M.SC. (MATHEMATICS) PART-II
Algebra and Field Theory
(Rev.) (P-I) (JUNE - 2019)

(3 Hours) **[Total Marks:80**

Instructions:
- Attempt **any two** questions from **each section**
- **All** questions carry **equal marks**.
- Answer to **section I** and **II** should be written on the **same answer book**

SECTION I (Attempt any two questions)

1. (a) Prove that subgroup of solvable group is solvable
   (b) Prove that a group of order 42 cannot be simple
2. (a) Calculate the character tables of $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3$
   (b) State and prove Jordan-Holder theorem
3. (a) (i) State (without proof) the Hilbert basis theorem. Define the terms: Noetherian ring, Noetherian module.
   (ii) Prove that any Artinian ring has finitely many maximal ideals.
   (b) Show that a submodule of a free module over a PID is free
4. (a) Let M be an R-module. Prove that a subset N of a M is a submodule of M if and only if N is non-empty and closed under addition as well as scalar multiplication from R.
   (b) (i) Find rational canonical form of $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}$
   (ii) Find Jordan canonical form of $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

SECTION II (Attempt any two questions)

5. (a) Define algebraic extension. Prove that if F is a field of characteristic 0 and a, b are algebraic over F then there is an element c in $F(a, b)$ such that $F(a, b) = F(c)$
   (b) Define a splitting field. Find splitting field for $f(x) = x^4 - x^2 - 2$ over $\mathbb{Q}$
6. (a) Define a splitting field of a polynomial $f(x)$ over a field K. If $f(x)$ is a monic polynomial over a field K , prove that there exist a splitting field of $f(x)$ over K
   (b) Determine splitting field and its degree over $\mathbb{Q}$ for the polynomial $x^{11} - 1$
7. (a) Prove that the field of complex number is algebraically closed

[TURN OVER

(b) Find the Galois group of $x^3 - 2$ over a field $\mathbb{Q}$

8. (a) Prove that it is impossible by ruler and compass to trisect the angle $\frac{\pi}{3}^c$

   (b) Define constructible number. If a and b are constructible numbers then prove that $a + b$ and $a - b$ are constructible

M.SC. (MATHEMATICS) PART-II
Advanced Analysis and Fourier Analysis (Rev.) (P-II) (JUNE - 2019)

[Total marks: 80]

Instructions:

1) **Attempt any two questions from each section.**
2) **All questions carry equal marks.**
3) **Answer to Section I and section II should be written in the same answer book.**

SECTION-I (Attempt any two questions)

Q.1]

A) Prove that " Let A be a rectangle in $\mathbb{R}^n$, A bounded function $f: A \to \mathbb{R}$ is integrable if and only if $\forall \in > 0$, there is a partition P such that $U(f,P) - L(f,P) < \in$ ". [10]

B) Let $A \subset \mathbb{R}^n$ be a rectangle show that A does not have measure zero. [10]

Q.2]

A) Show that exterior measure of an open rectangle in $\mathbb{R}^n$ is its volume. [10]

B) Show that the intersection of a countable collection of measurable set is measurable. [10]

Q.3]

A) State and prove Lusin's theorem. [10]

B) Define $f(x) = \frac{1}{x^{2/3}} \quad 0 < x < 1$ [10]

$= 0 \quad x = 0$

Show that f is Lebesgue integrable on [0, 1] and $\int_0^1 \frac{1}{x^{2/3}} \, dx = 3$. Also find $f(x, 2)$.

Q.4]

A) Show that $L^1(\mathbb{R}^n)$ is complete in its metric. [10]

B) Use the dominated, convergence theorem to evaluate $\lim_{n\to\infty} \int_0^1 \left(\frac{1+x}{n}\right)^n e^{-2x} \, dx$. [10]

SECTION-II (Attempt any two questions)

Q.5]

A) Find the Fourier coefficient and hence the Fourier series of the function $f(x) = xsinx$ where $0 \le x \le 2\pi$. [06]

B) Let f be an integrable function on the circle which is differentiable at a point x_0 then show that $S_N(f)(x_0) \to f(x_0)$ as $N \to \infty$. [08]

C) Let f be an integrable periodic function with period 2π with $\hat{f}(0) = 0$. Let $F(t) = \int_0^t f(x)dx$. show that F is continuous, 2π periodic and $\hat{F}(n) = \frac{1}{in} \hat{f}(n), \ n \neq 0$. [06]

