Ans	(see below)

Statement : Let S be a non-empty open subset of $\mathbb{R}^2$ as $S = \{(a+th, b+tk) \,/\, t \in [0,1]\}$.
Suppose $f: S \longrightarrow \mathbb{R}$ has continuous partial derivatives till order $n+1$ then

$$f(a+th, b+tk) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) \qquad \text{2M}$$

$$+\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b) + \cdots\cdots\cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a,b)$$

$$+\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(a+\theta h, b+\theta k), 0 < \theta < 1$$

Proof : Define $g : [0,1] \longrightarrow \mathbb{R}$ $by\ g(t) = f(a+th, b+tk)$ $\therefore g(0) = f(a,b)$ $and\ g(1) = f(a+h, b+k)$ ---------(1) 1M

Since f has continuous partial derivatives till order $n+1 \Longrightarrow g$ is $n+1$ times differentiable.

By Taylor's theorem of one variable $\exists\ 0 < \theta < 1$ such that

$$g(t) = g(0) + t\,g'(0) + \frac{t^2}{2!}g''(0) + \cdots\cdots\cdots + \frac{t^n}{n!}g^n(0) + \frac{t^{n+1}}{(n+1)!}g^{n+1}(\theta) \quad ----(*)$$

1M

Let $r(t) = (a+h, b+k) \Longrightarrow g(t) = f(r(t))$

By chain rule,

$$g'(t) = \nabla f(r(t)) \cdot r'(t) \Longrightarrow g'(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(r(t)) \Longrightarrow g'(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) \qquad \text{1M}$$

By differentiating $g'(t)$ again w.r.t $'t'$ we get $g''(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a+th, b+tk)$

$\therefore g''(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b),$ $Similarly\ g^m(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^m f(a,b),$ $m = 1\ to\ n$

and $g^{(n+1)}(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a+th, b+tk)$ 1M

put in (*) with $t = 1$ we get

$$g(1) = g(0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b)$$

$$+\cdots\cdots\cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a,b)$$

$$+\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(a+\theta h, b+\theta k)$$

Hence, $f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b) +$

$$\cdots\cdots\cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a,b)$$

$$+\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(a+\theta h, b+\theta k) \qquad \text{1M}$$

	ii.	Let $Q(x,y) = Ax^2 + 2Bxy + Cy^2$ be a function of two variables and $\Delta = AC - B^2$. Then prove that (1) if $\Delta > 0$ and $A > 0$ then $Q(x,y) > 0\ \forall\ (x,y) \in \mathbb{R}^2, (x,y) \neq (0,0)$. (2) if $\Delta > 0$ and $A < 0$ then $Q(x,y) < 0\ \forall\ (x,y) \in \mathbb{R}^2, (x,y) \neq (0,0)$. if $\Delta < 0$, then in every open ball around origin there exist points (x,y) such that $Q(x,y) < 0$ and also there exist points (x,y) such that $Q(x,y) > 0$.
	Ans	$AQ(x,y) = A^2x^2 + 2ABxy + CAy^2 = (Ax + By)^2 + \Delta y^2$. If $\Delta > 0, AQ(x,y) > 0$. So $A > 0$ then $Q(x,y) > 0$ and $A < 0$ then $Q(x,y) < 0, \forall\ (x,y) \in \mathbb{R}^2, (x,y) \neq (0,0)$. If $\Delta < 0$, three cases. Case 1: $A \neq 0$. $A > 0$ gives $A\Delta < 0$ and $A < 0$ gives $A\Delta > 0$. $Q(Bt, -At) = A(Bt)^2 + 2B(Bt)(-At) + C(-At)^2 = t^2 A\Delta$. $Q(t, 0) = At^2$. Since A and $A\Delta$ have opposite signs $Q(Bt, -At)$ and $Q(t, 0)$ will have opposite signs. Hence by taking t small enough we can see that every open ball around origin contains some points for which $Q(x,y) < 0$ and some points where $Q(x,y) > 0$. Case 2: $A = 0, C \neq 0$. $\Delta = AC - B^2 = -B^2 < 0$. Thus C and $C\Delta$ have opposite signs. $Q(Ct, -Bt) = A(Ct)^2 + 2B(Ct)(-Bt) + C(-Bt)^2 = Ct^2\Delta$. $Q(0, t) = Ct^2$. Since C and $C\Delta$ have opposite signs $Q(Ct, -Bt)$ and $Q(0, t)$ will have opposite signs. Hence by taking t small enough we can see that every open ball around origin contains some points for which $Q(x,y) < 0$ and also some points where $Q(x,y) > 0$. Case 3: $A = 0 = C$. $\Delta = AC - B^2 = -B^2 < 0$. $Q(t,t) = 2Bt^2$. $Q(t, -t) = -2Bt^2$. Hence $Q(t,t)$ and $Q(t,-t)$ have opposite signs. Hence by taking t small enough we can see that every open ball around origin contains some points for which $Q(x,y) < 0$ and some points where $Q(x,y) > 0$.