Q.6]

A) Show that any orthonormal subset of a Hilbert space is linearly independent. [08]

B) Prove that foe any $f, g \in H$, $\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2)$. [06]

C) Show that $S^\perp$ is a closed subspace of H, and $S \cap S^\perp = \{0\}$. [06]

Q.7]

A) Show that $L^2[-\pi,\pi]$ is a separable space. [08]

B) If $f \in L^2[-\pi,\pi]$, then show that $\sum_{-\infty}^{\infty}\left|\hat{f}(n)\right|^2 \leq \|f\|^2$. [06]

C) Show that any two infinite dimensional Hilbert spaces are unitary isomorphic. [06]

Q.8]

A) Show that if the series $\sum_{k=0}^{\infty} c_k$ of complex numbers converges to a finite limit s, then the series is Abel summable to s. [10]

B) If $0 \leq r \leq 1$ then prove that $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta) = 1$. [10]

[Marks: 80]

N.B. 1) All questions are compulsory and carry equal marks.
2) Solve any **TWO** from Section A.
3) Solve any **TWO** from Section B.

Section A

1. (a) (i) Define a hyperplane in $I\!R^n$ and reflection with respect to a hyperplane in $I\!R^n$. Is a reflection an isometry? Is it an orthogonal transformation? Justify you answers. (5)

(ii) Find the equation of the plane through the points $(1, -1, 3)$, $(2, 0, 5)$ and $(-1, 0, 7)$. Find the normal vector and distance of the origin from this plane. (5)

(i) Classify and derive expression for all isometries of plane $I\!R^2$. (5)

(ii) Define a rotation in $I\!R^3$ and prove that it is an element of $SO(3)$. (5)

2. (a) Calculate the curvature and torsion of the curve $\gamma(s) = (s - s^2, s^2, 2s)$. (10)

(b) (i) Prove that a plane curve $\gamma : [a, b] \to I\!R^2$ with constant curvature k_0 is either a straight line segment or an arc of a circle. (5)

(ii) Define a regular parametrized curve. Prove that a curve has a unit speed reparametrization if and only if it is a regular curve. (5)

3. (a) (i) Define a regular surface. Show that sphere is a regular surface by exhibiting an atlas of six coordinate charts (surface patches). (5)

(ii) Define a tangent vector and a tangent space of a regular surface. Prove that the dimension of tangent space of a regular surface is two. (5)

(b) (i) Derive the equation of a surface of revolution generated by a regular curve and prove that it is a regular surface. (5)

(ii) Prove that $C : \{(x, y, z) \in I\!R^3 : x^2 + z^2 = 4\}$ is a regular surface. Find the tangent space to C at a point $(2, 1, 0)$. (5)

4. (a) (i) Define the Gauss map and the Shape operator for a regular surface. Show that the Shape operator is self adjoint. (4)

(ii) Calculate the Gauss map and the shape operator for the following surfaces and: (6)
(a) Plane $ax + by + cz = d$
(b) Sphere $(x - 1)^2 + y^2 + (z - 2)^2 = 4$
(c) Cylinder $y^2 + z^2 = 9$.

(b) (i) Define a geodesic and a normal section. Prove that a normal section of a regular surface is a geodesic. (5)

(ii) Hence show that great circles are geodesics of Sphere. (3)

(iii) Prove that a straight line segment on any regular surface is a geodesic. (2)

Section B

1. (a) State and prove tha Arzela Ascoli theorem. (10)

   (b) Define a nowhere dense set. Show that if $\{A_n\}$ is a sequence of nowhere dense sets in a complete metric space X, then there exists a point in X which is not in any of the A_n's. (10)

2. (a) Prove that $L^2[a,b]$ is a Banach space under a suitable norm. (10)

   (i) State and derive expression for the Hölder and Minkowski's inequality. (6)

   (ii) Let $(X, ||\cdot||)$ be a finite dimensional Banach space. Prove or disprove with justification that $\{x \in X : ||x|| \leq 1\}$ is compact. (4)

3. (a) Let X be a normed linear space and $B(X, I\!R^n)$ denote the space of bounded linear transformations from X to $I\!R^n$. Prove that $B(X, I\!R^n)$ is a Banach space with a suitable norm. (10)

   (i) Define dual space of a Banach space. Find the dual space of $C[a,b]$ the set of continuous functions on $[a,b]$. (5)

   (ii) Find the dual space of the sequence space l^1. (5)

4. (a) (i) State and prove the closed graph theorem. (5)

   (ii) Let B be a Banach space and N a normed linear space. If $\{T_n\}$, $T_n : B \to N$ is a sequence of bounded linear transformations such that $T(x) = \lim T_n(x)$ exists for each x in B, then prove that T is a continuous linear transformation. (5)

   (b) State and prove the unifrom boundedness principle. (10)

M.SC. (MATHEMATICS) PART-II
Numerical Analysis (Rev)
(P-IV) (JUNE - 2019)

[Total Marks:80

Instructions:

(1) Attempt any two questions from each section.

(2) All questions carry equal marks. Scientific calculator can be used.

(3) Answer to Section-I and Section-II should be written in the same answer book

Section-I

Que. 1 (a) Define: Absolute error, Relative error and Percentage error.
Find the number of terms of the cosine series such that their sum gives the value of $\cos x$ correct to at least two significant digits at $x = \frac{\pi}{4}$.

(b) Convert the decimal fraction $(469.828125)_{10}$ to the binary form and then convert to the octal form.

Que. 2 (a) Let $x = \xi$ be a root of $f(x) = 0$ and let I be an interval containing ξ. Let $\phi(x)$ and $\phi'(x)$ be continuous in I, where $\phi(x)$ is defined by the equation $x = \phi(x)$ which is equivalent to $f(x) = 0$. Prove that if $|\phi'(x)| < 1$ for all $x \in I$, the sequence of approximations $x_{n+1} = \phi(x_n)$ converges to the unique root ξ, provided that the initial approximation $x_0 \in I$.

(b) Perform three iterations of the Newton-Raphson method to obtain the complex root of the equation $f(z) = z^3 + 1 = 0$. Use initial approximation $z_0 = 0.25 + 0.25i$.

Que. 3 (a) Use Given's method to find the eigenvalues of the following tridiagonal matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

(b) Describe Crout's method to solve the following system of linear equations:
$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$; $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$; $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$.

Que. 4 (a) Determine the step size h that can be used in the tabulation of a function $f(x)$, $a \leq x \leq b$, at equally spaced nodal points so that the truncation error of the cubic interpolation is less than ϵ.

(b) From the following table, find x, correct to two decimal places, for which y is maximum and find this value of y.

x:	1.2	1.3	1.4	1.5	1.6
y=f(x):	0.9320	0.9636	0.9855	0.9975	0.9996

Section-II

Que. 5 (a) Estimate the error in the Simpson's $1/3^{\text{rd}}$ rule.

(b) Evaluate the integral $\displaystyle\int_0^2 \frac{dx}{x^2+9}$ by Trapezoidal rule taking $h = 1.0, 0.5, 0.25$ and then use Romberg's method to get more accurate result correct to four decimal places.

Que. 6 (a) Let $y = f(x)$ be continuous function on $[a, b]$. Let the function $y(x)$ be approximated by $Y(x) = a_0 f_0(x) + a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x)$, where $f_j(x),\ j = 0, 1, 2, 3$ are orthogonal polynomials on $[a, b]$ of degree j with respect to the weight function $W(x)$. Find the unknown parameters a_0, a_1, a_2 and a_3 by least squares method.

(b) Using the Gram-Schmidt orthogonalization process obtain the first four orthogonal polynomials on $[0, 1]$ with respect to the weight function $W(x) = 1$.

Que. 7 (a) Derive the Adams-Moulton corrector formula to solve the differential equation $\dfrac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.

(b) Using Runge-Kutta method of fourth order, find an approximate value of $y(0.1)$ and $z(0.1)$, given that $\dfrac{dy}{dx} = x + yz,\ \dfrac{dz}{dx} = xz + y$ with $y(0) = 1,\ z(0) = -1$. Take $h = 0.1$.

Que. 8 (a) Derive a numerical method (Crank-Nicolson's method) to obtain the numerical solution of one dimensional heat equation with initial and boundary conditions.

(b) The hyperbolic equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$ satisfies the conditions $u(0, t) = 0,\ u(1, t) = 0$ for $t > 0$ and $u(x, 0) = \sin^3(\pi x)$ for $0 \le x \le 1$, $u_t(x, 0) = 0$ for $0 \le x \le 1$. Take $h = 0.25, k = 0.2$ and use explicit method to compute the values of $u(x, t) = u(ih, jk)$ for one time steps.