Q4. Attempt any **TWO** questions from the following: (12)

b) i. Given $z = f(x,y)$ where f has continuous partial derivatives of second order, $x = u + v, y = u - v$, show that $\frac{\partial^2 z}{\partial u\,\partial v} = \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}$.

Ans

$\frac{\partial x}{\partial u} = \frac{\partial x}{\partial v} = 1$, $\frac{\partial y}{\partial u} = 1$, $\frac{\partial y}{\partial v} = -1$ 1M

using chain rule, $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}$ 1M

diff $\frac{\partial z}{\partial v}$ w. r. t $'u'$

$$\frac{\partial^2 z}{\partial u\,\partial v} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right)$$

$= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial u} + \frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial y}{\partial u} - \frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial u} - \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial u}$ 2M

$= \frac{\partial^2 z}{\partial x^2}(1) + \frac{\partial^2 z}{\partial x\,\partial y}(1) - \frac{\partial^2 z}{\partial x\,\partial y}(1) - \frac{\partial^2 z}{\partial y^2}(1)$ 1M

$= \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}$ 1M

ii. a) If $f(x,y,z) = xi + yj + zk$ then prove that the Jacobian matrix $Df(x,y,z)$ is the identity matrix of order 3.

b) find all differentiable vector fields $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ for which the Jacobian matrix $Df(x,y,z)$ is a diagonal matrix of form $diag\ (p(x), q(y), r(z))$ where p, q, r are given continuous functions.

Ans

a) $f(x,y,z) = x\,\bar{\imath} + y\,\bar{\jmath} + z\,\bar{k}$

$Df(x,y,z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which is identity matrix of order 3. 2M

b) Let $f(x,y,z) = (f_1, f_2, f_3)$ where f_1, f_2, f_3 are functions of x, y, z.
Since $Df(x,y,z)$ is diagonal matrix of the $diag(p(x),\ q(y),\ r(z))$.

$\frac{\partial f_1}{\partial x} = p(x)$, $\frac{\partial f_2}{\partial y} = q(y)$, $\frac{\partial f_3}{\partial z} = r(z)$

$f_1 = P(x) + a$, $f_2 = Q(x) + b$, $f_3 = R(x) + c$ where $a, b, c \in \mathbb{R}$

and $\frac{d}{dx}(P(x)) = p(x)$, $\frac{d}{dy}(Q(y)) = q(y)$, $\frac{d}{dz}(R(z)) = r(z)$

$\therefore f(x,y,z) = (P(x) + a, Q(y) + b, R(z) + c)$ 4M

iii. Find the critical points, saddle points and local extrema if any for the function $f(x,y) = x^3 + xy^2 - 12x^2 - 2y^2 + 21x$.

Ans

Critical points	Δ	f_{xx}	Extrema	$f(x,y)$
(1,0)	36	-18	maxima	10

(7,0)	240	24	Minima	−98
$(2,\sqrt{15})$	-24		Saddle point	
$(2,-\sqrt{15})$	-24		Saddle point	

iv. Find the points on the surface $z^2 = xy + 1$ nearest to the origin. Also find the distance.

Ans $h(x,y) = x^2 + y^2 + z^2 + \lambda(z^2 - xy - 1)$. $\frac{\partial h}{\partial x} = 2x - \lambda y = 0, \frac{\partial h}{\partial y} = 2y - \lambda x = 0, \frac{\partial h}{\partial z} = 2z + 2z\,\lambda = 0$. Substituting in $z^2 = xy + 1$ we will get $\lambda = -1$. So $x = y = 0, z = 1, -1$. $f(0,0,\pm 1) = 1$.

Q5. Attempt any **FOUR** questions from the following: (20)

a) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \lim_{(x,y)\to(0,0)} \frac{e^{-\left(\frac{1}{x^2+y^2}\right)}}{x^2+y^2} \quad \text{if } (x,y) \neq (0,0).$$

Define $f(0,0)$ so that f is continuous at origin.