M.SC. (MATHEMATICS) PART-II
Graph Theory (Rev.) (P-V)
(JUNE - 2019)

(3 Hours) [Total Marks: 80]

N.B
1) Both the Sections are **Compulsory**.
2) Attempt **ANY TWO** questions from each Section.
3) Figures to the right indicate full marks.
4) Answers to section I and section II should be written in same answer book.

Duration : 3hrs

Section - I

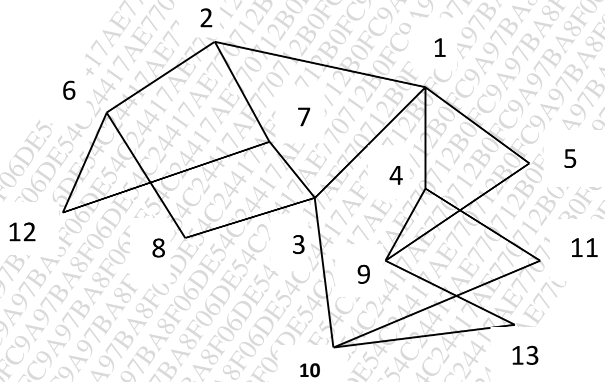
Q.1 20M

a) Prove that if G is a bipartite K -regular graph such that $K \geq 2$ then G has no bridge. 10M

b) For any graph G , if K is vertex connectivity, K' is edge connectivity and δ is minimum degree of the graph then prove that $K \leq K' \leq \delta$. 10M

Q.2 20M

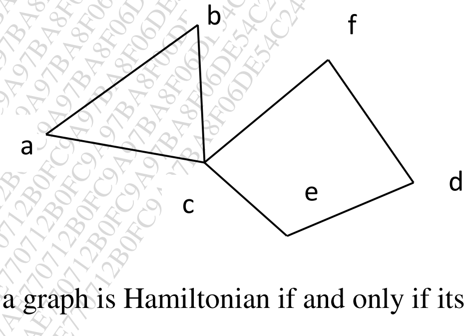
a) Write the steps of Breadth first search (BFS) algorithm and Implement it on the given graph. 10M



b) Write Algorithm for Huffman code and use it to encode the symbol with given frequency A: 0.10, B: 0.25, C: .05, D: 0.15 ,E: 0.30, F:0.07, G: 0.08. what is the average number of bits required to encode a symbol? 10M

Q.3 20M

a) Write an Flueury's algorithm and use it to find Eulerian circuit for the given graph. 10M



b) Prove that a graph is Hamiltonian if and only if its closure is Hamiltonian. 10M

Q.4 20M

a) For any graph G prove that $\alpha + \beta = p$. 10M
Where α : number of Independent set., β : number of independent covering.
and P: number of vertices.

b) Prove that i) $R(2, k) = k$ for all $k \geq 2$, ii) $R(3, 3) = 6$, where $R(s, t)$ is Ramsey number. 10M

Section II

Q.5 20M

a) If two graphs are isomorphic then prove that their corresponding line graphs are also isomorphic. What can you say about the converse? Give justification. 10M

b) Prove that there exist K- colouring of a graph G if and only if V(G) can be partitioned into k-subsets $V_1, V_2 ---- , V_K$ Such that no two vertices in V_i $i = 1,2,3,----,k$ are adjacent. 10M

Q.6 20M

a) Prove that there are exactly five regular polyhedron. 10M

b) Prove that every simple outer planar graph has a vertex of degree less than or equal to 2. 10M

Q.7 20M

a) Prove that a diagraph D is unilaterally connected if and only if there is a directed walk not necessarily closed, containing all its vertices. 10M

b) Prove that every tournament D contains a directed Hamiltonian path. 10M

Q.8 20M

a) Define Eigen value of a graph G and prove that if G be a connected graph with k distinct eigen values and let d be the diameter of G then k > d. 10M

b) Prove that the following are equivalent statements about a graph G 10M

(i) G is bipartite.
(ii) The non-zero eigenvalues of G occurs in pairs λ_i, λ_j such that $\lambda_i + \lambda_j = 0$ (with the same multiplicity).
(iii) p(G, x) is a polynomial in x^2 after factoring out the largest common power of x.
(iv) $\sum \lambda_i^{2t+1} = 0 \ \forall t \in N$.