12

Ans	Q:5 i) Let $f:\mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \frac{e^{-\frac{1}{x^2+y^2}}}{x^2+y^2}$ if $(x,y) \neq (0,0)$. Define $f(0,0)$ so that f is continuous at origin. Ans: Let $x = r\cos\alpha$, $y = r\sin\alpha$ $\therefore \lim_{(x,y)\to(0,0)} \frac{e^{-\frac{1}{x^2+y^2}}}{x^2+y^2} = \lim_{r\to 0} \frac{e^{-\frac{1}{r^2}}}{r^2}$ (02) Marks $= \lim_{r\to 0} \frac{1/r^2}{e^{1/r^2}}$ ($\frac{\infty}{\infty}$ form) $= \lim_{r\to 0} \frac{(-2)\times\frac{1}{r^3}}{e^{1/r^2}\times\frac{-2}{r^3}}$ $= \lim_{r\to 0} \frac{1}{e^{1/r^2}}$ $= 0$ (03) marks
b)	Find the real value of $\theta \in (0,1)$ if it exists, satisfying $f(b) - f(a) = < \nabla f(a + \theta(b-a)), b - a >$ For the following function : $f(x,y,z) = x^2 + y^2 + 2xz,\ \ a = (0,0,0),\ \ b = \left(1,\frac{1}{2},\frac{1}{3}\right)$
Ans	To find $\nabla f\big(a + \theta(b-a)\big)$: $f_x(x,y,z) = 2x + 2z; f_y(x,y,z) = 2y\ ; f_z(x,y,z) = 2x$ $a + \theta(b-a) = (0,0,0) + \theta\left(1,\frac{1}{2},\frac{1}{3}\right) = \left(\theta,\frac{\theta}{2},\frac{\theta}{3}\right)$ $Therefore\ \nabla f\big(a+\theta(b-a)\big) = \left(2\theta + \frac{2\theta}{3},\theta,2\theta\right) = \left(\frac{8\theta}{3},\theta,2\theta\right)$ Substituting this in $f(b) - f(a) = \nabla f\big(a+\theta(b-a)\big)\cdot(b-a)$ We have, $f\left(1,\frac{1}{2},\frac{1}{3}\right) - f(0,0,0) = \langle\left(\frac{8\theta}{3},\theta,2\theta\right),\left(1,\frac{1}{2},\frac{1}{3}\right)\rangle$

	$\frac{23}{12} = \frac{23\theta}{6} \Rightarrow \theta = \frac{1}{2}$.
$c)$	Find level surfaces of $f(x,y,z) = x^2 + y^2 + z^2$ for the constants $K = 1,9$.
Ans	For $K = 1$,we get equation of sphere of radius 1 and center (0,0,0) For $K = 9$,we get equation of sphere of radius 3 and center (0,0,0)
$d)$	Let $f(x,y) = x^2y^3 + 2y^5$, find $f_{x,}, f_y, f_{xy}, f_{xx}, f_{yy}$.
Ans	$f_x = 2xy^3 f_y = 3x^2y^2 + 10y^4$ $f_{xy} = 6xy^2 f_{xx} = 2y^3 f_{yy} = 6x^2y + 40y^3$
$e)$	Using Taylors formula find the quadratic approximation for the quantities $\alpha = (0.99)^3 + (2.01)^3 - 6(0.99)(2.01)$
Ans	Let $f(x,y,z) = x^3 + y^3 - 6xy$, $a = 1$, $b = 2$, $h = -0.01$, $k = 0.01$ $f_x = 3x^2 - 6y, f_y = 3y^2 - 6x$, $f_{xx} = 6x$, $f_{xy} = -6$, $f_{yy} = 6y$ Let $p = (1,2)$, $f(p) = -3,\ f_x = -9$, $f_y = 6$, $f_{xx} = 6$, $f_{xy} = -6$, $f_{yy} = 12$ $\alpha \doteq f(p) + hf_x(p) + kf_y(p) + \frac{1}{2!}\left[h^2 f_{xx}(p) + 2hkf_{xy}(p) + k^2 f_{yy}(p)\right]$ put above values and simplify $\alpha \doteq -2.8485$
$f)$	Find the Hessian matrix of $f: \mathbb{R}^3 \to \mathbb{R}$ given by $f(x,y,z) = x^3 + y^3 + z^3 + 3xyz + 3x^2y + 3y^2x + 3z^2x + 3x^2z$ at (1,1,1).
Ans	$\begin{bmatrix} 18 & 15 & 15 \\ 15 & 12 & 3 \\ 15 & 3 & 12 \end{bmatrix}$

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